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Some Pseudo-Order of Fuzzy Sets on $\mathbb{R}^n$

Abstract

The aim of this paper is to define an order on a class of fuzzy sets which is extending a pseudo-order for fuzzy numbers, and its characterization and several relations of the previous results are discussed. The idea comes from a set-relation in $n$-dimensional Euclid space given by Kuroiwa, Tanaka and Ha (1997). We induce the order of a class of fuzzy sets by a closed convex cone and characterize it by using the projection into the dual cone. Especially, a structure of the lattice is described for the class of rectangle-type fuzzy sets.

Keywords: Pseudo-order, fuzzy max order, multidimensional fuzzy sets, rectangle-type fuzzy sets.

1. Introduction and notations

In the theory of optimization it is a quite important problem how to induce a natural definition of order on the class of considering systems. Since it isn’t a simple problem about a order on the fuzzy set theory, many author tried to consider its natural extension.

Ramík and Římánek [8] has introduced a partial order on the set of fuzzy numbers, called the fuzzy max order. The present authors also tried to optimize the dynamic fuzzy system [4]. Also there are various types of order relations on the class of fuzzy numbers. See [3], [11] and their references. Congxin and Cong [1] have described the fuzzy number lattice.

This paper is to extend the fuzzy max order of fuzzy numbers to a class of fuzzy sets defined on $\mathbb{R}^n$. The pseudo order for fuzzy sets is induced by a closed convex cone $K$ in $\mathbb{R}^n$ and characterized by the projection in the dual cone $K^+$. Also, the structure of a lattice is discussed for the class of rectangle-type fuzzy sets. By our works we can imagine the much wider application to the fuzzy optimization problem. Our idea of the motivation originates from a set-relation in $\mathbb{R}^n$ given by Kuroiwa, Tanaka and Ha [5] and Kuroiwa [6], in which various types of set-relations in $\mathbb{R}^n$ are used in set-valued optimizations.

In the remainder of this section, we will give some notations and review a vector ordering of $\mathbb{R}^n$ by a convex cone. Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}^n$ an $n$-dimensional Euclidean space. We write fuzzy sets on $\mathbb{R}^n$ by their membership functions $\tilde{s} : \mathbb{R}^n \rightarrow [0,1]$ (see Novák [7] and Zadeh [10]). The $\alpha$-cut ($\alpha \in [0,1]$) of the fuzzy set $\tilde{s}$ on $\mathbb{R}^n$ is defined as

$$\tilde{s}_\alpha := \{x \in \mathbb{R}^n \mid \tilde{s}(x) \geq \alpha\} (\alpha > 0) \quad \text{and} \quad \tilde{s}_0 := \text{cl}\{x \in \mathbb{R}^n \mid \tilde{s}(x) > 0\},$$
where cl denotes the closure of the set. A fuzzy set \( \tilde{s} \) is called convex if

\[
\tilde{s}(ax + (1 - \lambda)y) \geq \tilde{s}(x) \land \tilde{s}(y) \quad x, y \in \mathbb{R}^n, \ \lambda \in [0, 1],
\]

where \( a \land b = \min\{a, b\} \). Note that \( \tilde{s} \) is convex iff the \( \alpha \)-cut \( \tilde{s}_\alpha \) is a convex set for all \( \alpha \in [0, 1] \). Let \( \mathcal{F}(\mathbb{R}^n) \) be the set of all convex fuzzy sets whose membership functions \( \tilde{s} : \mathbb{R}^n \rightarrow [0, 1] \) are upper-semicontinuous and normal \((\sup_{x \in \mathbb{R}^n} \tilde{s}(x) = 1)\) and have a compact support. When the one-dimensional case \( n = 1 \), the fuzzy sets are called fuzzy numbers and \( \mathcal{F}(\mathbb{R}) \) denotes the set of all fuzzy numbers.

Let \( C(\mathbb{R}^n) \) be the set of all compact convex subsets of \( \mathbb{R}^n \), and \( \mathcal{C}_r(\mathbb{R}^n) \) be the set of all rectangles in \( \mathbb{R}^n \). For \( \tilde{s} \in \mathcal{F}(\mathbb{R}^n) \), we have \( \tilde{s}_\alpha \in C(\mathbb{R}^n) \) \((\alpha \in [0, 1])\). We write a rectangle in \( \mathcal{C}_r(\mathbb{R}^n) \) by

\[
[x, y] = [x_1, y_1] \times [x_2, y_2] \times \cdots [x_n, y_n]
\]

for \( x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n \) with \( x_i \leq y_i \) \((i = 1, 2, \cdots, n)\). For the case of \( n = 1 \), \( C(\mathbb{R}) = C_r(\mathbb{R}) \) and it denotes the set of all bounded closed intervals. When \( \tilde{s} \in \mathcal{F}(\mathbb{R}^n) \) satisfies \( \tilde{s}_\alpha \in \mathcal{C}_r(\mathbb{R}^n) \) for all \( \alpha \in [0, 1] \), \( \tilde{s} \) is called a rectangle-type. We denote by \( \mathcal{F}_r(\mathbb{R}^n) \) the set of all rectangle-type fuzzy sets on \( \mathbb{R}^n \). Obviously \( \mathcal{F}_r(\mathbb{R}) = \mathcal{F}(\mathbb{R}) \).

The definitions of addition and scalar multiplication on \( \mathcal{F}(\mathbb{R}) \) are as follows: For \( \tilde{m}, \tilde{n} \in \mathcal{F}(\mathbb{R}) \) and \( \lambda \geq 1 \),

\[
(1.1) \quad (\tilde{m} + \tilde{n})(x) := \sup_{x_1 + x_2 = x} \{\tilde{m}(x_1) \land \tilde{n}(x_2)\},
\]

\[
(1.2) \quad (\lambda \tilde{m})(x) := \begin{cases} \tilde{m}(x/\lambda) & \text{if } \lambda > 0 \\ I_{[0]}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbb{R}),
\]

where \( I_{[\cdot]}(\cdot) \) is an indicator. By using set operations \( A + B : = \{x + y \mid x \in A, y \in B\} \) and \( \lambda A : = \{\lambda x \mid x \in A\} \) for any non-empty sets \( A, B \subset \mathbb{R} \), the following holds immediately.

\[
(1.3) \quad (\tilde{m} + \tilde{n})_\alpha := \tilde{m}_\alpha + \tilde{n}_\alpha \quad \text{and} \quad (\lambda \tilde{m})_\alpha = \lambda \tilde{m}_\alpha \quad (\alpha \in [0, 1]).
\]

Let \( K \) be a non-empty cone of \( \mathbb{R}^n \). Using this \( K \), we can define a pseudo-order relation \( \leq_K \) on \( \mathbb{R}^n \) by \( x \leq_K y \iff y - x \in K \). Let \( \mathbb{R}^+_n \) be the subset of entrywise non-negative elements in \( \mathbb{R}^n \). When \( K = \mathbb{R}^+_n \), the order \( \leq_K \) will be denoted by \( \leq_n \) and \( x \leq_n y \) means that \( x_i \leq y_i \) for all \( i = 1, 2, \cdots, n \), where \( x = (x_1, x_2, \cdots, x_n) \) and \( y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n \).

In Section 2, we will introduce a pseudo-order relation on \( \mathcal{F}(\mathbb{R}^n) \) which is characterized by the scalarization technique. In section 3, the lattice structure is discussed for the class of rectangle-type fuzzy sets.

### 2. A pseudo-order on \( \mathcal{F}(\mathbb{R}^n) \)

First we introduce a binary relation on \( C(\mathbb{R}^n) \), by which a pseudo-order on \( \mathcal{F}(\mathbb{R}^n) \) is given. Henceforth we assume that the convex cone \( K \subset \mathbb{R}^n \) is given.

We define a binary relation \( \preceq_K \) on \( C(\mathbb{R}^n) \): For \( A, B \in C(\mathbb{R}^n) \), \( A \preceq_K B \) means the following (C.a) and (C.b) (c.f. [5], [6]):
(C.a) For any $x \in A$, there exists $y \in B$ such that $x \leq_{K} y$.

(C.b) For any $y \in B$, there exists $x \in A$ such that $x \leq_{K} y$.

**Lemma 2.1.** The relation $\leq_{K}$ is a pseudo-order on $C(\mathbb{R}^{n})$.

**Proof.** It is trivial that $A \leq_{K} B$ for all $A, B \in C(\mathbb{R}^{n})$. Let $A, B, C \in C(\mathbb{R}^{n})$ such that $A \leq_{K} B$ and $B \leq_{K} C$. We will check $A \leq_{K} C$ by two cases (c.a) and (c.b). Case (c.a): Since $A \leq_{K} B$ and $B \leq_{K} C$, for any $x \in A$ there exists $y \in B$ such that $x \leq_{K} y$, and there exists $z \in C$ such that $y \leq_{K} z$. Since $\leq_{K}$ is a pseudo-order on $\mathbb{R}^{n}$, we have $x \leq_{K} z$. Therefore it holds that for any $x \in A$ there exists $z \in C$ such that $x \leq_{K} z$. Case (c.b): Since $A \leq_{K} B$ and $B \leq_{K} C$, for any $x \in A$ there exists $y \in B$ such that $y \leq_{K} z$ and there exists $x \in A$ such that $x \leq_{K} y$. Since $\leq_{K}$ is a pseudo-order on $\mathbb{R}^{n}$, we have $x \leq_{K} z$. Therefore it holds that for any $z \in C$ there exists $x \in A$ such that $x \leq_{K} z$.

From the above (a) and (b), we obtain $A \leq_{K} C$. Thus the lemma holds. Q.E.D.

When $K = \mathbb{R}^{n}_{+}$, the relation $\leq_{K}$ on $C(\mathbb{R}^{n})$ will be written simply by $\leq_{n}$ and for $[x, y], [x', y'] \in C_{r}(\mathbb{R}^{n})$, $[x, y] \leq_{n} [x', y']$ means $x \leq_{n} x'$ and $y \leq_{n} y'$.

Next, we introduce a binary relation $\preceq_{K}$ on $\mathcal{F}(\mathbb{R}^{n})$: Let $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^{n})$. The relation $\tilde{s} \preceq_{K} \tilde{r}$ means the following (F.a) and (F.b):

(F.a) For any $x \in \mathbb{R}^{n}$, there exists $y \in \mathbb{R}^{n}$ such that $x \leq_{K} y$ and $\tilde{s}(x) \leq \tilde{r}(y)$.

(F.b) For any $y \in \mathbb{R}^{n}$, there exists $x \in \mathbb{R}^{n}$ such that $x \leq_{K} y$ and $\tilde{s}(x) \geq \tilde{r}(y)$.

**Lemma 2.2.** The relation $\preceq_{K}$ is a pseudo-order on $\mathcal{F}(\mathbb{R}^{n})$.

**Proof.** It is trivial that $\tilde{s} \preceq_{K} \tilde{s}$ for $\tilde{s} \in \mathcal{F}(\mathbb{R}^{n})$. Let $\tilde{s}, \tilde{r}, \tilde{p} \in \mathcal{F}(\mathbb{R}^{n})$ such that $\tilde{s} \preceq_{K} \tilde{r}$ and $\tilde{r} \preceq_{K} \tilde{p}$. We will check $\tilde{s} \preceq_{K} \tilde{p}$ by two cases (F.a) and (F.b). Case (F.a): Since $\tilde{s} \preceq_{K} \tilde{r}$ and $\tilde{r} \preceq_{K} \tilde{p}$, for any $x \in \mathbb{R}^{n}$ there exists $y \in \mathbb{R}^{n}$ such that $x \leq_{K} y$ and $\tilde{s}(x) \leq \tilde{r}(y)$, and there exists $z \in \mathbb{R}^{n}$ such that $y \leq_{K} z$ and $\tilde{r}(y) \leq \tilde{p}(z)$. Since $\preceq_{K}$ is a pseudo-order on $\mathbb{R}^{n}$, we have $x \leq_{K} z$ and $\tilde{s}(x) \leq \tilde{p}(z)$. Therefore it holds that for any $x \in \mathbb{R}^{n}$ there exists $z \in \mathbb{R}^{n}$ such that $x \leq_{K} z$ and $\tilde{s}(x) \leq \tilde{p}(z)$. Case (F.b) Since $\tilde{s} \preceq_{K} \tilde{r}$ and $\tilde{r} \preceq_{K} \tilde{p}$, for any $y \in \mathbb{R}^{n}$ there exists $z \in \mathbb{R}^{n}$ such that $y \leq_{K} z$ and $\tilde{s}(x) \geq \tilde{r}(y)$, and there exists $x \in \mathbb{R}^{n}$ such that $x \leq_{K} y$ and $\tilde{s}(x) \geq \tilde{r}(y)$. Since $\preceq_{K}$ is a pseudo-order on $\mathbb{R}^{n}$, we have $x \leq_{K} z$. Therefore it holds that for any $z \in \mathbb{R}^{n}$ there exists $x \in \mathbb{R}^{n}$ such that $x \leq_{K} z$ and $\tilde{s}(x) \geq \tilde{p}(z)$.

From the above (a) and (b), we obtain $\tilde{s} \preceq_{K} \tilde{p}$. Thus the lemma holds. Q.E.D.

The following lemma implies the correspondence between the pseudo-order on $\mathcal{F}(\mathbb{R}^{n})$ for fuzzy sets and the pseudo-order on $C(\mathbb{R}^{n})$ for the $\alpha$-cuts.

**Lemma 2.3.** Let $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^{n})$. $\tilde{s} \preceq_{K} \tilde{r}$ on $\mathcal{F}(\mathbb{R}^{n})$ if and only if $\tilde{s}_{\alpha} \preceq_{K} \tilde{r}_{\alpha}$ on $C(\mathbb{R}^{n})$ for all $\alpha \in (0, 1]$. 
Proof. Let $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n)$ and $\alpha \in (0, 1]$. Suppose $\tilde{s} \preceq_K \tilde{r}$ on $\mathcal{F}(\mathbb{R}^n)$. Then, Two cases (a) and (b) are considered. Case(a): Let $x \in \tilde{s}_\alpha$. Since $\tilde{s} \preceq_K \tilde{r}$, there exists $y \in \mathbb{R}^n$ such that $x \preceq_K y$ and $\alpha \leq \tilde{s}(x) \leq \tilde{r}(y)$. Namely $y \in \tilde{r}_\alpha$. Case(b): Let $y \in \tilde{r}_\alpha$. Since $\tilde{s} \preceq_K \tilde{r}$, there exists $x \in \mathbb{R}^n$ such that $x \preceq_K y$ and $\tilde{s}(x) \geq \tilde{r}(y) \geq \alpha$. Namely $x \in \tilde{s}_\alpha$.

Therefore we get $\tilde{s}_\alpha \preceq_K \tilde{r}_\alpha$ on $\mathcal{F}(\mathbb{R}^n)$ for all $\alpha \in (0, 1]$ from the above (a) and (b).

On the other hand, suppose $\tilde{s}_\alpha \preceq_K \tilde{r}_\alpha$ on $\mathcal{F}(\mathbb{R}^n)$ for all $\alpha \in (0, 1]$. Then, Two cases (a') and (b') are considered. Case(a'): Let $x \in \mathbb{R}^n$. Put $\alpha = \tilde{s}(x)$. If $\alpha = 0$, then $x \preceq_K x$ and $\tilde{s}(x) = 0 \leq \tilde{r}(x)$. While, if $\alpha > 0$, then $x \in \tilde{s}_\alpha$. Since $\tilde{s} \preceq_K \tilde{r}_\alpha$, there exists $y \in \tilde{r}_\alpha$ such that $x \preceq_K y$. And we have $\tilde{s}(x) = \alpha \leq \tilde{r}(y)$. Case(b'): Let $y \in \mathbb{R}^n$. Put $\alpha = \tilde{r}(y)$. If $\alpha = 0$, then $x \preceq_K x$ and $\tilde{s}(x) \geq 0 = \tilde{r}(y)$. While, if $\alpha > 0$, then $y \in \tilde{r}_\alpha$. Since $\tilde{s}_\alpha \preceq_K \tilde{r}_\alpha$, there exists $x \in \tilde{s}_\alpha$ such that $x \preceq_K y$. And we have $\tilde{s}(x) \geq \alpha = \tilde{r}(y)$.

Therefore we get $\tilde{s} \preceq_K \tilde{r}$ on $\mathcal{F}(\mathbb{R}^n)$ from the above Case (a') and (b'). Thus we obtain this lemma. Q.E.D.

Define the dual cone of a cone $K$ by

$$K^+ := \{ a \in \mathbb{R}^n \mid a \cdot x \geq 0 \text{ for all } x \in K \},$$

where $x \cdot y$ denotes the inner product on $\mathbb{R}^n$ for $x, y \in \mathbb{R}^n$. For a subset $A \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, we define

(2.1) $\quad a \cdot A := \{ a \cdot x \mid x \in A \} (\subset \mathbb{R})$.

The equation (2.1) means the projection of $A$ on the extended line of the vector $a$ if $a \cdot a = 1$. It is trivial that $a \cdot A \in \mathcal{C}(\mathbb{R})$ if $A \in \mathcal{C}(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$.

Lemma 2.4. Let $A, B \in \mathcal{C}(\mathbb{R}^n)$. $A \preceq_K B$ on $\mathcal{C}(\mathbb{R}^n)$ if and only if $a \cdot A \preceq a \cdot B$ on $\mathcal{C}(\mathbb{R})$ for all $a \in K^+$, where $\preceq_1$ is the natural order on $\mathcal{C}(\mathbb{R})$.

Proof. Suppose $A \preceq_K B$ on $\mathcal{C}(\mathbb{R}^n)$. Consider the two cases (a) and (b). Case(a): For any $a \cdot x \in a \cdot A$, there exists $y \in B$ such that $x \preceq_K y$. Then $y - x \in K$. If $a \in K^+$, then $a \cdot (y - x) \geq 0$ and i.e. $a \cdot x \leq a \cdot y$. Case(b): For any $a \cdot y \in a \cdot B$, there exists $x \in A$ such that $x \preceq_K y$. Then $y - x \in K$. If $a \in K^+$, then $a \cdot (y - x) \geq 0$ and i.e. $a \cdot x \leq a \cdot y$. From the above cases (a) and (b), we have that $a \cdot A \preceq a \cdot B$.

On the other hand, to prove the inverse statement, we assume that $A \preceq_K B$ on $\mathcal{C}(\mathbb{R}^n)$ does not hold. Then we have the following two cases (i) and (ii). Case(i): There exists $x \in A$ such that $y - x \not\in K$ for all $y \in B$. Then $B \cap (x + K) = \emptyset$. Since $B$ and $x + K$ are closed convex, by the separation theorem there exists $a \in \mathbb{R}$ ($a \neq 0$) such that $a \cdot y < a \cdot x + a \cdot z$ for all $y \in B$ and all $z \in K$. Hence we suppose that there exists $z \in K$ such that $a \cdot z \geq 0$. Then $\lambda z \in K$ for all $\lambda \geq 0$ since $K$ is a cone, and so we have $a \cdot x + a \cdot \lambda z = a \cdot x + a \cdot z \to -\infty$ as $\lambda \to \infty$. This contradicts $a \cdot y < a \cdot x + a \cdot z$. Therefore we obtain $a \cdot z \geq 0$ for all $z \in K$. Especially taking $z = 0 \in K$, we get $a \cdot y < a \cdot x$ for all $y \in B$. This contradicts $a \cdot A \preceq a \cdot B$. Case(ii): There exists $y \in B$ such that $y - x \not\in K$ for all $x \in A$. Then we derive the contradiction in a similar way to the case (i).
Therefore the inverse statement holds from the results of the above (i) and (ii). The proof of this lemma is completed. Q.E.D.

For \( a \in \mathbb{R}^n \) and \( \tilde{s} \in \mathcal{F}(\mathbb{R}^n) \), we define a fuzzy number \( a \cdot \tilde{s} \in \mathcal{F}(\mathbb{R}) \) by

\[
(2.2) \quad a \cdot \tilde{s}(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{a \cdot \tilde{s}_\alpha}(x)\}, \quad x \in \mathbb{R}.
\]

where \( 1_D(\cdot) \) is the classical indicator function of a closed interval \( D \in \mathcal{C}(\mathbb{R}) \).

We define a partial relation \( \preceq_M \) on \( \mathcal{F}(\mathbb{R}) \) as follows ([8]): For \( \tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n) \), \( \tilde{s} \preceq_M \tilde{r} \) means that \( \tilde{s}_\alpha \preceq_{1} \tilde{r}_\alpha \) for all \( \alpha \in [0,1] \).

The following theorem gives the correspondence between the pseudo-order \( \preceq_K \) on \( \mathcal{F}(\mathbb{R}^n) \) and the fuzzy max order \( \preceq_M \) on \( \mathcal{F}(\mathbb{R}) \).

**Theorem 2.1.** For \( \tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n) \), \( \tilde{s} \preceq_K \tilde{r} \) iff \( a \cdot \tilde{s} \preceq_M a \cdot \tilde{r} \) for all \( a \in K^+ \).

**Proof.** From Lemmas 2.3 and 2.4, \( \tilde{s} \preceq_K \tilde{r} \) iff \( a \cdot \tilde{s}_\alpha \preceq_{1} a \cdot \tilde{r}_\alpha \) for all for all \( a \in K^+ \) and \( \alpha \in (0,1] \). Is equivalent to \( a \cdot \tilde{s} \preceq_M a \cdot \tilde{r} \) for all \( a \in K^+ \). Q.E.D.

For \( \{\tilde{s}_k\}_{k=1}^\infty \subset \mathcal{F}(\mathbb{R}^n) \) and \( \tilde{s} \in \mathcal{F}(\mathbb{R}^n) \), \( \lim_{k \to \infty} \tilde{s}_k = \tilde{s} \) means that \( \sup_{\alpha \in [0,1]} \rho(\tilde{s}_k, \tilde{s}) \to 0 \) \( (k \to \infty) \), where \( \tilde{s}_{k,\alpha} \) is the \( \alpha \)-cut of \( \tilde{s}_k \) and \( \rho \) is the Hausdorff metric on \( \mathcal{C}(\mathbb{R}^n) \).

**Lemma 2.5.** Let \( \{\tilde{s}_k\}_{k=1}^\infty \subset \mathcal{F}(\mathbb{R}) \) and \( \tilde{s} \in \mathcal{F}(\mathbb{R}) \) such that \( \tilde{s}_k \preceq_M \tilde{s}_{k+1} \) \( (k \geq 1) \) and \( \lim_{k \to \infty} \tilde{s}_k = \tilde{s} \). Then \( \tilde{s}_1 \preceq_M \tilde{s} \).

**Proof.** Trivial. Q.E.D.

**Theorem 2.2.** Let \( \{\tilde{s}_k\}_{k=1}^\infty \subset \mathcal{F}(\mathbb{R}^n) \) and \( \tilde{s} \in \mathcal{F}(\mathbb{R}^n) \) such that \( \tilde{s}_k \preceq_K \tilde{s}_{k+1} \) \( (k \geq 1) \) and \( \lim_{k \to \infty} \tilde{s}_k = \tilde{s} \). Then \( \tilde{s}_1 \preceq_K \tilde{s} \).

**Proof.** From Theorem 2.1, for all \( a \in K^+ \) we have \( a \cdot \tilde{s}_k \preceq_M a \cdot \tilde{s}_{k+1} \) \( (k \geq 1) \) and \( \lim_{k \to \infty} a \cdot \tilde{s}_k = a \cdot \tilde{s} \). By Lemma 2.3, \( a \cdot \tilde{s}_1 \preceq_M a \cdot \tilde{s} \) all \( a \in K^+ \). From Theorem 2.1, \( \tilde{s}_1 \preceq_K \tilde{s} \). Q.E.D.

**Remark.** Let the map \( g : [0,1] \to \mathcal{F}(\mathbb{R}^n) \) be continuous. A point \( x_0 \) is said to be efficient if \( x_0 \in [0,1] \) and \( g(x_0) \preceq_K g(x) \) for some \( x \in [0,1] \) implies \( g(x) = g(x_0) \). Then, by applying the same idea as in Lemma 3.2 of Furukawa [2], we observe that there exists at least one efficient point in \( [0,1] \). In fact, considering, if necessary, a partial order \( \preceq_K \) on the class of the quotient sets with respect to the equivalence relation \( \sim_K \) defined by \( \tilde{s} \sim_K \tilde{r} \) iff \( \tilde{s} \preceq_K \tilde{r} \) and \( \tilde{r} \preceq_K \tilde{s} \), we can assume that \( \preceq_K \) is a partial order on \( \mathcal{F}(\mathbb{R}^n) \). By theorem 2.2 and the continuity of \( g \), \( \{g(x) \mid x \in [0,1] \} \) can be proved to be an inductively ordered set. So, by Zorn's lemma \( \{g(x) \mid x \in [0,1] \} \) has an efficient element.

### 3. Further results

In this section, we investigate a pseudo-order \( \preceq_K \) on \( \mathcal{F}_r(\mathbb{R}^n) \) for a polyhedral cone \( K \) with \( K^+ \subset \mathbb{R}^n \). To this end, we need the following lemma.
Lemma 3.1. Let $a, b \in \mathbb{R}^n_+$ and $A \in C_r(\mathbb{R}^n)$. Then for any scalars $\lambda_1, \lambda_2 \geq 0$, it holds

$$ (\lambda_1 a + \lambda_2 b) \cdot A = \lambda_1 (a \cdot A) + \lambda_2 (b \cdot A), $$

where the arithmetic in (3.1) is defined in (2.1).

Proof. Let $\lambda_1 a \cdot x + \lambda_2 b \cdot y \in (\lambda_1 a + \lambda_2 b) \cdot A$. Define $z = (z_1, z_2, \cdots, z_n)$ by

$$ z_i = (\lambda_1 a_i x_i + \lambda_2 b_i y_i) / (\lambda_1 a_i + \lambda_2 b_i) \quad \text{if} \quad (\lambda_1 a_i + \lambda_2 b_i) > 0 $$

$$ = x_i \quad \text{if} \quad (\lambda_1 a_i + \lambda_2 b_i) = 0 \quad (i = 1, 2, \cdots, n) $$

Then, clearly $(\lambda_1 a + \lambda_2 b) \cdot z = \lambda_1 a \cdot x + \lambda_2 b \cdot y$. Since $A \in C_r(\mathbb{R}^n), z \in A$, so that $\lambda_1 a \cdot x + \lambda_2 b \cdot y = (\lambda_1 a + \lambda_2 b) \cdot A$. Q.E.D.

Henceforth, we assume that $K$ is a polyhedral convex cone with $K^+ \subset \mathbb{R}^n$, i.e., there exist vectors $b^i \in \mathbb{R}^n_+ (i = 1, 2, \cdots, m)$ such that

$$ K = \{ x \in \mathbb{R}^n \mid b^i \cdot x \leq 0 \text{ for all } i = 1, 2, \cdots, m \}. $$

Then, it is well-known (c.f. [9]) that $K^+$ is expressed as

$$ K^+ = \{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^{m} \lambda_i b_i, \lambda_i \geq 0 \text{ (i = 1, 2, \cdots, m)} \}. $$

The above dual cone $K^+$ is denoted simply by

$$ K^+ = \text{conv}\{b^1, b^2, \cdots, b^m\}. $$

The pseudo-order $\preceq_K$ on $C_r(\mathbb{R}^n)$ is characterized in the following.

Corollary 3.1. Let $K^+ = \text{conv}\{b^1, b^2, \cdots, b^m\}$ with $b^i \in \mathbb{R}^n_+$. Then, for $A, B \in C_r(\mathbb{R}^n)$, $A \preceq_K B$ if and only if $b^i \cdot A \preceq_1 b^i \cdot B$ for all $i = 1, 2, \cdots, m$, where $\preceq_1$ is a pseudo-order on $C_r(\mathbb{R})$.

Proof. We assume that $b^i \cdot A \preceq_1 b^i \cdot B$ for all $i = 1, 2, \cdots, m$. For any $a \in K^+$, there exists $\lambda_i \geq 0$ with $a = \sum_{i=1}^{m} \lambda_i b^i$. From Lemma 3.1 we have:

$$ a \cdot A = \sum_{i=1}^{m} \lambda_i (b^i \cdot A) \preceq_1 = \sum_{i=1}^{m} \lambda_i (b^i \cdot B) = a \cdot B. $$

Thus, by Lemma 2.4, $A \preceq_K B$ follows. By applying Lemma 2.4 again, the ‘only if’ part of Corollary holds. Q.E.D.

Lemma 3.2. Let $a, b \in \mathbb{R}^n_+$ and $\tilde{s} \in F_r(\mathbb{R}^n)$. Then, for any $\lambda_1, \lambda_2 \geq 0$, (3.2)

$$ (\lambda_1 a + \lambda_2 b) \cdot \tilde{s} = \lambda_1 (a \cdot \tilde{s}) + \lambda_2 (b \cdot \tilde{s}), $$
where the arithmetic in (3.2) are given in (1.1), (1.2) and (2.2).

**Proof.** For any $\alpha \in [0,1]$, it follows from the definition and Lemma 3.1 that

$$[(\lambda_1 a + \lambda_2 b) \cdot \tilde{s}]_\alpha = (\lambda_1 a + \lambda_2 b) \cdot \tilde{s}_\alpha = \lambda_1 (a \cdot \tilde{s}_\alpha) + \lambda_2 (b \cdot \tilde{s}_\alpha) = [\lambda_1 (a \cdot \tilde{s}) + \lambda_2 (b \cdot \tilde{s})]_\alpha.$$  

The last equality follows from (1.3). The above shows that (3.3) holds. Q.E.D.

The main results in this section are given in the following.

**Theorem 3.1.** Let $K^+ = \text{conv}\{b^1, b^2, \ldots, b^m\}$ with $b^i \in \mathbb{R}^n$. Then, for $\tilde{s}, \tilde{r} \in \mathcal{F}_r(\mathbb{R}^n)$,

$$\tilde{s} \preceq_K \tilde{r} \quad \text{if and only if} \quad b^i \cdot \tilde{s} \preceq_M b^i \cdot \tilde{r} \quad \text{for} \quad i = 1, 2, \ldots, m.$$

**Proof.** It suffices to prove the 'if' part of Theorem 3.1. For any $a \in K^+$, there exists $\lambda_i \geq 0$ with $a = \sum_{i=1}^{m} \lambda_i b^i$. Applying Lemma 3.2, we have

$$a \cdot \tilde{s} = \sum_{i=1}^{m} \lambda_i (b^i \cdot \tilde{s}) \preceq_M \sum_{i=1}^{m} \lambda_i (b^i \cdot \tilde{r}) = a \cdot \tilde{r},$$

From Theorem 2.1, $\tilde{s} \preceq_k \tilde{r}$ follows. Q.E.D.

![Figure 1: $\hat{v} = \sup(\tilde{s}, \tilde{r})$](image)

When $K = \mathbb{R}^n$, the pseudo-order $\preceq_K$ on $\mathcal{F}_r(\mathbb{R}^n)$ will be simply written by $\preceq_n$. Obviously, $\preceq_1$ and $\preceq_M$ are the same.
Congxin and Cong [1] described the structure of the fuzzy number lattice \( (\mathcal{F}_r(\mathbb{R}), \preceq_1) \).
When \( K = \mathbb{R}^n, K^+ = \mathbb{R}^n \) and \( K^+ = \text{conv}\{e^1, e^2, \cdots, e^m\} \). So that, by Theorem 3.1, we see that for \( \tilde{s}, \tilde{r} \in \mathcal{F}_r(\mathbb{R}^n) \), \( \tilde{s} \preceq_n \tilde{r} \) means \( e^i\tilde{s}_1 \preceq e^i\tilde{r} \) for all \( i = 1, 2, \cdots, n \). Therefore, by applying the same method as [1], we can describe the structure of the fuzzy set lattice \( (\mathcal{F}_r(\mathbb{R}^n), \preceq_n) \). Figure 1 illustrates \( \sup(\tilde{s}, \tilde{r}) \) for \( \tilde{s}, \tilde{r} \in \mathcal{F}_r(\mathbb{R}^2) \).

References


