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Kyoto University
STRONG CONVERGENCE TO FIXED POINTS OF
NON-LIPSCHITZIAN MAPPINGS IN BANACH SPACES

Gang-Eun Kim

Abstract. In this paper, we study the strong convergence of the modified Ishikawa and Das-Debata iteration process of non-Lipschitzian mappings which satisfies the property (K) type in a Banach spaces.

1. Introduction

Let $C$ be a nonempty bounded closed convex subset of a Banach space $E$ and let $T$ be a mapping of $C$ into itself. Then $T$ is said to be asymptotically nonexpansive [5] if there exists a sequence $\{k_n\}$ of real numbers with $\lim_{n\to \infty} k_n = 1$ such that

$$
\|T^n x - T^n y\| \leq k_n \|x - y\|
$$

for $x, y \in C$ and $n = 1, 2, \ldots$. In particular, if $k_n = 1$ for all $n \geq 1$, $T$ is said to be nonexpansive. The weaker definition (cf., Kirk [10]) requires that

$$
\limsup_{n \to \infty} \sup_{y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0
$$

for each $x \in C$, and that $T^N$ be continuous for some $N \geq 1$. Consider a definition somewhere between these two: $T$ is said to be weakly asymptotically nonexpansive provided $T$ is continuous and

$$
\limsup_{n \to \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.
$$

Compare with the definition of asymptotically nonexpansive mappings in the intermediate sense initiated by Bruck et al. [1]. For two mappings $S, T$ of $C$ into itself, we consider the following modified Das-Debata iteration scheme (cf. Das-Debata [3]): $x_1 \in C$,

$$
x_{n+1} = \alpha_n S^\gamma [\beta_n T^n x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n
$$

(\star)

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$. In this case of $S = T$, such an iteration scheme was considered by Tan-Xu [17]; see also Ishikawa [7], Mann [11], Schu [14]. Reich [12], using Mann iteration procedure in a uniformly convex Banach space whose norm is Fréchet differentiable, proved that the iterates $\{x_n\}$ defined by

$$
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,
$$

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for all $n \geq 1$, converge weakly to a fixed point of nonexpansive mappings $T : C \to C$ under $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Tan-Xu [16] improved a result of Reich [12] to the case of the Ishikawa type iteration. On the other hand, Takahashi-Tamura [15] studied the weak convergence of iterates $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n S[\beta_n Tx_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n$$

for all $n \geq 1$, in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable. Recently Verma [18] proved the following interesting result using modified iterative algorithm: Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. Let $T : C \to C$ be a relaxed Lipschitz (see Definition below) and Lipschitz continuous operator on $C$. Let $r \geq 0$ and $s \geq 1$ be constants for relaxed Lipschitzity and Lipschitz continuity of $T$, respectively. Let $F = \{x \in C : Tx = x\}$ be nonempty, and let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then for any $x_0$ in $C$ the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[(1 - t)x_n + tTx_n]$$

for $n \geq 0$, $0 < k = ((1 - t)^2 - 2t(1 - t)r + t^2s^2)^{1/2} < 1$ for all $t$ such that $0 < t < \frac{2(1+r)}{(1+2r+s^2)}$, and $r \leq s$, converges to a fixed point of $T$.

In this paper, we first show how to construct (in a uniformly convex Banach space which neither satisfies the Opial property nor has a Fréchet differentiable norm) a unique fixed point of a non-Lipschitzian mapping $T : C \to C$ which satisfies the property (K) type (see Definition 2.2 below) as the strong limit of a sequence $\{x_n\}$ defined by a modified Ishikawa iteration of the form

$$x_{n+1} = \alpha_n T^n[\beta_n T^n x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1]$ are chosen so that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $0 \leq \beta_n < b$ for some $b$ with $0 < b < 1$. Next, we consider the sequence $\{x_n\}$ defined by $(*)$ converges strongly to a common fixed point of $T$ and $S$ under another conditions, that is, in cases when $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in (a, b]$ for some $a, b$ with $0 < a \leq b < 1$. Finally, we consider the sequence $\{x_n\}$ defined by $(*)$ converges strongly to a common fixed point of $T$ and $S$ under another parameter conditions, that is, in cases when $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 \leq \beta_n \leq 1$ for all $n \geq 1$.

2. Preliminaries and Some Examples

Let $H$ be a real Hilbert space. We denote by $\langle x, y \rangle$ and $\|x\|$ the inner product and the norm on $H$ for $x, y \in H$, respectively. An operator $T : H \to H$ is said to be relaxed Lipschitz [18] if, for all $x, y \in H$, there exists a constant $r > 0$ such that

$$\langle Tx - Ty, x - y \rangle \leq -r\|x - y\|^2.$$ 

Throughout this paper, let $E$ be a Banach space. Recall that $E$ is said to be uniformly convex if the modulus of convexity $\delta_E = \delta_E(\epsilon), 0 < \epsilon \leq 2$, of $E$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : x, y \in E, \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\}$$
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satisfies the inequality $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^*: \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

where $\langle x, x^* \rangle$ denotes the value of $x^*$ at $x$. Then $J$ is said to be the duality mapping of $E$.

Let $C$ be a nonempty closed convex subset of $E$ and let $T$ be a mapping from $C$ into itself. Then we denote by $F(T)$ the set of all fixed points of $T$, i.e., $F(T) = \{x \in C : Tx = x\}$. When $\{x_n\}$ is a sequence in $E$, then $x_n \to x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to $x$. We denote by $\mathbb{R}$ the set of all real numbers.

Let $C$ be a nonempty closed convex subset of $E$. If $F(T) \neq \emptyset$, the mapping $T : C \to E$ is said to be strictly hemicontractive [2] if there exists $t > 1$ such that for all $x \in C$ and $y \in F(T)$ there exists $j \in J(x - y)$ such that

$$\text{Re}(Tx - y, j) \leq \frac{1}{t} \|x - y\|^2.$$

**Definition 2.1** [8]. Let $C$ be a nonempty subset of $E$. Let $T$ be a mappings of $C$ into itself with $F(T) \neq \emptyset$. Then $T$ is said to be of (H) type if there exists $t > 1$ such that for each $x \in C$ and $y \in F(T)$, there exists $j \in J(x - y)$ such that

$$\limsup_{n \to \infty} \text{Re}(T^n x - y, j) \leq \frac{1}{t} \|x - y\|^2.$$

Here we need the following stronger concept than (H) type for constructing an approximating fixed point of a non-Lipschitzian self-mapping in a Banach space.

**Definition 2.2.** Let $C$ be a nonempty subset of $E$. Let $T$ be a mappings of $C$ into itself with $F(T) \neq \emptyset$. Then $T$ is said to be of (K) type if, for each $x \in C$ and $y \in F(T)$, there exists $j \in J(x - y)$ such that

$$\limsup_{n \to \infty} \text{Re}(T^n x - y, j) \leq 0.$$

It is obvious that if $T : C \to C$ is mapping with $F(T) = \{y\}$ and $T^n x \to y$ as $n \to \infty$ for each $x \in C$, then $T$ is of (K) type. Every relaxed Lipschitz mappings are obviously of (K) type.

**Example 2.1** [2]. Take $E = C = \mathbb{R}$ with the usual norm $|\cdot|$. Let $T : C \to C$ be defined by

$$Tx = \frac{2}{3} x \cos x$$

for all $x \in C$. Clearly $F(T) = \{0\}$ and, since $T^n x \to 0$ for each $x \in C$, $T$ is of (K) type.

**Example 2.2.** Take $E = C = \mathbb{R}$ with the usual norm $|\cdot|$ and let $0 < k < 1$. Let $T : C \to C$ be defined by

$$Tx = kx$$

for all $x \in C$. Clearly $F(T) = \{0\}$. Since $T^n x \to 0$ for each $x \in C$, $T$ is also of (K) type.

**Example 2.3.** Take $E = \mathbb{R}$ with the usual norm $|\cdot|$ and let $C = (0, 2]$. Let $T : C \to C$ be defined by

$$Tx = \sqrt{x}$$

$\forall x \in C$. Clearly $F(T) = \{1\}$ and, since $T^n x \to 1$ as $n \to \infty$ for each $x \in C$, $T$ is weakly asymptotically nonexpansive which is of (K) type but not Lipschitz mapping.
3. STRONG CONVERGENCE THEOREMS

We first begin with the following:

**Lemma 3.1 [1].** Suppose \( \{v_n\} \) is a bounded sequence of real numbers and \( \{a_{n,m}\} \) is a doubly-indexed sequence of real numbers which satisfy \( \lim \sup_{n \to \infty} \lim \sup_{m \to \infty} a_{n,m} \leq 0 \), \( v_{n+m} \leq v_n + a_{n,m} \) for each \( n, m \geq 1 \). Then \( \{v_n\} \) converges to an \( v \in \mathbb{R} \); \( a_{n,m} \) can be taken to be independent of \( n \), \( a_{n,m} = a_m \), then \( v \leq v_n \) for each \( n \).

**Lemma 3.2 [6].** For any \( x, y \in E \) and \( j \in J(x + y) \), we obtain
\[
\|x + y\|^2 \leq \|x\|^2 + 2 \text{Re}(y, j).
\]

From the proof of Lemma 3 of [16], we note

**Lemma 3.3.** Let \( a_n, \ b_n > 0 \) for \( n \geq 1 \). If \( \sum_{n=1}^{\infty} a_n = \infty \) and \( \sum_{n=1}^{\infty} a_n b_n < \infty \), then \( \lim \inf_{n \to \infty} b_n = 0 \).

Using Lemma 3.1-3.3, we obtain the following Theorem 3.1.

**Theorem 3.1 [9].** Let \( E \) be a uniformly convex Banach space and let \( C \) be a nonempty bounded closed convex subset of \( E \). Suppose that \( T : C \to C \) is both weakly asymptotically nonexpansive and of \( (K) \) type. Put
\[
c_n = \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0,
\]
so that \( \sum_{n=1}^{\infty} c_n < \infty \). Then for any \( x_1 \) in \( C \), the sequence \( \{x_n\} \) defined by
\[
x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n) x_n, \quad y_n = \beta_n T^n x_n + (1 - \beta_n) x_n,
\]
which \( \{\alpha_n\} \) and \( \{\beta_n\} \) are chosen so that \( \alpha_n \in [0, 1] \) and \( \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty \) and \( 0 \leq \beta_n < b < 1 \) for all \( n \geq 1 \), converge strongly to the unique fixed point of \( T \).

**Remark.** If \( \{\alpha_n\} \) is a sequence in \( [0, 1] \) which is bounded away from 0 and 1, i.e., \( a \leq \alpha_n \leq b \) for some \( a, b \) with \( 0 < a \leq b < 1 \), then \( \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty \).

As a direct consequence of Theorem 3.1 with \( \beta_n = 0 \), we have the following result.

**Corollary 3.1.** Let \( E \) be a uniformly convex Banach space and \( C \) be a nonempty bounded closed convex subset of \( E \). Let \( T : C \to C \) be both weakly asymptotically nonexpansive and of \( (K) \) type. Put
\[
c_n = \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0,
\]
so that \( \sum_{n=1}^{\infty} c_n < \infty \). Then for any \( x_1 \) in \( C \), the sequence \( \{x_n\} \) defined by
\[
x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n,
\]
which \( \{\alpha_n\} \) is chosen so that \( \alpha_n \in [0, 1] \) and \( \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty \) for all \( n \geq 1 \), converge strongly to the unique fixed point of \( T \).
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Lemma 3.4 [13]. Let $E$ be a uniformly convex Banach space, $0 < b \leq t_n \leq c < 1$ for all $n \geq 1$, $a \geq 0$. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences of $E$ such that
\[
\limsup_{n \to \infty} \|x_n\| \leq a, \quad \limsup_{n \to \infty} \|y_n\| \leq a, \quad \text{and} \quad \lim_{n \to \infty} \|t_n x_n + (1-t_n)y_n\| = a.
\]
Then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

By using Lemma 3.4, we obtain the following Theorem 3.2.

Theorem 3.2. Let $E$ be a uniformly convex Banach space and $C$ be a nonempty bounded closed convex subset of $E$. Let $T, S : C \to C$ be both weakly asymptotically nonexpansive and of $(K)$ type with $F(T) \cap F(S) \neq \emptyset$. Put
\[
c_n = \max(0, \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|), \sup_{x,y \in C} (\|S^n x - S^n y\| - \|x - y\|)),
\]
so that $\sum_{n=1}^{\infty} c_n < \infty$. Then for any $x_1$ in $C$, the sequence $\{x_n\}$ defined by $(\ast)$, which $\{\alpha_n\}$ and $\beta_n$ are chosen so that $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$ for some $a, b$ with $0 < a \leq b < 1$, converge strongly to a common fixed point of $T$ and $S$.

The following lemma is very useful to prove the convergence of a sequence to 0. Compare with Lemma 1 due to Dunn [4].

Lemma 3.5 [19]. Let $\beta_n$ be a nonnegative sequence satisfying
\[
\beta_{n+1} \leq (1 - \delta_n) \beta_n + \sigma_n
\]
with $\delta_n \in [0, 1]$, $\sum_{i=1}^{\infty} \delta_i = \infty$, and $\sigma_n = o(\delta_n)$. Then $\lim_{n \to \infty} \beta_n = 0$.

Theorem 3.3. Let $C$ be a nonempty bounded closed convex subset of a Banach space $E$. Let $T, S : C \to C$ be both weakly asymptotically nonexpansive and of $(K)$ type with $F(T) \cap F(S) \neq \emptyset$. Put
\[
c_n = \max(0, \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|), \sup_{x,y \in C} (\|S^n x - S^n y\| - \|x - y\|)),
\]
so that $\sum_{n=1}^{\infty} c_n < \infty$. Then for any $x_1$ in $C$, the sequence $\{x_n\}$ defined by $(\ast)$, which $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 \leq \beta_n \leq 1$ for all $n \geq 1$, converge strongly to a common fixed point of $T$ and $S$.

As a direct consequence of Theorem 3.3 with $\beta_n = 0$, we have the following result.

Corollary 3.2. Let $C$ be a nonempty bounded closed convex subset of a Banach space $E$. Let $T, S : C \to C$ be both weakly asymptotically nonexpansive and of $(K)$ type with $F(T) \cap F(S) \neq \emptyset$. Put
\[
c_n = \max(0, \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|), \sup_{x,y \in C} (\|S^n x - S^n y\| - \|x - y\|)),
\]
so that $\sum_{n=1}^{\infty} c_n < \infty$. Then for any $x_1$ in $C$, the sequence $\{x_n\}$ defined by
\[
x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n,
\]
which $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ for all $n \geq 1$, converge strongly to a common fixed point of $T$ and $S$. 

G.E. KIM

REFERENCES

8. T. H. Kim and E. S. Kim, Remarks on approximation of fixed points of strictly pseudocontractive mappings, submitted.

PUKYONG NATIONAL UNIVERSITY, DEPARTMENT OF APPLIED MATHEMATICS, BUSAN 608-737, KOREA