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SPATIAL NUMERICAL RANGES OF ELEMENTS OF C*-ALGEBRAS

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1. INTRODUCTION AND RESULTS

A を複素ノルム環, A* をその双対空間, a を A の元とする。もし A が単位的であれば、集合

$$V(A, a) \equiv \{f(a) : f \in A^*, \|f\| = f(1) = 1\}$$

は, a の (algebra) numerical range と呼ばれ, それは複素平面 C 上の空でないコンパクト凸部分集合であることが知られている ([1, p. 52] 参照) 。しかしながら A が非単位的であれば, この定義は意味をなさない。この場合我々は次の二つの集合を導入する:

$$V_1(A, a) = \{f(xa) : \textit{there exist } f \in A^* \textit{ and } x \in A \textit{ such that } \|f\| = \|x\| = f(x) = 1\}$$

and

$$V_2(A, a) = \{f(ax) : \textit{there exist } f \in A^* \textit{ and } x \in A \textit{ such that } \|f\| = \|x\| = f(x) = 1\} .$$

勿論 A が単位的であれば $V(A, a) = V_1(A, a) = V_2(A, a)$ となっている。 A. K. Gaur and T. Husain [3] は $V_2(A, a)$ を特に spatial numerical range と呼び, この立場から研究を進めている。その中で, A が可換 C*-環であるときは、

$$\mathrm{co}\{\hat{a}(\varphi) : \varphi \in \Phi_A\} \subseteq V_2(A, a) \subseteq \overline{\mathrm{co}}\{\hat{a}(\varphi) : \varphi \in \Phi_A\}$$

が成り立つことを示している。ここに $\hat{a}$ は a の Gelfand 変換を表し, Φ_A は A の極大イデアル空間を表す ([3, Theorem 4.1] 参照) 。

本講演での我々の主目的は, C*-環の部分環における spatial numerical range は正汎函数の言葉で特徴付けられること, そしてその応用として Gaur - Husain の結果の非可換版が成立することを示すことにある。先ず主定理は次のように述べられる:

Theorem 1. Let A be a C*-algebra and B a subalgebra of A. Let $b \in B$. Then

$$V_1(B, b) = \{|f|(b) : \textit{there exist } f \in A^* \textit{ and } x \in B \textit{ such that } \|f\| = \|x\| = f(x) = 1\}$$

and

$$V_2(B, b) = \{|f|(b) : \textit{there exist } f \in A^* \textit{ and } x \in B \textit{ such that } \|f\| = \|x\| = f(x^*) = 1\},$$

where $|f|$ denotes the absolute value of f (cf. [, Definition 12.2.8]).

If B is a *-subalgbera of A, then $V_1(B, b) = V_2(B, b)$.

主定理の系として, Gaur - Husain [3, Theorem 4. 1] の非可換への拡張となっている次のような結果を得る：

Corollary 2. Let A be a C*-algbera and $a \in A$. Then

$$\mathrm{co}\{f(a) : f \in P(A)\} \subseteq V_1(A, a) = V_2(A, a) \subseteq \overline{\mathrm{co}}\{f(a) : f \in P(A)\},$$

where P(A) denotes the set of all pure states of A.

問題。いつ $\mathrm{co}\{f(a) : f \in P(A)\} = V_1(A, a)$ $(=V_2(A, a))$ が成立するか？またいつ $\overline{\mathrm{co}}\{f(a) : f \in P(A)\} = V_1(A, a)$ $(=V_2(A, a))$ が成立するか？

2. PROOFS OF THEOREM 1 AND COROLLARY 2

Proof of Theorem 1. Set

$$W_1 = \{|f|(b) : \textit{there exist } f \in A^* \textit{ and } x \in B \textit{ such that } \|f\| = \|x\| = f(x) = 1\}$$

and let $\lambda \in V_1(B, b)$. Then there exist $g \in B^*$ and $x \in B$ such that $\lambda = g(xb)$ and $\|g\| = \|x\| = g(x) = 1$. Take a functional $f \in A^*$ such that $f \mid B = g$ and $\|f\| = \|g\|$ and let $f = u \cdot |f|$ be the enveloping polar decomposition of f (cf. [2, Definition 12.2.8]). Then

$$1 = f(x) = |f|(ux) = (x \mid u^*)_{|f|} \leq \|x\|_{|f|} \|u^*\|_{|f|} \leq 1 \cdot 1 = 1, \tag{1}$$

so that we can find a scalar α satisfying

$$\|u^* - \alpha x\|_{|f|} = 0 \tag{2}$$

since the equality of the Cauchy-Schwarz inequality in (1) holds. Note that (1) implies

$$(u^* \mid x)_{|f|} = (x \mid u^*)_{|f|} = (u^* \mid u^*)_{|f|} = (x \mid x)_{|f|} = 1 \tag{3}$$

and hence $1 - \bar{\alpha} - \alpha + |\alpha|^2 = 0$ by (2). Therefore, α must be equal to 1, and so

$\|u^*-x\|_{|f|}=0$, that is u^*-x belongs to the left kernel (in the envelopig von Neumann algebra of A) $N_{|f|}=\{x\in A^{**}:|f|(x^*x)=0\}$ of $|f|$. Also since $|f|(x^*x)=(x \mid x)_{|f|}=\|x\|_{|f|}{}^2=1$ by (1), it follows that $1-x^*x\in N_{|f|}$, where 1 denotes the identity element of A** .
Therefore we have

$$\lambda=f(xb)=|f|(uxb)=(xb\mid u^*)_{|f|}=(xb\mid x)_{|f|}=|f|(x^*xb)=|f|(b)$$

(the 4th-equality follows from $u^*-x\in N_{|f|}$ and the 6th-equality follows from $1-x^*x\in N_{|f|}$) and hence $\lambda\in W_1$, so $V_1(B, b)\subseteq W_1$.

Conversely suppose $\lambda\in W_1$. Then there exist $f\in A^*$ and $x\in B$ such that $\lambda=|f|(b)$ and $\|f\|=\|x\|=f(x)=1$. Let $f=u\cdot|f|$ be the enveloping polar decomposition of f. Then we can apply directly the above arguments for f, x and u. Consequently, we have $f(xb)=|f|(b)$ and hence $\lambda\in V_1(B, b)$, so $W_1\subseteq V_1(B, b)$. We thus obatain $V_1(B, b)=W_1$.

We next set

$$W_2=\{|f|(b): \textit{there exist } f\in A^* \textit{ and } x\in B \textit{ such that } \|f\|=\|x\|=f(x^*)=1\}.$$

and let $\lambda\in V_2(B, b)$. Then there exist $g\in B^*$ and $x\in B$ such that $\lambda=g(bx)$ and $\|g\|=\|x\|=g(x)=1$. Take a functional $f\in A^*$ such that $f\mid B=g$ and $\|f\|=\|g\|$. Then

$$\|f^*\|=\|f\|=\|x\|=\|x^*\| \textit{ and } 1=f(x)=f^*(x^*),$$

so that $\bar{\lambda}=\overline{f(bx)}=f^*(x^*b^*)$, $\|f^*\|=\|f\|=\|x\|=\|x^*\|$ *and* $1=f(x)=f^*(x^*)$, and hence $\bar{\lambda}\in V_1(\bar{B}, b^*)$, where $\bar{B}=\{x\in A: x^*\in B\}$. Therefore by the preceding argument, we can find $h\in A^*$ and $y\in B$ such that $\bar{\lambda}=|h|(b^*)$ and $\|h\|=\|y\|=h(y^*)=1$. This means that $\lambda\in W_2$, so we have $V_2(B, b)\subseteq W_2$.

The inverse inclusion $W_2\subseteq V_2(B, b)$ can be easily obtained by tracing the converse of the above argument.

Set

$$A^*_{1,B}=\{f\in A^*: \|f\|=1 \textit{ and there exists } x\in B \textit{ such that } \|x\|=f(x)=1\}$$

and

$$A^*_{2,B}=\{f\in A^*: \|f\|=1 \textit{ and there exists } x\in B \textit{ such that } \|x\|=f(x^*)=1\}.$$

If B is a *-subalgebra, then $f\to f^*$ is a bijection of $A^*_{1,B}$ onto $A^*_{2,B}$ and hence we have

$$V_1(B, b)=\{|f|(b): f\in A^*_{1,B}\}=\{|f|(b): f\in A^*_{2,B}\}=V_2(B, b).$$

Q. E. D.

Proof of Corollary 2. Let A be a C*-algbera and $a \in A$. Then we have $V_1(A, a) = V_2(A, a)$ by Theorem 1. We next show that $\mathrm{co}\{f(a) : f \in P(A)\} \subseteq V_1(A, a)$. To do this, let $\alpha \in \mathrm{co}\{f(a) : f \in P(A)\}$. Then there exist $f_{11}, \dots, f_{1m_1}, \dots, f_{n1}, \dots, f_{nm_n} \in P(A)$ and $\lambda_{11}, \dots, \lambda_{1m_1}, \dots, \lambda_{n1}, \dots, \lambda_{nm_n} \geq 0$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij} = 1, \quad \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij} f_{ij}(a) = \alpha,$$

$$\pi_{f_{11}} \cong \dots \cong \pi_{f_{1m_1}}, \dots, \pi_{f_{n1}} \cong \dots \cong \pi_{f_{nm_n}} \text{ and } \pi_{f_{i1}} \neq \pi_{f_{j1}} \ (i \neq j).$$

Let $\pi_1 \cong \pi_{f_{11}} \cong \dots \cong \pi_{f_{1m_1}}, \dots, \pi_n \cong \pi_{f_{n1}} \cong \dots \cong \pi_{f_{nm_n}}$. For each $i, j \ (1 \leq i \leq n,\ 1 \leq j \leq m_i)$, choose an isomorphism U_{ij} of the Hilbert space H_{π_i} onto the Hilbert space $H_{\pi_{f_{ij}}}$ which transforms $\pi_i(x)$ into $\pi_{f_{ij}}(x)$ for every $x \in A$, and set $\xi_{ij} = U_{ij}^*(\xi_{f_{ij}})$. Also set $f = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij} f_{ij}$. Then we have $\|f\| = 1$, $f = |f|$, $\alpha = f(a)$ and

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij}(\pi_{f_{ij}}(x)\xi_{f_{ij}} \mid \xi_{f_{ij}}) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij}(\pi_i(x)\xi_{ij} \mid \xi_{ij}) \qquad (*)$$

for every $x \in A$. Furthermore since $\pi_1, \dots, \pi_n$ are mutually inequivalent, it follows that there exists a hermitian element $y \in A$ such that $\pi_i(y)\xi_{ij} = \xi_{ij} \ (1 \leq i \leq n,\ 1 \leq j \leq m_i)$ by [2, Theorem 2.8.3, (i)].

Consider the continuous function $h(t)$ on $[0, \infty)$ defined by

$$h(t) = \begin{cases} t, & if\ 0 \leq t \leq 1 \\ 1, & if\ t > 1 \end{cases},$$

and set $z = h(y^2)$. Then z is a positive element of A with $\|z\| \leq 1$. Moreover, we assert that

$$\pi_i(z)\xi_{ij} = \xi_{ij} \ (1 \leq i \leq n,\ 1 \leq j \leq m_i). \qquad (**)$$

In fact, let $\varepsilon > 0$ be arbitrary and take a polymonial $p(t)$ such that $p(0) = 0$ and $\sup\{|p(t) - h(t)| : 0 \leq t \leq \|z\|\} < \varepsilon / 2$. Let $1 \leq i \leq n$ and $1 \leq j \leq m_i$. Then

$$\begin{aligned} \|\pi_i(z)\xi_{ij} - \xi_{ij}\| &\leq \|\pi_i(h(y^2))\xi_{ij} - \pi_i(p(y^2))\xi_{ij}\| + \|p(\pi_i(y^2))\xi_{ij} - \xi_{ij}\| \\ &\leq \|h(y^2) - p(y^2)\| + |p(1) - 1| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and hence we obtain (**) since ε is arbitrary. By (*) and (**), we have

$$f(z) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij}(\pi_i(z)\xi_{ij} \mid \xi_{ij}) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij} = 1.$$

Consequently we have $\alpha \in V_1(A, a)$ and so $\mathrm{co}\{f(a) : f \in P(A)\} \subseteq V_1(A, a)$.

We next show that $V_1(A, a) \subseteq \overline{\mathrm{co}}\{f(a) : f \in P(A)\}$. To do this, let $\alpha \in V_1(A, a)$ and so there exist $f \in A^*$ and $x \in A$ such that $\alpha = |f|(a)$ and $\|f\| = \|x\| = f(x) = 1$. Note that $|f|(x^*x) = 1$ as observed in the proof of the main theorem and consider the following set :

$$S = \{g \in A^* : g \geq 0 \text{ and } \|g\| = g(x^*x) = 1\}.$$

Then $|f| \in S$ and S is weak*-closed. Moreover, we can easily see that any extreme point of S is also an extreme point of $\{g \in A^* : g \geq 0 \text{ and } \|g\| \leq 1\}$. But since the extreme points of $\{g \in A^* : g \geq 0 \text{ and } \|g\| \leq 1\}$ consist of 0 and $P(A)$ (cf. Proposition 2.5.5), it follows by the Krein-Milman theorem that $S \subseteq \overline{\mathrm{co}}\, P(A)$. Then $\alpha = |f|(a) = \lim_\lambda g_\lambda(a)$ for some net $\{g_\lambda\}$ in $\mathrm{co}P(A)$, and hence $\alpha \in \overline{\mathrm{co}}\{f(a) : f \in P(A)\}$. Q. E. D.

3. COMMUTATIVE CASES

X を局所コンパクト Hausdorff 空間, $C_0(X)$ を無限遠点でゼロとなる X 上の連続関数のつくる可換 C*-環, A を $C_0(X)$ の部分環, f を A に属する関数とする。このとき, 勿論 $V_1(A, f) = V_2(A, f)$ が成り立っているが, この spatial numerical range に関しては次のようにもう少し詳しい情報を得る。

Theorem 3. Let A be a subalgebra of $C_0(X)$ and $f \in A$. Then

$$V_1(A, f) = \{\int f d|\mu| : \text{there exist } \mu \in M(X) \text{ and } g \in A \text{ such that } \|\mu\| = \|g\|_\infty = \int g d\mu = 1\}$$
$$\subseteq \overline{\mathrm{co}}\, R(f),$$

where $M(X)$ denotes the space of all bounded regular Borel measures on X and $|\mu|$ denotes the total variation of μ. Moreover, $\mathrm{co}\, R(f) \subseteq V(A, f)$ if A has the following property : For any finite set $\{x_1, \dots, x_n\}$ in X, there exists $g \in A$ such that $\|g\|_\infty = 1$ and $g(x_1) = \dots = g(x_n) = 1$.

また A が * を保存する場合は, 次のようにもっと詳しい情報を得る。

Corollary 4. Let A be a *-subalgebra of $C_0(X)$ and $f \in A$. Then

$$V(A, f) = \{\int f d\mu : \text{there exist } \mu \in M(X) \text{ and } g \in A \text{ such that } \|\mu\| = 1,\ \mu \geq 0,\ 0 \leq g \leq 1 \text{ and } \int g d\mu = 1\}.$$

Moreover,

$$V(A, f) = \{\int f d\mu : 0 \leq \mu \in M(X),\ \|\mu\| = 1 \text{ and } \operatorname{supp}(\mu) \text{ is compact}\},$$

if A has the following property : For any compact set $E \subseteq X$, there exists $g \in A$ such that $0 \leq g \leq 1$ and $g(x) = 1$ for all $x \in E$. Here $\operatorname{supp}(\mu)$ denotes the support of μ.

最後に実例を出してこの節を終わろう。

Let $X = (0, 1]$, the half open interval and let $h \in C_0(X)$ be such that $h(x) \neq 0$ for all $x \in X$. Set

$$A = \{hg : g \in C_0(X)\}.$$

Then A is an ideal (and hence subalgebra) of $C_0(X)$. In this case, A is neighter closed or unital. Also A has the desired property : For any compact set $E \subseteq X$, there exists $g \in A$ such that $\|g\|_\infty = 1$ and $g(x) = 1$ for all $x \in E$, and so by Theorem 3, we have

$$V(A, f) = \{\int f d|\mu| : \text{there exist } \mu \in M(X) \text{ and } g \in A \text{ such that } \|\mu\| = \|g\|_\infty = \int g d\mu = 1\}$$

and

$$\operatorname{co} R(f) \subseteq V(A, f) \subseteq \overline{\operatorname{co}}\, R(f)$$

for every $f \in A$. In particular, if $f \in A$ is real-valued, then we have

$$V(A, f) = \begin{cases} [\alpha, \beta] & \text{if } f \text{ has a zero point} \\ (0, \beta] \ \textit{or} \ [\alpha, 0) & \text{if } f \text{ does not have a zero point}, \end{cases}$$

where $\alpha = \inf\{f(x) : x \in X\}$ and $\beta = \sup\{f(x) : x \in X\}$.

Of course, this holds even if $A = C_0(X)$, so we have the spatial numerical range of the function $f(x) = x$ $(x \in X)$ with respect to $C_0(X)$ is equal to $X = (0, 1]$. This fact has been observed in [3, Example 4.2].

Also, A is not generally a *-subalgebra of $C_0(X)$. But if h is real-valued, then A becomes a *-subalgebra of $C_0(X)$ and so A has the property : For any compact set $E \subseteq X$, there exists $g \in A$ such that $0 \leq g \leq 1$ and $g(x) = 1$ for all $x \in E$.

この節で述べた結果の証明及び実例に関する詳細は, 筆者[4] を参照されたい。

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