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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1998), 1068: 142-149</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1998-10</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62517">http://hdl.handle.net/2433/62517</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>Textversion</td>
<td>publisher</td>
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Some Pseudo-Order of Fuzzy Sets on $\mathbb{R}^n$

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Abstract

The aim of this paper is to define an order on a class of fuzzy sets which is extending a pseudo-order for fuzzy numbers, and its characterization and several relations of the previous results are discussed. The idea comes from a set-relation in $n$-dimensional Euclid space given by Kuroiwa, Tanaka and Ha (1997). We induce the order of a class of fuzzy sets by a closed convex cone and characterize it by using the projection into the dual cone. Especially, a structure of the lattice is described for the class of rectangle-type fuzzy sets.

Keywords: Pseudo-order, fuzzy max order, multidimensional fuzzy sets, rectangle-type fuzzy sets.

1. Introduction and notations

In the theory of optimization it is a quite important problem how to induce a natural definition of order on the class of considering systems. Since it isn’t a simple problem about a order on the fuzzy set theory, many author tried to consider its natural extension.

Ramík and Římanek [8] has introduced a partial order on the set of fuzzy numbers, called the fuzzy max order. The present authors also tried to optimize the dynamic fuzzy system [4]. Also there are various types of order relations on the class of fuzzy numbers. See [3], [11] and their references. Congxin and Cong [1] have described the fuzzy number lattice.

This paper is to extend the fuzzy max order of fuzzy numbers to a class of fuzzy sets defined on $\mathbb{R}^n$. The pseudo order for fuzzy sets is induced by a closed convex cone $K$ in $\mathbb{R}^n$ and characterized by the projection in the dual cone $K^+$. Also, the structure of a lattice is discussed for the class of rectangle-type fuzzy sets. By our works we can imagine the much wider application to the fuzzy optimization problem. Our idea of the motivation originates from a set-relation in $\mathbb{R}^n$ given by Kuroiwa, Tanaka and Ha [5] and Kuroiwa [6], in which various types of set-relations in $\mathbb{R}^n$ are used in set-valued optimizations.

In the remainder of this section, we will give some notations and review a vector ordering of $\mathbb{R}^n$ by a convex cone. Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}^n$ an $n$-dimensional Euclidean space. We write fuzzy sets on $\mathbb{R}^n$ by their membership functions $\tilde{s}: \mathbb{R}^n \rightarrow [0, 1]$ (see Novák [7] and Zadeh [10]). The $\alpha$-cut ($\alpha \in [0, 1]$) of the fuzzy set $\tilde{s}$ on $\mathbb{R}^n$ is defined as

$$\tilde{s}\alpha := \{x \in \mathbb{R}^n \mid \tilde{s}(x) \geq \alpha\} \ (\alpha > 0) \quad \text{and} \quad \tilde{s}_0 := \text{cl}\{x \in \mathbb{R}^n \mid \tilde{s}(x) > 0\}.$$
where cl denotes the closure of the set. A fuzzy set \(\tilde{s}\) is called convex if
\[
\tilde{s}(\lambda x + (1 - \lambda)y) \geq \tilde{s}(x) \wedge \tilde{s}(y) \quad x, y \in \mathbb{R}^n, \quad \lambda \in [0, 1],
\]
where \(a \wedge b = \min\{a, b\}\). Note that \(\tilde{s}\) is convex iff the \(\alpha\)-cut \(\tilde{s}_\alpha\) is a convex set for all \(\alpha \in [0, 1]\). Let \(\mathcal{F}(\mathbb{R}^n)\) be the set of all convex fuzzy sets whose membership functions \(\tilde{s}: \mathbb{R}^n \to [0, 1]\) are upper-semicontinuous and normal (\(\sup_{x \in \mathbb{R}^n} \tilde{s}(x) = 1\)) and have a compact support. When the one-dimensional case \(n = 1\), the fuzzy sets are called fuzzy numbers and \(\mathcal{F}(\mathbb{R})\) denotes the set of all fuzzy numbers.

Let \(\mathcal{C}(\mathbb{R}^n)\) be the set of all compact convex subsets of \(\mathbb{R}^n\), and \(\mathcal{C}_r(\mathbb{R}^n)\) be the set of all rectangles in \(\mathbb{R}^n\). For \(\tilde{s} \in \mathcal{F}(\mathbb{R}^n)\), we have \(\tilde{s}_\alpha \in \mathcal{C}(\mathbb{R}^n)\) (\(\alpha \in [0, 1]\)). We write a rectangle in \(\mathcal{C}_r(\mathbb{R}^n)\) by
\[
[x, y] = [x_1, y_1] \times [x_2, y_2] \times \cdots [x_n, y_n]
\]
for \(x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n\) with \(x_i \leq y_i\) \((i = 1, 2, \cdots, n)\). For the case of \(n = 1\), \(\mathcal{C}(\mathbb{R}) = \mathcal{C}_r(\mathbb{R})\) and it denotes the set of all bounded closed intervals. When \(\tilde{s} \in \mathcal{F}(\mathbb{R}^n)\) satisfies \(\tilde{s}_\alpha \in \mathcal{C}_r(\mathbb{R}^n)\) for all \(\alpha \in [0, 1]\), \(\tilde{s}\) is called a rectangle-type. We denote by \(\mathcal{F}_r(\mathbb{R}^n)\) the set of all rectangle-type fuzzy sets on \(\mathbb{R}^n\). Obviously \(\mathcal{F}_r(\mathbb{R}) = \mathcal{F}(\mathbb{R})\).

The definitions of addition and scalar multiplication on \(\mathcal{F}(\mathbb{R})\) are as follows: For \(\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbb{R})\) and \(\lambda \geq 1\),
\[
(\tilde{m} + \tilde{n})(x) := \sup_{x_1, x_2 \in \mathbb{R}; \ x_1 + x_2 = x} \{\tilde{m}(x_1) \wedge \tilde{n}(x_2)\},
\]
\[
(\lambda \tilde{m})(x) := \begin{cases} 
\tilde{m}(x/\lambda) & \text{if } \lambda > 0 \\
I_{\{0\}}(x) & \text{if } \lambda = 0
\end{cases} \quad (x \in \mathbb{R}),
\]
where \(I_{\{\cdot\}}(\cdot)\) is an indicator. By using set operations \(A + B := \{x + y \mid x \in A, y \in B\}\) and \(\lambda A := \{\lambda x \mid x \in A\}\) for any non-empty sets \(A, B \subset \mathbb{R}\), the following holds immediately.
\[
(\tilde{m} + \tilde{n})_\alpha := \tilde{m}_\alpha + \tilde{n}_\alpha \quad \text{and} \quad (\lambda \tilde{m})_\alpha = \lambda \tilde{m}_\alpha \quad (\alpha \in [0, 1]).
\]

Let \(K\) be a non-empty cone of \(\mathbb{R}^n\). Using this \(K\), we can define a pseudo-order relation \(\preceq_K\) on \(\mathbb{R}^n\) by \(x \preceq_K y\) iff \(y - x \in K\). Let \(\mathbb{R}^+_n\) be the subset of entrywise non-negative elements in \(\mathbb{R}^n\). When \(K = \mathbb{R}^+_n\), the order \(\preceq_K\) will be denoted by \(\leq_n\) and \(x \leq_n y\) means that \(x_i \leq y_i\) for all \(i = 1, 2, \cdots, n\), where \(x = (x_1, x_2, \cdots, x_n)\) and \(y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n\).

In Section 2, we will introduce a pseudo-order relation on \(\mathcal{F}(\mathbb{R}^n)\) which is characterized by the scalarization technique. In section 3, the lattice structure is discussed for the class of rectangle-type fuzzy sets.

2. A pseudo-order on \(\mathcal{F}(\mathbb{R}^n)\)

First we introduce a binary relation on \(\mathcal{C}(\mathbb{R}^n)\), by which a pseudo-order on \(\mathcal{F}(\mathbb{R}^n)\) is given. Henceforth we assume that the convex cone \(K \subset \mathbb{R}^n\) is given.

We define a binary relation \(\preceq_K\) on \(\mathcal{C}(\mathbb{R}^n)\): For \(A, B \in \mathcal{C}(\mathbb{R}^n)\), \(A \preceq_K B\) means the following (C.a) and (C.b) (c.f. [5], [6]):
(C.a) For any $x \in A$, there exists $y \in B$ such that $x \leq_{K} y$.

(C.b) For any $y \in B$, there exists $x \in A$ such that $x \leq_{K} y$.

Lemma 2.1. The relation $\leq_{K}$ is a pseudo-order on $C(\mathbb{R}^{n})$.

Proof. It is trivial that $A \leq_{K} A$ for $A \in C(\mathbb{R}^{n})$. Let $A, B, C \in C(\mathbb{R}^{n})$ such that $A \leq_{K} B$ and $B \leq_{K} C$. We will check $A \leq_{K} C$ by two cases (c.a) and (c.b). Case(C.a): Since $A \leq_{K} B$ and $B \leq_{K} C$, for any $x \in A$ there exists $y \in B$ such that $x \leq_{K} y$ and there exists $z \in C$ such that $y \leq_{K} z$. Since $\leq_{K}$ is a pseudo-order on $\mathbb{R}^{n}$, we have $x \leq_{K} z$. Therefore it holds that for any $x \in A$ there exists $z \in C$ such that $x \leq_{K} z$. Case(C.b): Since $A \leq_{K} B$ and $B \leq_{K} C$, for any $z \in C$ there exists $y \in B$ such that $y \leq_{K} z$ and there exists $x \in A$ such that $x \leq_{K} y$. Since $\leq_{K}$ is a pseudo-order on $\mathbb{R}^{n}$, we have $x \leq_{K} z$. Therefore it holds that for any $z \in C$ there exists $x \in A$ such that $x \leq_{K} z$.

From the above (a) and (b), we obtain $A \leq_{K} C$. Thus the lemma holds. Q.E.D.

When $K = \mathbb{R}^{n}_{+}$, the relation $\leq_{K}$ on $C(\mathbb{R}^{n})$ will be written simply by $\leq_{n}$ and for $[x, y], [x', y'] \in C_{r}(\mathbb{R}^{n})$, $[x, y] \leq_{n} [x', y']$ means $x \leq_{n} x'$ and $y \leq_{n} y'$.

Next, we introduce a binary relation $\leq_{K}$ on $\mathcal{F}(\mathbb{R}^{n})$: Let $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^{n})$. The relation $\tilde{s} \leq_{K} \tilde{r}$ means the following (F.a) and (F.b):

(F.a) For any $x \in \mathbb{R}^{n}$, there exists $y \in \mathbb{R}^{n}$ such that $x \leq_{K} y$ and $\tilde{s}(x) \leq \tilde{r}(y)$.

(F.b) For any $y \in \mathbb{R}^{n}$, there exists $x \in \mathbb{R}^{n}$ such that $x \leq_{K} y$ and $\tilde{s}(x) \geq \tilde{r}(y)$.

Lemma 2.2. The relation $\leq_{K}$ is a pseudo-order on $\mathcal{F}(\mathbb{R}^{n})$.

Proof. It is trivial that $\tilde{s} \leq_{K} \tilde{s}$ for $\tilde{s} \in \mathcal{F}(\mathbb{R}^{n})$. Let $\tilde{s}, \tilde{r}, \tilde{p} \in \mathcal{F}(\mathbb{R}^{n})$ such that $\tilde{s} \leq_{K} \tilde{r}$ and $\tilde{r} \leq_{K} \tilde{p}$. We will check $\tilde{s} \leq_{K} \tilde{p}$ by two cases (F.a) and (F.b). Case(F.a):Since $\tilde{s} \leq_{K} \tilde{r}$ and $\tilde{r} \leq_{K} \tilde{p}$, for any $x \in \mathbb{R}^{n}$ there exists $y \in \mathbb{R}^{n}$ such that $x \leq_{K} y$ and $\tilde{s}(x) \leq \tilde{r}(y)$, and there exists $z \in \mathbb{R}^{n}$ such that $y \leq_{K} z$ and $\tilde{r}(y) \leq \tilde{p}(z)$. Since $\leq_{K}$ is a pseudo-order on $\mathbb{R}^{n}$, we have $x \leq_{K} z$ and $\tilde{s}(x) \leq \tilde{p}(z)$. Therefore it holds that for any $x \in \mathbb{R}^{n}$ there exists $z \in \mathbb{R}^{n}$ such that $x \leq_{K} z$ and $\tilde{s}(x) \leq \tilde{p}(z)$. Case(F.b) Since $\tilde{s} \leq_{K} \tilde{r}$ and $\tilde{r} \leq_{K} \tilde{p}$, for any $y \in \mathbb{R}^{n}$ there exists $x \in \mathbb{R}^{n}$ such that $y \leq_{K} x$ and $\tilde{s}(x) \geq \tilde{r}(y)$, and there exists $x \in \mathbb{R}^{n}$ such that $x \leq_{K} y$ and $\tilde{s}(x) \geq \tilde{r}(y)$. Since $\leq_{K}$ is a pseudo-order on $\mathbb{R}^{n}$, we have $x \leq_{K} z$. Therefore it holds that for any $z \in \mathbb{R}^{n}$ there exists $x \in \mathbb{R}^{n}$ such that $x \leq_{K} z$ and $\tilde{s}(x) \geq \tilde{p}(z)$.

From the above (a) and (b), we obtain $\tilde{s} \leq_{K} \tilde{p}$. Thus the lemma holds. Q.E.D.

The following lemma implies the correspondence between the pseudo-order on $\mathcal{F}(\mathbb{R}^{n})$ for fuzzy sets and the pseudo-order on $C(\mathbb{R}^{n})$ for the $\alpha$-cuts.

Lemma 2.3. Let $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^{n})$. $\tilde{s} \leq_{K} \tilde{r}$ on $\mathcal{F}(\mathbb{R}^{n})$ if and only if $\tilde{s}_{\alpha} \leq_{K} \tilde{r}_{\alpha}$ on $C(\mathbb{R}^{n})$ for all $\alpha \in (0, 1]$. 
**Proof.** Let \( \tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n) \) and \( \alpha \in (0, 1] \). Suppose \( \tilde{s} \ll_{K} \tilde{r} \) on \( \mathcal{F}(\mathbb{R}^n) \). Then, Two cases (a) and (b) are considered. Case(a): Let \( x \in \tilde{s}_\alpha \). Since \( \tilde{s} \ll_{K} \tilde{r} \), there exists \( y \in \mathbb{R}^n \) such that \( x \ll_{K} y \) and \( \alpha \leq \tilde{s}(x) \leq \tilde{r}(y) \). Namely \( y \in \tilde{r}_\alpha \). Case(b): Let \( y \in \tilde{r}_\alpha \). Since \( \tilde{s} \ll_{K} \tilde{r} \), there exists \( x \in \mathbb{R}^n \) such that \( x \ll_{K} y \) and \( \tilde{s}(x) \geq \tilde{r}(y) \geq \alpha \). Namely \( x \in \tilde{s}_\alpha \).

Therefore we get \( \tilde{s}_\alpha \ll_{K} \tilde{r}_\alpha \) on \( \mathcal{F}(\mathbb{R}^n) \) for all \( \alpha \in (0, 1] \) from the above (a) and (b).

On the other hand, suppose \( \tilde{s}_\alpha \ll_{K} \tilde{r}_\alpha \) on \( \mathcal{F}(\mathbb{R}^n) \) for all \( \alpha \in (0, 1] \). Then, Two cases (a') and (b') are considered. Case(a'): Let \( x \in \mathbb{R}^n \). Put \( \alpha = \tilde{s}(x) \). If \( \alpha = 0 \), then \( x \ll_{K} x \) and \( \tilde{s}(x) = 0 \leq \tilde{r}(x) \). While, if \( \alpha > 0 \), then \( x \in \tilde{s}_\alpha \). Since \( \tilde{s}_\alpha \ll_{K} \tilde{r}_\alpha \), there exists \( y \in \tilde{r}_\alpha \) such that \( x \ll_{K} y \). And we have \( \tilde{s}(x) = \alpha \leq \tilde{r}(y) \). Case(b'): Let \( y \in \mathbb{R}^n \). Put \( \alpha = \tilde{r}(y) \). If \( \alpha = 0 \), then \( x \ll_{K} x \) and \( \tilde{s}(x) \geq 0 = \tilde{r}(y) \). While, if \( \alpha > 0 \), then \( y \in \tilde{r}_\alpha \). Since \( \tilde{s}_\alpha \ll_{K} \tilde{r}_\alpha \), there exists \( x \in \tilde{s}_\alpha \) such that \( x \ll_{K} y \). And we have \( \tilde{s}(x) \geq \alpha = \tilde{r}(y) \).

Therefore we get \( \tilde{s} \ll_{K} \tilde{r} \) on \( \mathcal{F}(\mathbb{R}^n) \) from the above Case (a') and (b'). Thus we obtain this lemma. Q.E.D.

Define the dual cone of a cone \( K \) by
\[
K^+ := \{ a \in \mathbb{R}^n \mid a \cdot x \geq 0 \text{ for all } x \in K \},
\]
where \( x \cdot y \) denotes the inner product on \( \mathbb{R}^n \) for \( x, y \in \mathbb{R}^n \). For a subset \( A \subset \mathbb{R}^n \) and \( a \in \mathbb{R}^n \), we define
\[
(2.1) \quad a \cdot A := \{ a \cdot x \mid x \in A \} \subset \mathbb{R}.
\]
The equation (2.1) means the projection of \( A \) on the extended line of the vector \( a \) if \( a \cdot a = 1 \). It is trivial that \( a \cdot A \subset C(\mathbb{R}) \) if \( A \subset C(\mathbb{R}^n) \) and \( a \in \mathbb{R}^n \).

**Lemma 2.4.** Let \( A, B \in C(\mathbb{R}^n) \). \( A \ll_{K} B \) on \( C(\mathbb{R}^n) \) if and only if \( a \cdot A \ll_{1} a \cdot B \) on \( C(\mathbb{R}) \) for all \( a \in K^+ \), where \( \ll_{1} \) is the natural order on \( C(\mathbb{R}) \).

**Proof.** Suppose \( A \ll_{K} B \) on \( C(\mathbb{R}^n) \). Consider the two cases (a) and (b). Case(a): For any \( a \cdot x \in a \cdot A \), there exists \( y \in B \) such that \( x \ll_{K} y \). Then \( y - x \in K \). If \( a \in K^+ \), then \( a \cdot (y - x) \geq 0 \) and i.e. \( a \cdot x \leq a \cdot y \). Case(b): For any \( a \cdot y \in a \cdot B \), there exists \( x \in A \) such that \( x \ll_{K} y \). Then \( y - x \in K \). If \( a \in K^+ \), then \( a \cdot (y - x) \geq 0 \) and i.e. \( a \cdot x \leq a \cdot y \). From the above cases (a) and (b), we have that \( a \cdot A \ll_{1} a \cdot B \).

On the other hand, to prove the inverse statement, we assume that \( A \ll_{K} B \) on \( C(\mathbb{R}^n) \) does not hold. Then we have the following two cases (i) and (ii). Case(i): There exists \( x \in A \) such that \( y - x \nexists K \) for all \( y \in B \). Then \( B \cap (x + K) = \emptyset \). Since \( B \) and \( x + K \) are closed convex, by the separation theorem there exists \( a \in \mathbb{R} \) \( a \neq 0 \) such that \( a \cdot y < a \cdot x + a \cdot z \) for all \( y \in B \) and all \( z \in K \). Hence we suppose that there exists \( z \in K \) such that \( a \cdot z \geq 0 \). Then \( \lambda z \in K \) for all \( \lambda \geq 0 \) since \( K \) is a cone, and so we have \( a \cdot x + a \cdot \lambda z = a \cdot x + \lambda a \cdot z \to -\infty \) as \( \lambda \to \infty \). This contradicts \( a \cdot y < a \cdot x + a \cdot z \). Therefore we obtain \( a \cdot z \geq 0 \) for all \( z \in K \). Especially taking \( z = 0 \in K \), we get \( a \cdot y < a \cdot x \) for all \( y \in B \). This contradicts \( a \cdot A \ll_{1} a \cdot B \). Case(ii): There exists \( y \in B \) such that \( y - x \nexists K \) for all \( x \in A \). Then we derive the contradiction in a similar way to the case (i).
Therefore the inverse statement holds from the results of the above (i) and (ii). The proof of this lemma is completed. Q.E.D.

For \( a \in \mathbb{R}^n \) and \( \tilde{s} \in \mathcal{F}(\mathbb{R}^n) \), we define a fuzzy number \( a \cdot \tilde{s} \in \mathcal{F}(\mathbb{R}) \) by
\[
(2.2) \quad a \cdot \tilde{s}(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{a \cdot \tilde{s}}(x)\}, \quad x \in \mathbb{R},
\]
where \( 1_D(\cdot) \) is the classical indicator function of a closed interval \( D \in \mathcal{C}(\mathbb{R}) \).

We define a partial relation \( \preceq_M \) on \( \mathcal{F}(\mathbb{R}) \) as follows (\cite{8}): For \( \tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n) \), \( \tilde{s} \preceq_M \tilde{r} \) means that \( \tilde{s}_\alpha \preceq_M \tilde{r}_\alpha \) for all \( \alpha \in [0,1] \).

The following theorem gives the correspondence between the pseudo-order \( \preceq_K \) on \( \mathcal{F}(\mathbb{R}^n) \) and the fuzzy max order \( \preceq_M \) on \( \mathcal{F}(\mathbb{R}) \).

**Theorem 2.1.** For \( \tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n) \), \( \tilde{s} \preceq_K \tilde{r} \) iff \( a \cdot \tilde{s} \preceq_M a \cdot \tilde{r} \) for all \( a \in K^+ \).

**Proof.** From Lemmas 2.3 and 2.4, \( \tilde{s} \preceq_K \tilde{r} \) iff \( a \cdot \tilde{s}_\alpha \preceq_K a \cdot \tilde{r}_\alpha \) for all \( a \in K^+ \) and \( \alpha \in (0,1] \). Is equivalent to \( a \cdot \tilde{s} \preceq_M a \cdot \tilde{r} \) for all \( a \in K^+ \). Q.E.D.

For \( \{\tilde{s}_k\}_{k=1}^\infty \subset \mathcal{F}(\mathbb{R}^n) \) and \( \tilde{s} \in \mathcal{F}(\mathbb{R}^n) \), \( \lim_{k \to \infty} \tilde{s}_k = \tilde{s} \) means that \( \sup_{\alpha \in [0,1]} \rho(\tilde{s}_{k,\alpha}, \tilde{s}_\alpha) \to 0 \) \((k \to \infty)\), where \( \tilde{s}_{k,\alpha} \) is the \( \alpha \)-cut of \( \tilde{s}_k \) and \( \rho \) is the Hausdorff metric on \( \mathcal{C}(\mathbb{R}^n) \).

**Lemma 2.5.** Let \( \{\tilde{s}_k\}_{k=1}^\infty \subset \mathcal{F}(\mathbb{R}) \) and \( \tilde{s} \in \mathcal{F}(\mathbb{R}) \) such that \( \tilde{s}_k \preceq_M \tilde{s}_{k+1} \) \((k \geq 1)\) and \( \lim_{k \to \infty} \tilde{s}_k = \tilde{s} \). Then \( \tilde{s}_1 \preceq_M \tilde{s} \).

**Proof.** Trivial. Q.E.D.

**Theorem 2.2.** Let \( \{\tilde{s}_k\}_{k=1}^\infty \subset \mathcal{F}(\mathbb{R}^n) \) and \( \tilde{s} \in \mathcal{F}(\mathbb{R}^n) \) such that \( \tilde{s}_k \preceq_K \tilde{s}_{k+1} \) \((k \geq 1)\) and \( \lim_{k \to \infty} \tilde{s}_k = \tilde{s} \). Then \( \tilde{s}_1 \preceq_K \tilde{s} \).

**Proof.** From Theorem 2.1, for all \( a \in K^+ \) we have \( a \cdot \tilde{s}_k \preceq_K a \cdot \tilde{s}_{k+1} \) \((k \geq 1)\) and \( \lim_{k \to \infty} a \cdot \tilde{s}_k = a \cdot \tilde{s} \). By Lemma 2.3, \( a \cdot \tilde{s}_1 \preceq_K a \cdot \tilde{s} \) all \( a \in K^+ \). From Theorem 2.1, \( \tilde{s}_1 \preceq_K \tilde{s} \). Q.E.D.

**Remark.** Let the map \( g : [0,1] \to \mathcal{F}(\mathbb{R}^n) \) be continuous. A point \( x_0 \in [0,1] \) is said to be efficient if \( x_0 \in [0,1] \) and \( g(x_0) \preceq_K g(x) \) for some \( x \in [0,1] \) implies \( g(x) = g(x_0) \). Then, by applying the same idea as in Lemma 3.2 of Furukawa [2], we observe that there exists at least one efficient point in \([0,1] \). In fact, considering, if necessary, a partial order \( \preceq_K \) on the class of the quotient sets with respect to the equivalence relation \( \sim_K \) defined by \( \tilde{s} \sim_K \tilde{r} \) iff \( \tilde{s} \preceq_K \tilde{r} \) and \( \tilde{r} \preceq_K \tilde{s} \), we can assume that \( \preceq_K \) is a partial order on \( \mathcal{F}(\mathbb{R}^n) \). By theorem 2.2 and the continuity of \( g \), \( \{g(x) \mid x \in [0,1]\} \) can be proved to be an inductively ordered set. So, by Zorn's lemma \( \{g(x) \mid x \in [0,1]\} \) has an efficient element.

3. Further results

In this section, we investigate a pseudo-order \( \preceq_K \) on \( \mathcal{F}_r(\mathbb{R}^n) \) for a polyhedral cone \( K \) with \( K^+ \subset \mathbb{R}^n \). To this end, we need the following lemma.
Lemma 3.1. Let $a, b \in \mathbb{R}_{+}^{n}$ and $A \in C_{r}(\mathbb{R}^{n})$. Then for any scalars $\lambda_{1}, \lambda_{2} \geq 0$, it holds

\begin{equation}
(\lambda_{1}a + \lambda_{2}b) \cdot A = \lambda_{1}(a \cdot A) + \lambda_{2}(b \cdot A),
\end{equation}

where the arithmetic in (3.1) is defined in (2.1).

Proof. Let $\lambda_{1}a \cdot x + \lambda_{2}b \cdot y \in \lambda_{1}(a \cdot A) + \lambda_{2}(b \cdot B)$ with $x, y \in A$. It suffices to show that $\lambda_{1}a \cdot x + \lambda_{2}b \cdot y \in (\lambda_{1}a + \lambda_{2}b) \cdot A$. Define $z = (z_{1}, z_{2}, \ldots, z_{n})$ by

\begin{align*}
z_{i} &= (\lambda_{1}a_{i}x_{i} + \lambda_{2}b_{i}y_{i})/((\lambda_{1}a_{i} + \lambda_{2}b_{i}) \text{ if } (\lambda_{1}a_{i} + \lambda_{2}b_{i}) > 0) \\
&= x_{i} \quad \text{if } (\lambda_{1}a_{i} + \lambda_{2}b_{i}) = 0 \quad (i = 1, 2, \ldots, n)
\end{align*}

Then, clearly $(\lambda_{1}a + \lambda_{2}b) \cdot z = \lambda_{1}a \cdot x + \lambda_{2}b \cdot y$. Since $A \in C_{r}(\mathbb{R}^{n}), z \in A$, so that $\lambda_{1}a \cdot x + \lambda_{2}b \cdot y = (\lambda_{1}a + \lambda_{2}b) \cdot A$. Q.E.D.

Henceforth, we assume that $K$ is a polyhedral convex cone with $K^{+} \subset \mathbb{R}^{n}$, i.e., there exist vectors $b^{i} \in \mathbb{R}_{+}^{n}(i = 1, 2, \ldots, m)$ such that

\begin{equation*}
K = \{x \in \mathbb{R}^{n} | b^{i} \cdot x \leq 0 \text{ for all } i = 1, 2, \ldots, m\}.
\end{equation*}

Then, it is well-known (c.f. [9]) that $K^{+}$ is expressed as

\begin{equation*}
K^{+} = \{x \in \mathbb{R}^{n} | x = \sum_{i=1}^{m} \lambda_{i}b^{i}, \lambda_{i} \geq 0 (i = 1, 2, \ldots, m)\}.
\end{equation*}

The above dual cone $K^{+}$ is denoted simply by

\begin{equation*}
K^{+} = \text{conv}\{b^{1}, b^{2}, \ldots, b^{m}\}.
\end{equation*}

The pseudo-order $\preceq_{K}$ on $C_{r}(\mathbb{R}^{n})$ is characterized in the following.

Corollary 3.1. Let $K^{+} = \text{conv}\{b^{1}, b^{2}, \ldots, b^{m}\}$ with $b^{i} \in \mathbb{R}_{+}^{n}$. Then, for $A, B \in C_{r}(\mathbb{R}^{n})$, $A \preceq_{K} B$ if and only if $b^{i} \cdot A \preceq_{1} b^{i} \cdot B$ for all $i = 1, 2, \ldots, m$, where $\preceq_{1}$ is a pseudo-order on $C_{r}(\mathbb{R})$.

Proof. We assume that $b^{i} \cdot A \preceq_{1} b^{i} \cdot B$ for all $i = 1, 2, \ldots, m$. For any $a \in K^{+}$, there exists $\lambda_{i} \geq 0$ with $a = \sum_{i=1}^{m} \lambda_{i}b^{i}$. From Lemma 3.1 we have:

\begin{equation*}
a \cdot A = \sum_{i=1}^{m} \lambda_{i}(b^{i} \cdot A) \preceq_{1} \sum_{i=1}^{m} \lambda_{i}(b^{i} \cdot B) = a \cdot B.
\end{equation*}

Thus, by Lemma 2.4, $A \preceq_{K} B$ follows. By applying Lemma 2.4 again, the 'only if' part of Corollary holds. Q.E.D.

Lemma 3.2. Let $a, b \in \mathbb{R}_{+}^{n}$ and $\tilde{s} \in \mathcal{F}_{r}(\mathbb{R}^{n})$. Then, for any $\lambda_{1}, \lambda_{2} \geq 0$, a

\begin{equation}
(\lambda_{1}a + \lambda_{2}b) \cdot \tilde{s} = \lambda_{1}(a \cdot \tilde{s}) + \lambda_{2}(b \cdot \tilde{s}),
\end{equation}

where the arithmetic in (3.2) is defined in (2.1).
where the arithmetic in (3.2) are given in (1.1), (1.2) and (2.2).

Proof. For any $\alpha \in [0,1]$, it follows from the definition and Lemma 3.1 that

$$ [(\lambda_1 a + \lambda_2 b) \cdot \tilde{s}]_\alpha = (\lambda_1 a + \lambda_2 b) \cdot \tilde{s}_\alpha = \lambda_1 (a \cdot \tilde{s}_\alpha) + \lambda_2 (b \cdot \tilde{s}_\alpha) $$

$$ = \lambda_1 (a \cdot \tilde{s})_\alpha + \lambda_2 (b \cdot \tilde{s})_\alpha = [\lambda_1 (a \cdot \tilde{s}) + \lambda_2 (b \cdot \tilde{s})]_\alpha. $$

The last equality follows from (1.3). The above shows that (3.3) holds. Q.E.D.

The main results in this section are given in the following.

Theorem 3.1. Let $K^+ = \text{conv}\{b^1, b^2, \ldots, b^m\}$ with $b^i \in \mathbb{R}^n$. Then, for $\tilde{s}, \tilde{r} \in \mathcal{F}_r(\mathbb{R}^n)$,

$$ \tilde{s} \preceq_K \tilde{r} \quad \text{if and only if} \quad b^i \cdot \tilde{s} \preceq_M b^i \cdot \tilde{r} \quad \text{for} \quad i = 1, 2, \ldots, m. $$

Proof. It suffices to prove the 'if' part of Theorem 3.1. For any $a \in K^+$, there exists $\lambda_i \geq 0$ with $a = \sum_{i=1}^{m} \lambda_i b^i$. Applying Lemma 3.2, we have

$$ a \cdot \tilde{s} = \sum_{i=1}^{m} \lambda_i (b^i \cdot \tilde{s}) \preceq_M \sum_{i=1}^{m} \lambda_i (b^i \cdot \tilde{r}) = a \cdot \tilde{r}, $$

From Theorem 2.1, $\tilde{s} \preceq_k \tilde{r}$ follows. Q.E.D.

Figure 1: $\hat{v} = \sup(\tilde{s}, \tilde{r})$

When $K = \mathbb{R}^n$, the pseudo-order $\preceq_K$ on $\mathcal{F}_r(\mathbb{R}^n)$ will be simply written by $\preceq_n$. Obviously, $\preceq_1$ and $\preceq_M$ are the same.
Congxin and Cong [1] described the structure of the fuzzy number lattice \((\mathcal{F}_r(\mathbb{R}), \prec_1)\).

When \(K = \mathbb{R}^n\), \(K^+ = \mathbb{R}^n\) and \(K^+ = \text{conv}\{e^1, e^2, \cdots, e^m\}\). So that, by Theorem 3.1, we see that for \(\tilde{s}, \tilde{r} \in \mathcal{F}_r(\mathbb{R}^n)\), \(\tilde{s} \prec_n \tilde{r}\) means \(e^i \tilde{s} \prec e^i \tilde{r}\) for all \(i = 1, 2, \cdots, n\). Therefore, by applying the same method as [1], we can describe the structure of the fuzzy set lattice \((\mathcal{F}_r(\mathbb{R}^n), \prec_n)\).

Figure 1 illustrates \(\sup(\tilde{s}, \tilde{r})\) for \(\tilde{s}, \tilde{r} \in \mathcal{F}_r(\mathbb{R}^2)\).

References


