Lagrange Duality of Set-Valued Optimization with Natural Criteria*

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Abstract

Set optimization problems with objective set-valued maps are considered, and some criteria of solutions are defined. Also, cone lower semicontinuities of set-valued maps are introduced, and existence theorems of solutions of such problems are established. Moreover, some duality results of these problems are investigated.

1 Introduction

We observe a set-valued optimization problem (SP) as follows:

(SP) \quad \text{Minimize } F(x) \\
{} \quad \text{subject to } x \in S

where $X$ is a set, $(Y, \leq)$ an ordered vector space, $F$ a map from $X$ to $2^Y$, and $S$ a nonempty subset of $\text{Dom}(F) = \{x \in X \mid F(x) \neq \emptyset\}$. This type of set-valued optimization problem has been developed as a generalization of vector-valued optimization problems for around twenty years. In many paper concerned with set-valued optimization (for example [2, 5, 4, 6, 7, 11]), we can see that a minimal solution $x_0 \in S$ is defined such as:

\[ F(x_0) \cap \text{Min } \bigcup_{x \in S} F(x) \neq \emptyset \]

and this problem are often called ‘vector optimization with set-valued maps.’ However the criterion of solutions is sometimes not suitable for set-valued optimization because it is only based on simple comparisons between vectors though our problem is set-valued optimization.

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Our aim of this paper is to introduce some natural, suitable, and proper criteria based on comparisons between values of the (set-valued) objective map for set-valued optimization, and investigate some properties concerned with the problem. In this paper, we call such criteria based on the idea above natural criteria for set-valued optimization, see [9].

The organization of this paper is as follows: in Section 2, we formulate our set-valued optimization problem and define two types of notions of solutions. In Section 3, we introduce (natural) lower semicontinuities for set-valued maps, characterize such continuities, and derive some existence theorems of solutions by using the lower semicontinuities. Finally, we show some duality theorems for our set-valued optimization in Section 4.

## 2 Natural Criteria of Set-Valued Optimization

First, we redefine our set-valued minimization problem (SP). Let $X$ be a topological space, $(Y, \leq_Y)$ an ordered topological space with an ordering convex cone $K$, $F$ a map from $X$ to $2^Y$, and $S \subset \text{Dom}(F)(= \{x \in X \mid F(x) \neq \emptyset\})$. Our set-valued minimization problem is the following:

(SP) Minimize $F(x)$
subject to $x \in S$.

To define notions of solutions for our problem, we introduce some relations between two nonempty sets which like the order relation in topological vector spaces; though the number types of such relations is six, we treat two important relations of them, see [8].

In this paper, we define

$$A \leq^l B \overset{\text{def}}{\iff} A + K \supset B,$$

$$A \leq^u B \overset{\text{def}}{\iff} A \subset B - K,$$

for nonempty subsets $A, B$ of $Y$. In these cases, $A$ is said to be smaller than $B$ with $l$-inequality (resp. $u$-inequality) if $A \leq^l B$ (resp. $A \leq^u B$).

In the above notations, $l$ means lower and $u$ means upper. Note that $A \leq^l B$ is equivalent to $\text{Min} \ A = \text{Min} \ B$ and $A \leq^u B$ is equivalent to $\text{Max} \ A = \text{Max} \ B$.

By using the set relations above, we introduce two types criteria of minimal solutions in the following definition. In this paper, when we consider $l$-minimal solution, we assume that $F$ is $l$-closed map, that is $F(x)$ is $l$-closed for each $x \in X$ for simple consideration. Also we assume similar assumption, $u$-closedness of $F$, when we consider $u$-minimal solution.

**Definition 2.1** [9] An element $x_0 \in S$ is said to be

(i) $l$-minimal solution of (SP) if
for any $x \in S$ with $F(x) \leq^l F(x_0), F(x_0) \leq^l F(x)$ is satisfied;

(ii) $u$-minimal solution of (SP) if
for any $x \in S$ with $F(x) \leq^u F(x_0), F(x_0) \leq^u F(x)$ is satisfied.
3  Semicontinuities of Set-Values Maps and Existence Theorems

To consider existence of solutions of (SP) for our solutions, some cone semicontinuities were introduced in [5, 9].

Definition 3.1 [9] A set-valued map $F$ is said to be $l$-type lower semicontinuous on $S$ if for any $l$-closed subset $A$ of $Y$, $\mathcal{L}^l(A) = \{x \in S | F(x) \leq^l A\}$ is closed.

Definition 3.2 [9] A set-valued map $F$ is said to be $l$-type quasi lower semicontinuous at $x_0 \in S$ if for each net $\{x_\lambda\}$ converges to $x_0$ with $\{F(x_\lambda)\}$ is $l$-decreasing, that is, $F(x_\lambda') \leq^l F(x_\lambda)$ for $\lambda < \lambda'$, $F(x_0) \leq^l \limsup_\lambda (F(x_\lambda) + K)$ is satisfied. A set-valued map $F$ is said to be $l$-type quasi lower semicontinuous on $S$ if it is $l$-type quasi lower semicontinuous at each point of $S$.

Definition 3.3 [5] A set-valued map $F$ is said to be upper $K$-semicontinuous at $x_0 \in S$ if for any open set $V$ with $V \leq^l F(x_0)$, there exists a neighborhood $U$ of $x_0$ such that $x \in U$ implies $V \leq^l F(x)$; A set-valued map $F$ is said to be upper $K$-semicontinuous on $S$ if it is upper $K$-semicontinuous at each point of $S$.

Now we can see some characterization with respect to these lower semicontinuities.

Proposition 3.1 [9] Let $F$ be a $l$-closed set-valued map. Then we have the following:

(i) upper $K$-semicontinuity on $S$ implies $l$-type lower semicontinuity on $S$;
(ii) $l$-type lower semicontinuity on $S$ implies $l$-type quasi lower semicontinuity on $S$.

Also, if $X$ and $Y$ are finite dimensional and $F$ is locally bounded, then we have

(iv) $l$-type lower semicontinuity on $S$ implies upper $K$-semicontinuity on $S$.

Moreover, $Y$ is the real-field, and $F$ is a singleton map, then $l$-type lower semicontinuity and upper $K$-semicontinuity are equivalent to the ordinary lower semicontinuity of real-valued functions.

Note that quasi lower semicontinuity is more weaker than another semicontinuities.

Now, we investigate $u$-type lower semicontinuities of set-valued maps.

Definition 3.4 [9] A set-valued map $F$ is said to be $u$-type lower semicontinuous on $S$ if for any $u$-closed subset $A$ of $Y$, $\mathcal{L}^u(A) = \{x \in S | F(x) \leq^u A\}$ is closed.

Definition 3.5 [9] A set-valued map $F$ is said to be $u$-type quasi lower semicontinuous at $x_0 \in S$ if for each net $\{x_\lambda\}$ converges to $x_0$ with $\{F(x_\lambda)\}$ is $u$-decreasing, that is, $F(x_\lambda') \leq^u F(x_\lambda)$ for $\lambda < \lambda'$, $F(x_0) \leq^u \limsup_\lambda (F(x_\lambda) + K)$ is satisfied. A set-valued map $F$ is said to be $u$-type quasi lower semicontinuous on $S$ if it is $u$-type quasi lower semicontinuous at each point of $S$. 
Definition 3.6 [5] A set-valued map $F$ is said to be lower $K$-semicontinuous at $x_0 \in S$ if for any open set $V$ with $V \cap F(x_0) \neq \emptyset$, there exists a neighborhood $U$ of $x_0$ such that $x \in U$ implies $V \cap (F(x) - K) \neq \emptyset$; A set-valued map $F$ is said to be lower $K$-semicontinuous on $S$ if it is lower $K$-semicontinuous at each point of $S$.

Now we can see some characterization with respect to these lower semicontinuities.

Proposition 3.2 [9] Let $F$ be a $u$-closed set-valued map. Then we have the following:

(i) $u$-type lower semicontinuity on $S$ implies $u$-type quasi lower semicontinuity on $S$.

Also, if $X$ and $Y$ are finite dimensional and $F$ is locally bounded, then we have

(iii) $u$-type lower semicontinuity on $S$ is equivalent to lower $K$-semicontinuity on $S$.

Moreover, $Y$ is the real-field, and $F$ is a singleton map, then $u$-type lower semicontinuity and lower $K$-semicontinuity are equivalent to to the ordinary lower semicontinuity of real-valued functions.

Now, we consider existence theorems for $l$-type and $u$-type minimal solutions.

Theorem 3.1 [9] Let $X$ be a topological space and $Y$ an ordered topological vector space. If $S$ is a nonempty compact subset of $X$ and $F : S \rightarrow 2^Y$ is a $l$-type quasi lower semicontinuous and $l$-closed set-valued map, then there exists a $l$-type minimal solution of (SP).

Theorem 3.2 [9] Let $X$ be a topological space and $Y$ an ordered topological vector space. If $S$ is a nonempty compact subset of $X$ and $F : S \rightarrow 2^Y$ is a $u$-type quasi lower semicontinuous and $u$-closed set-valued map, then there exists a $u$-type minimal solution of (SP).

By using one of the above theorems, we can prove the following:

Corollary 3.1 Let $X$ be a topological space, $S$ a nonempty compact subset of $X$, and $f : S \rightarrow 2^Y$ is a lower semicontinuous, then there exists an element $x_0 \in S$ such that $f(x_0) = \inf_{x \in S} f(x)$.

Let $Y^*$ be the topological dual space of $Y$, $\theta^*$ the null vector of $Y^*$, and $K^+$ the positive polar cone of $K$, that is, $K^+ = \{y^* \in Y^*| \langle y^*, k \rangle \geq 0, \forall k \in K\}$.

Theorem 3.3 [9] Let $(X, d)$ be a complete metric space, $Y$ an ordered locally convex space with the cone $K$. Also, $F$ be a map from $X$ to $2^Y$ satisfying the following conditions:

- there exists $y^* \in K^+ \setminus \{\theta^*\}$ such that

  $\inf \langle y^*, F(x) \rangle$ is finite for each $x \in S$

  $F(x_1) \preceq^l F(x_2), x_1, x_2 \in S \Rightarrow \inf \langle y^*, F(x_2) \rangle - \inf \langle y^*, F(x_1) \rangle \geq d(x_2, x_1)$
• $F : S \rightarrow 2^Y$ is $l$-type lower semicontinuous and $l$-closed.

Then, there exists a $l$-type minimal solution of (SP).

**Theorem 3.4** [9] Let $(X, d)$ be a complete metric space, $Y$ an ordered locally convex space with the cone $K$. Also, $F$ be a map from $X$ to $2^Y$ satisfying the following conditions:

• there exists $y^* \in K^+ \setminus \{\theta^*\}$ such that
  \[
  \text{inf} \langle y^*, F(x) \rangle \text{ is finite for each } x \in S
  \]
  \[
  F(x_1) \leq^u F(x_2), x_1, x_2 \in S \Rightarrow \sup \langle y^*, F(x_2) \rangle - \sup \langle y^*, F(x_1) \rangle \geq d(x_2, x_1)
  \]

• $F : S \rightarrow 2^Y$ is $u$-type lower semicontinuous and $u$-closed.

Then, there exists a $u$-type minimal solution of (SP).

Using one of the above theorems, we can show Phelps’ theorem, see [1]:

**Corollary 3.2** Let $(Y, \|\cdot\|)$ be a Banach space, $D$ a closed nonempty subset of $Y$, and $K$ a convex cone of $Y$. If there exist $y^* \in K^+$ and $\alpha > 0$ such that

(i) $\langle y^*, \cdot \rangle$ is bounded from below on $D$ and

(ii) $K \subset \{y \in Y | \langle y^*, y \rangle + \alpha \|y\| \leq 0\}$.

Then $\text{Min} D = \text{Ext}_K D \neq \emptyset$.

### 4 Duality of Set-Valued Optimization

In this section, we consider a $l$-type set-valued minimization problem with an inequality constraint (SP) and its dual problem (SD).

\[
\begin{align*}
\text{(SP)} & \quad l-\text{Minimize} & F(x) \\
& \text{subject to} & G(x) \leq^l \theta
\end{align*}
\]

\[
\begin{align*}
\text{(SD)} & \quad l-\text{Maximize} & \Phi(T) \\
& \text{subject to} & T \in \mathcal{L}_+(Y, Z)
\end{align*}
\]

where, $X$ is a nonempty set, $(Y, \leq_K), (Z, \leq_L)$ ordered vector spaces with ordering cones $K, L$, respectively, $F : X \rightarrow 2^Z, G : X \rightarrow 2^Y, \mathcal{L}(Y, Z) = \{T : Y \rightarrow Z \mid T \text{ is linear}\}$, $\mathcal{L}_+(Y, Z) = \{T \in \mathcal{L}(Y, Z) \mid T(K) \subset L\}$, $\text{Gr}(G) = \{(x, y) \in X \times Y \mid y \in G(x)\}$ and $\Phi : \mathcal{L}(Y, Z) \rightarrow 2^Z$ defined by $\Phi(T) = l-\text{Min}\{F(x) + T(y) \mid (x, y) \in \text{Gr}(G)\}$.

**Definition 4.1** (Solutions) $x_0$ is said to be

(i) an $l$-feasible solution of (SP) if $G(x) \leq^l \theta$;
(ii) an \(l\)-solution of (SP) if \(x_0\) is \(l\)-feasible and
\[
x \in X, G(x) \leq^l \theta, F(x) \leq^l F(x_0) \text{ implies } F(x_0) \leq^l F(x).
\]

\(T_0\) is said to be

(i) a feasible solution of (SD) if
\[
T_0 \in \mathcal{L_+}(Y, Z) \text{ and } \Phi(T) \neq \emptyset;
\]

(ii) an \(l\)-solution of (SD) if \(T_0\) is feasible and there exists \(A_0 \in \Phi(T_0)\) such that
\[
T_1 \in \mathcal{L_+}(Y, Z), A_1 \in \Phi(T), A_0 \leq^l A_1 \text{ implies } A_1 \leq^l A_0.
\]

**Proposition 4.1 (Weak Duality)**

Let \(x_0\) be an \(l\)-feasible solution of (SP), \(T_1\) an \(l\)-feasible solution of (SD), and \((x_1, y_1)\) an element of \(\text{Gr}(G)\) satisfying \(F(x_1) + T_1(y_1) \in \Phi(T_1)\). Then,
\[
F(x_0) \leq^l F(x_1) + T_1(y_1) \text{ implies } F(x_1) + T_1(y_1) \leq^l F(x_0).
\]

**Definition 4.2 (Lagrangian Function)** For \(x \in X, y \in Y, T \in \mathcal{L}(Y, Z)\),
\[
L(x, y, T) = F(x) + T(y).
\]

In usual, \(y\) is an element of \(G(x)\).

**Definition 4.3 (Saddle Point)**

\((x_0, y_0, T_0) \in \text{Gr}(G) \times \mathcal{L_+}(Y, Z)\) is said to be an \(l\)-saddle point of \(L\) if

(i) \(L(x, y, T_0) \leq^l L(x_0, y_0, T_0), (x, y) \in \text{Gr}(G) \Rightarrow L(x_0, y_0, T_0) \leq^l L(x, y, T_0)\)

(ii) \(L(x_0, y_0, T_0) \leq^l L(x_0, y_0, T), T \in \mathcal{L_+}(Y, Z) \Rightarrow L(x_0, y_0, T_0) \leq^l L(x_0, y_0, T)\)

**Theorem 4.1** Assume that \(K\) is closed, \(L\) is solid, and \(F\) satisfies the following bounded condition: for each \(x \in \text{Dom}(F)\) there exists \(y^* \in K^+\) such that

- \(\langle y^*, y \rangle > 0\) for each \(y \in K \setminus \{\theta\}\);
- \(\inf_{y \in F(x)} \langle y^*, y \rangle > -\infty\).

If \((x_0, y_0, T_0) \in \text{Gr}(G) \times \mathcal{L_+}(Y, Z)\) is an \(l\)-saddle point of \(L\), then we have

(i) \(y_0 \leq \theta\) and \(T_0(y_0) = \theta\);

(ii) \(x_0\) is an \(l\)-solution of (SP);

(iii) \(T_0\) is an \(l\)-solution of (SD).

**Theorem 4.2** \((x_0, y_0, T_0) \in \text{Gr}(G) \times \mathcal{L_+}(Y, Z)\) is an \(l\)-saddle point of \(L\) if and only if

(i) \(L(x, T_0) \leq^l L(x_0, y_0, T_0), (x, y) \in \text{Gr}(G) \Rightarrow L(x_0, y_0, T_0) \leq^l L(x, y, T_0)\)

(ii) \(y_0 \leq \theta\) and \(T_0(y_0) = \theta\).
References


