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Polytopes of linear programming relaxation for triangulations

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Abstract

Universal polytope is the polytope defined as the convex hull of the characteristic vectors of all triangulations for a given point configuration. The equality system defining this polytope was found, but the system of inequalities are not known yet. Larger polytopes, corresponding to linear programming relaxations, have been used in practice. We show that (1) the universal polytope, the polytope of relaxation for (2) clique, (3) cocircuit and (4) chamber conditions have inclusion relation in this order. Examples of point configurations for which these polytopes coincide and differ are given. We also discuss briefly on the difficulty of giving inequalities for the universal polytope.

1 Introduction

Triangulation has been an important subject in several areas such as computational geometry and mathematics. One natural approach for handling all possible triangulations of a given point configuration is to investigate the polytope made as the convex hull of the characteristic vectors of the triangulations.

This polytope, the universal polytope, was first studied by Billera et al. with relation to the secondary polytope [2]. Let the configuration be one of n points spanning the d dimensional space. They showed that this polytope has dimension $\binom{n-1}{d+1}$, lying in the space of d-simplices which has dimension $\binom{n}{d+1}$. De Loera et al. studied the relation of this universal polytope and some larger polytopes, and gave explicit descriptions for the equality conditions of this polytope [4]. However, the description for the inequality conditions have not been found yet.

In computational geometry, an important subject is to find the optimal triangulation according to some cost. For example, the minimum weight triangulation is among those problems. These problems can be thought of as optimization problems on the universal polytope. However, neither description of this polytope, by inequalities or by vertices, can be obtained easily.

Practical approaches taken recently begin the computation by a polytope larger than the universal polytope [5] [11]. These polytopes are the polytopes of linear programming relaxation of the universal polytope.

In this paper, we will show set inclusions of the universal polytope and several polytopes of relaxation (section 2). We also discuss briefly on the difficulty of enumerating inequalities for the universal polytope (section 3).

2 Polytopes of relaxation

Let $\mathcal{A} \subset \mathbb{R}^d$ be a point configuration of n points spanning the d dimensional space. A set of points $\sigma \subset \mathcal{A}$ is a *simplex* if the points are affinely independent. Its dimension is $\#\sigma - 1$. A simplex of dimension i is called an i-simplex in short. We represent the set of d-simplices by $\mathcal{S}_d$. The polytopes we consider will be polytopes in $\mathbb{R}^{\mathcal{S}_d}$.

We define two simplices σ, τ to *intersect properly* if $\mathrm{conv}(\sigma) \cap \mathrm{conv}(\tau) = \mathrm{conv}(\sigma \cap \tau)$. If not, they *intersect improperly.*

For a d-simplex σ, we denote its volume by $v_\sigma = \mathrm{vol}(\sigma)$, and the volume of the whole convex hull by $V = \mathrm{vol}(\mathrm{conv}(\mathcal{A}))$. We name the inequality $\sum_{\sigma \in \mathcal{S}_d} v_\sigma x_\sigma \leq V$ in $\mathbb{R}^{\mathcal{S}_d}$ the *volume inequality*, and the equality obtained by replacing the inequality by equality the *volume equality.* We can also define volume (in)equality for sets of d-simplices by setting the characteristic vector as the value of the variable.

A set of d-simplices is a *triangulation* if they (1) intersect properly and (2) satisfy the volume equality. The *universal polytope* defined by Billera et al. [2] is the convex hull of the characteristic vectors of triangulations:

$$P_{\text{universal}} = \text{conv}\{\chi_T \in \mathbb{R}^{S_d} : T \text{ is a triangulation}\}.$$

The *intersection graph* of d-simplices is a graph with the d-simplices the vertices and edges between pairs of improperly intersecting d-simplices. A set of d-simplices is a triangulation if and only if (1) it is a (maximal) independent set of the intersection graph and (2) suffices the volume equality. A maximal independent set is not necessarily a triangulation. Such example can be made by Schönhardt's polytope [10] [12]. The *independent set polytope* is the convex hull of the characteristic vectors of independent sets:

$$P_{\text{independent}} = \text{conv}\{\chi_I : I \text{ is an independent set of the intersection graph}\}.$$

We immediately have the following lemma.

Lemma 2.1

The volume inequality is valid on the independent set polytope, and the face defined by this becomes the universal polytope:

$$P_{\text{universal}} = P_{\text{independent}} \cap \{\boldsymbol{x} : \sum_{\sigma \in S_d} v_\sigma x_\sigma = V\}$$

Proof. The d-simplices in an independent set only have overlap with volume zero. Thus, any independent set does not have more volume than the whole convex hull, and satisfies the volume inequality. Since the independent set polytope was the convex hull of the characteristic vectors of the independent sets, it also satisfies the volume inequality.

Any triangulation is an independent set satisfying the volume equality. Thus, its characteristic vector belongs to the polytope in the right side. The universal polytope was the convex hull of such vectors, thus is included in the right side.

Conversely, take a point from the right side. It is a convex combination of independent sets. Since the whole combination satisfies the volume equality, all of the independent sets making the combination should satisfy the equality, thus are triangulations. Hence, this combination is a point of the left side. □

Any triangulation includes at most one element of a clique of the intersection graph. It satisfies the *clique inequality* $\sum_{\sigma \in \text{clique}} x_\sigma \leq 1$. The set of points satisfying this condition for all cliques becomes the polytope

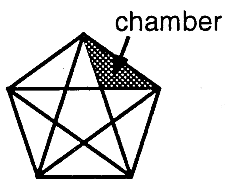
$$P_{\text{clique} \leq 1} = \{\boldsymbol{x} \geq \mathbf{0} \ : \ \forall \text{ (maximal) clique of the intersection graph } \sum_{\sigma \in \text{clique}} x_\sigma \leq 1\}.$$

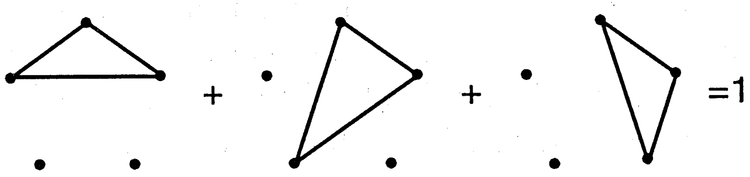
In the proof of theorem 2.3 we will show that the volume inequality is valid for this polytope. Then its face

$$P_{\text{clique} \leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma \in \mathcal{S}_d} v_\sigma x_\sigma = V\}$$

becomes the feasible points of the linear programming relaxation by clique conditions.

We next define polytopes described by chamber conditions [1] [3] [4]. Consider the intersections of the convex hulls of d-simplices. A chamber is such set having positive volume and minimal with respect to inclusion as point sets. Take a chamber. Any triangulation has exactly one d-simplex whose convex hull includes that chamber.





The (in)equality $\sum_{\sigma:\, \text{chamber} \subset \text{conv}(\sigma)} x_\sigma = (\leq) 1$ is called the *chamber (in)equality*. Polytopes defined by these chamber conditions are

$$P_{\text{chamber} \leq 1} = \{\boldsymbol{x} \geq \mathbf{0} : \forall \text{ chamber} \sum_{\sigma:\, \text{chamber} \subset \text{conv}(\sigma)} x_\sigma \leq 1\},$$

$$P_{\text{chamber}=1} = \{\boldsymbol{x} \geq \mathbf{0} : \forall \text{ chamber} \sum_{\sigma:\, \text{chamber} \subset \text{conv}(\sigma)} x_\sigma = 1\}.$$

$P_{\text{chamber}=1}$ is the feasible points of the linear programming relaxation by chamber conditions. We have a lemma for these polytopes.

Lemma 2.2

The volume inequality is valid on $P_{\text{chamber}\leq 1}$, and it defines the face

$$P_{\text{chamber}=1} = P_{\text{chamber}\leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = V\}$$

Proof. The convex hull of each d-simplex can be divided into several chambers. We can sum up the volume by chambers:

$$\sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = \sum_{\text{chamber}} \text{vol(chamber)} \sum_{\sigma:\,\text{chamber}\subset\text{conv}(\sigma)} x_\sigma.$$

This shows that the volume inequality is valid on $P_{\text{chamber}\leq 1}$.

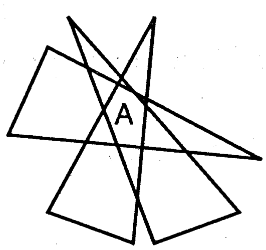
$P_{\text{chamber}=1}$ is included in $P_{\text{chamber}\leq 1}$, and the above rewriting shows that the volume equality holds on $P_{\text{chamber}=1}$. Thus the left side is included in the right side.

Conversely, take a point from $P_{\text{chamber}\leq 1}$ satisfying the volume equality. If there exists a chamber with $\sum_{\sigma:\,\text{chamber}\subset\text{conv}(\sigma)} x_\sigma < 1$ the volume sum cannot reach V. Thus all chambers must have $\sum_{\sigma:\,\text{chamber}\subset\text{conv}(\sigma)} x_\sigma = 1$. Hence, it is a point of $P_{\text{chamber}=1}$. □

Theorem 2.3

$$P_{\text{universal}} \subset P_{\text{clique}\leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = V\} \subset P_{\text{chamber}=1}$$

Proof. First, we show $P_{\text{independent}} \subset P_{\text{clique}\leq 1} \subset P_{\text{chamber}\leq 1}$. Any independent set satisfies the clique conditions. $P_{\text{independent}}$ is the convex hull of the characteristic vectors of independent sets, thus is included in $P_{\text{clique}\leq 1}$. For the second inclusion, observe that for any chamber the d-simplices including it are making a clique in the intersection graph.

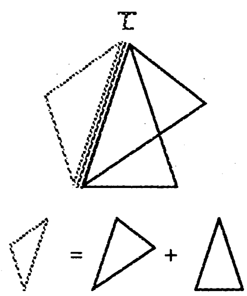


Thus the clique condition $\sum_{\sigma\in\text{clique}} x_\sigma \leq 1$ implies the chamber condition $\sum_{\sigma:\,\text{chamber}\subset\text{conv}(\sigma)} x_\sigma \leq 1$, and we have $P_{\text{clique}\leq 1} \subset P_{\text{chamber}\leq 1}$.

The volume inequality is valid on $P_{\text{chamber}\le 1}$ by lemma 2.2. The inclusion above shows that it is valid also on $P_{\text{independent}}$ and $P_{\text{clique}\le 1}$. Thus the volume inequality defines faces of these polytopes and we also have inclusion relation among them: $P_{\text{independent}} \cap \{\boldsymbol{x} : \sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = V\} \subset P_{\text{clique}\le 1} \cap \{\boldsymbol{x} : \sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = V\} \subset P_{\text{chamber}\le 1} \cap \{\boldsymbol{x} : \sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = V\}$. By lemmas 2.1, 2.2, this means $P_{\text{universal}} \subset P_{\text{clique}\le 1} \cap \{\boldsymbol{x} : \sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = V\} \subset P_{\text{chamber}=1}$ $\square$

The last conditions we consider are the cocircuit conditions by de Loera et al. [4]. A $(d-1)$-simplex is in interior if its convex hull is not included in the boundary of $\operatorname{conv}(\mathcal{A})$. For such interior $(d-1)$-simplex τ, the *interior cocircuit condition* is

$$\sum_{\sigma\in\mathcal{S}_d:\ \sigma \text{ is on one side of } \tau} x_\sigma = \sum_{\sigma\in\mathcal{S}_d:\ \sigma \text{ is on the other side of } \tau} x_\sigma.$$



Any triangulation satisfies these conditions. The last polytope is defined as

$$P_{\text{cocircuit}} = \operatorname{aff}(P_{\text{universal}}) \cap \{\boldsymbol{x} \ge \boldsymbol{0}\}.$$

De Loera et al. showed the following theorem.

Theorem 2.4 ([4, theorem 1.1])

$P_{\text{cocircuit}}$ is the polytope described by the interior cocircuit conditions and one non-homogeneous equality (e.g. the volume equality).

$P_{\text{cocircuit}}$ is the feasible points of the linear programming relaxation by cocircuit conditions.

The following is our second theorem.

Theorem 2.5

$$P_{\text{clique}\le 1} \cap \{\boldsymbol{x} : \sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = V\} \subset P_{\text{cocircuit}} \subset P_{\text{chamber}=1}$$

Proof. By theorem 2.3, all points in $P_{\text{universal}}$ satisfy $\sum_{\sigma:\,\text{chamber}\subset\text{conv}(\sigma)} x_\sigma = 1$ for all chambers. The points in $\text{aff}(P_{\text{universal}})$ also satisfy them. Thus $P_{\text{cocircuit}} \subset P_{\text{chamber}=1}$.

For the first inclusion, we have to check that any $\boldsymbol{x} \in P_{\text{clique}\leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = V\}$ satisfies the cocircuit conditions. Suppose it was not satisfied. We should have some interior $(d-1)$-simplex τ with $\sum_{\sigma:\text{ first side of }\tau} x_\sigma > \sum_{\sigma:\text{ second side of }\tau} x_\sigma$. Since we took $\boldsymbol{x}$ from $\boldsymbol{x} \in P_{\text{clique}\leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = V\}$, and this polytope is included in $P_{\text{chamber}=1}$ by theorem 2.3, $\boldsymbol{x}$ satisfies the chamber equality for all chambers.

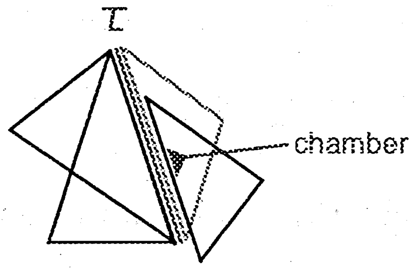
Take a chamber adjacent to τ on the second side. The set

$$
\begin{aligned}
&\{\sigma : \text{ adjacent to } \tau \text{ on the first side}\}\\
&\cup\,\{\sigma : \text{ adjacent to } \tau \text{ on the second side, but intersecting improperly,}\\
&\qquad\quad \tau \subset \text{conv}(\sigma)\}\\
&\cup\,\{\sigma : \text{ crossing } \tau, \tau \subset \text{conv}(\sigma)\}
\end{aligned}
$$

is making a clique in the intersection graph. However,

$$
\begin{aligned}
(*)\quad & \sum_{\substack{\sigma:\text{ adjacent to }\tau\text{ on}\\ \text{the first side}}} x_\sigma + \sum_{\substack{\sigma:\text{ adjacent to }\tau\text{ on the second side, but}\\ \text{intersecting improperly, }\tau\subset\text{conv}(\sigma)}} x_\sigma + \sum_{\substack{\sigma:\text{ crossing }\tau,\\ \tau\subset\text{conv}(\sigma)}} x_\sigma\\
& > \sum_{\substack{\sigma:\text{ adjacent to }\tau\text{ on}\\ \text{the second side}}} x_\sigma + \sum_{\substack{\sigma:\text{ adjacent to }\tau\text{ on the second side, but}\\ \text{intersecting improperly, }\tau\subset\text{conv}(\sigma)}} x_\sigma + \sum_{\substack{\sigma:\text{ crossing }\tau,\\ \tau\subset\text{conv}(\sigma)}} x_\sigma\\
& = \sum_{\sigma:\,\tau\subset\text{conv}(\sigma)} x_\sigma\\
& = 1,
\end{aligned}
$$

but since $\boldsymbol{x}$ was satisfying the clique conditions, $(*) \leq 1$, a contradiction.



□

Theorems 2.3, 2.5 lead the main theorem.

Theorem 2.6

$$P_{\text{universal}} \subset P_{\text{clique}\leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma \in \mathcal{S}_d} v_\sigma x_\sigma = V\} \subset P_{\text{cocircuit}} \subset P_{\text{chamber}=1}$$

Remark 2.7

For the polytope $P_{\text{clique}\leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma \in \mathcal{S}_d} v_\sigma x_\sigma = V\}$, we can get rid of the volume equality, which has coefficient other than 0/1, adding many 0/1 conditions instead:

$$\begin{aligned} & P_{\text{clique}\leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma \in \mathcal{S}_d} v_\sigma x_\sigma = V\} \\ &= P_{\text{clique}\leq 1} \cap P_{\text{chamber}=1} \cap \{\boldsymbol{x} : \sum_{\sigma \in \mathcal{S}_d} v_\sigma x_\sigma = V\} \\ &= P_{\text{clique}\leq 1} \cap P_{\text{chamber}=1} \\ &= P_{\text{clique}\leq 1} \cap P_{\text{chamber}\geq 1} \end{aligned}$$

Next we will give examples of point configurations to show that these inclusions can be either equal or proper.

Example 2.8 (polygon)

For the vertices of a polygon de Loera et al. [4, theorem 4.1] showed

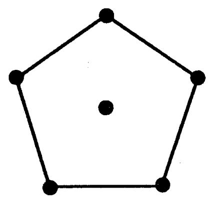
$$P_{\text{universal}} = P_{\text{clique}\leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma \in \mathcal{S}_d} v_\sigma x_\sigma = V\} = P_{\text{cocircuit}} = P_{\text{chamber}=1}.$$

Example 2.9 (regular pentagon with a point in the center)

For this example from [4, example 4.2], the inclusion becomes

$$P_{\text{universal}} \underset{\neq}{\subset} P_{\text{clique}\leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma \in \mathcal{S}_d} v_\sigma x_\sigma = V\} = P_{\text{cocircuit}} = P_{\text{chamber}=1},$$

and all the polytopes except $P_{\text{universal}}$ has a fractional vertex.



Example 2.10 (regular 9-gon with a point in the center)

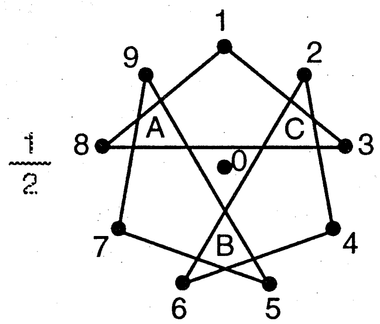
In this example, the regions A,B,C show that triangles $\langle 138\rangle$, $\langle 246\rangle$, $\langle 579\rangle$ form a clique in the intersection graph. Thus, the inequality

$$x_{\langle 138\rangle} + x_{\langle 246\rangle} + x_{\langle 579\rangle} \leq 1$$

must hold on $P_{\text{clique}\leq 1}$. On the other hand, there exists a point of $P_{\text{cocircuit}}$ with $x_{\langle 138\rangle} = x_{\langle 246\rangle} = x_{\langle 579\rangle} = 1/2$ violating that inequality:

$$\begin{aligned}
&x_{\langle 138\rangle} = x_{\langle 246\rangle} = x_{\langle 579\rangle} = x_{\langle 038\rangle} = x_{\langle 026\rangle} = x_{\langle 059\rangle} = x_{\langle 029\rangle} = x_{\langle 035\rangle} = x_{\langle 068\rangle}\\
&= x_{\langle 123\rangle} = x_{\langle 234\rangle} = x_{\langle 345\rangle} = x_{\langle 456\rangle} = x_{\langle 567\rangle} = x_{\langle 678\rangle} = x_{\langle 789\rangle} = x_{\langle 189\rangle} = x_{\langle 129\rangle}\\
&= 1/2,\\
&x_{\text{others}} = 0.
\end{aligned}$$

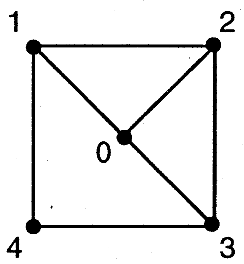
For this point configuration, $P_{\text{clique}\leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = V\} \subsetneq P_{\text{cocircuit}}$.



De Loera et al. showed that for points in general position, $P_{\text{cocircuit}} = P_{\text{chamber}=1}$ [4, proposition 2.5]. However, for degenerate point configurations, we cannot guarantee this.

Example 2.11 (square with a point in the center)

The point $x_{\langle 012\rangle} = x_{\langle 023\rangle} = x_{\langle 134\rangle} = 1$, $x_{\text{others}} = 0$ belongs to $P_{\text{chamber}=1} \setminus P_{\text{cocircuit}}$. Thus $P_{\text{cocircuit}} \subsetneq P_{\text{chamber}=1}$ for this point configuration. Further, $P_{\text{cocircuit}}$ has dimension 2, while $P_{\text{chamber}=1}$ has dimension 4.



Remark 2.12

The integers points of $P_{\text{clique}\leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = V\}$ and $P_{\text{cocircuit}}$ are the integer points of $P_{\text{universal}}$, thus correspond to the triangulations. However, as shown in example 2.11 $P_{\text{chamber}=1}$ can have extra integer vertices other than those.

Remark 2.13

By definition, $P_{\text{universal}}$ and $P_{\text{cocircuit}}$ have the same dimension. Thus, $P_{\text{clique}\leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = V\}$ also has the same dimension. However, the dimension of $P_{\text{chamber}=1}$ can be larger, as shown in example 2.11.

From the examples above, we obtain the following theorem.

Theorem 2.14

The polytopes, the inclusion of which we showed in theorem 2.6, can coincide or differ depending on the given point configuration.

The polytopes we handled correspond to linear relaxations of the universal polytope. Here we summarize briefly their efficiency in practice.

$P_{\text{clique}\leq 1} \cap \{\boldsymbol{x} : \sum_{\sigma\in\mathcal{S}_d} v_\sigma x_\sigma = V\}$ is the closest to the universal polytope. To describe this polytope, we have to enumerate all cliques of the intersection graph. This can be done by the generalized Paull-Unger procedure [8] with improvements by Tsukiyama et al. [7] [13]. This enumeration can be done in linear time with respect to the number of cliques, but with a large coefficient. The investigation of the computational complexity and the number of the cliques for the case of the intersection graph of d-simplices remains to be explored.

$P_{\text{chamber}=1}$ is not efficient, because (1) the number of chambers can be large and (2) there can be new integer points. Giving a base of the chamber conditions is done by Alekseyevskaya [1].

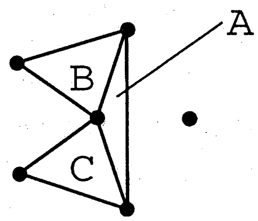
In practice, it has been shown that $P_{\text{cocircuit}}$ is the most efficient [5] [11].

3 Difficulty of enumerating the inequalities of $P_{\text{universal}}$

De Loera et al. gave descriptions for the equality system of the universal polytope. On the other hand, there are no results for giving descriptions of the *in*equalities using geometric information of the point configuration. Even partial results for this problem would be useful, because they can result in defining stricter relaxation polytopes for the universal polytope. However, the authors have doubts whether this problem is computationally easy. In this section, we would like to show some examples, which may have relation with the difficulty of this problem.

Example 3.1

For the example of regular pentagon with a point in the center (example 2.9), there is a facet $x_A \leq x_B + x_C$ in the universal polytope. This can be read as a condition "if $x_A = 1$ then either $x_B = 1$ or $x_C = 1$". However if there were other points in the left of the triangles A, B, $x_A = 1$, $x_B = x_C = 0$ can happen, and the condition does not hold. Thus, we can say that this facet is representing some "global" information of the point configuration, and this is one reason we think the problem might be difficult.



Ruppert et al. showed that deciding whether a nonconvex polyhedron can be triangulated or not without adding new vertices is NP-complete [9]. Triangulability can be judged by computing the intersection of the universal polytope and the hyperplanes $x_\sigma = 0$ for d-simplices σ not included in the polyhedron, which we cannot use. The polyhedron is triangulable if and only if this intersection is not empty. However, this connection does not immediately imply that giving inequalities is difficult. The triangulability problem is a decision problem and giving inequalities is an enumeration problem, And further, the connection between these problems are not straightforward.

The problem of computing the triangulation with minimum sum of edge lengths for a given point configuration in the plane is one of the famous problems in computational geometry. It is not know whether this problem is NP-complete or computable in polynomial time [6]. This problem can be solved as an optimization problem on the universal polytope. Even if the universal polytope has many facets, if we can generate (appropriate) inequalities of the universal polytope efficiently, we can use them as cutting planes, and compute this problem quickly.

Acknowledgments

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