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Kyoto University
BERNSTEIN-TYPE APPROXIMATION PROCESSES
FOR VECTOR-VALUED FUNCTIONS

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ABSTRACT. A sequence of the Bernstein-type operators for vector-valued functions is provided and its uniform convergence is considered by making use of a theorem of Korovkin type under certain requirements.

1. Introduction

Let $f$ be a real-valued continuous function on the unit $r$-cube

$$\mathbb{I}_r = \{x = (x_1, x_2, \cdots, x_r) \in \mathbb{R}^r : 0 \leq x_i \leq 1, i = 1, 2, \cdots, r\},$$

where $\mathbb{R}^r$ is the $r$-dimensional Euclidean space and let $n$ be a positive integer. Then the $n$-th Bernstein polynomial of $f$ is defined by

$$B_n(f)(x) = \sum_{k_1=0}^{n} \cdots \sum_{k_r=0}^{n} \prod_{i=1}^{r} \binom{n}{k_i} x_i^{k_i} (1-x_i)^{n-k_i} f(k_1/n, \cdots, k_r/n). \tag{1}$$

It is well-known that $\{B_n(f)\}$ converges uniformly to $f$ on $\mathbb{I}_r$ (cf. [6]). This result also remains true for a continuous function $f$ taking values in a normed linear space ([8]).

In this paper, we give a generalization of (1) and consider its uniform convergence in the context of normed vector lattices. For this we have to establish a theorem of Korovkin type for vector-valued functions (cf. [8], [9], [10]). For the background of the Korovkin-type
approximation theory, see the book of Altomare and Campiti [2], in which an excellent source and a vast literature of this theory can be found (cf. [3], [4], [5]).

2. A theorem of Korovkin type

Let $X$ be a compact Hausdorff space and let $E$ be a normed vector lattice with its positive cone $E_+ = \{ a \in E : a \geq 0 \}$. For the general notions and terminology needed from the theory of normed vector lattices, we refer to [12] (cf. [1], [7]). Let $B(X, E)$ denote the normed vector lattice of all $E$-valued norm bounded functions on $X$ with the usual pointwise addition, scalar multiplication, ordering and the supremum norm $\| \cdot \|$. We shall use the same symbol $\| \cdot \|$ for the underlying norms. $C(X, E)$ denotes the closed sublattice of $B(X, E)$ consisting of all $E$-valued continuous functions on $X$. In the case when $E$ is equal to $\mathbb{R}$, we simply write $B(X)$ and $C(X)$ instead of $B(X, E)$ and $C(X, E)$, respectively.

Throughout this paper we suppose that $E$ always contains an element $e$ such that $e > 0, \| e \| = 1$ and $|a| \leq \| a \| e$ for all $a \in E$. We call $e$ the normal order unit of $E$. We define $\rho(x) = e$ and $1_X(x) = 1$ for all $x \in X$. Notice that $\rho$ and $1_X$ are the normal order units of $C(X, E)$ and $C(X)$, respectively. For any $a \in E$ and $v \in B(X)$, the function $v \otimes a$ is defined by $(v \otimes a)(x) = v(x)a$ for all $x \in X$. Also, for any $v \in B(X)$ and $f \in B(X, E)$, we define $(vf)(x) = v(x)f(x)$ for all $x \in X$. Clearly, $v \otimes a$ and $vf$ belong to $B(X, E)$, and $\| v \otimes a \| = \| v \| \| a \|, \| vf \| \leq \| v \| \| f \|$ and $\rho = 1_X \otimes e$. We shall denote by $C(X) \otimes E$ the linear subspace of $C(X, E)$ consisting of all finite sums of functions of the form $v \otimes a$, where $v \in C(X)$ and $a \in E$. A bounded linear operator $L$ of $C(X, E)$ into $B(X, E)$ is said to be quasi-positive if $v, w \in C(X)$ and $|v| \leq w$, then $\| L(v \otimes a)(x) \| \leq \| L(w \otimes a)(x) \|$ for all $a \in E_+$ and all $x \in X$. (cf. [8], [9]). A typical example of such an
operator is given by

\[ T(f) = hf \quad \text{for every } f \in C(X, E), \quad (2) \]

where \( h \) is an arbitrary fixed function in \( B(X) \).

**Lemma 1.** If \( L \) is a positive linear operator of \( C(X, E) \) into \( B(X, E) \), then it is quasi-positive and \( \|L\| = \|L(\rho)\| \).

**Proof.** Let \( v, w \in C(X), |v| \leq w \) and \( a \in E_+ \). Then we have \( |v \otimes a| \leq w \otimes a \), and so \( |L(v \otimes a)| \leq L(w \otimes a) \). Thus for all \( x \in X, |L(v \otimes a)(x)| \leq L(w \otimes a)(x) \), which implies \( ||L(v \otimes a)(x)|| \leq ||L(w \otimes a)(x)|| \). Since, for all \( f \in C(X, E), |f| \leq ||f||\rho \), we have \( |L(f)| \leq ||f||L(\rho) \), and so \( ||L(f)|| \leq ||f||||L(\rho)|| \). Therefore, \( ||L|| \leq ||L(\rho)|| \). On the other hand, \( ||L(\rho)|| \leq ||L|| \) because of \( ||\rho|| = 1 \). \( \square \)

**Lemma 2.** ([8; Lemma 2]) \( C(X) \otimes E \) is dense in \( C(X, E) \).

In fact, this is an immediate consequence of [11; Theorem 1.15], since \( C(X) \) separates the points of \( X \).

Now, we have the following Korovkin-type theorem (cf. [8; Corollary 4 (i) and Remark]), which can be useful for later applications.

**Theorem 1.** Let \( \{L_\alpha\} \) be a net of quasi-positive linear operators of \( C(X, E) \) into \( B(X, E) \) such that there exists an element \( \alpha_0 \) for which

\[ \sup\{||L_\alpha|| : \alpha \geq \alpha_0\} < \infty \quad (3) \]

and let \( T \) be as in (2). Let \( G \) be a subset of \( C(X) \) separating the points of \( X \). Then the following statements are equivalent:

(a) For all \( g \in G, a \in E_+ \) and for \( j = 0, 1, 2 \),

\[ \lim_\alpha ||L_\alpha(g^j \otimes a) - T(g^j \otimes a)|| = 0, \quad (4) \]

where \( g^0 = 1_X \).
(b) For all $g \in G$ and all $a \in E_+$, (4) holds with $j = 0$ and
\[ \lim_{\alpha} \mu_{\alpha}(g, a) = 0, \]
where
\[ \mu_{\alpha}(g, a) = \sup \{ \| L_{\alpha}((g - g(y)1_X)^2 \otimes a)(y) \| : y \in X \}. \]

(c) For all $f \in C(X, E)$,
\[ \lim_{\alpha} \| L_{\alpha}(f) - T(f) \| = 0. \]

Proof. Since
\[ L_{\alpha}((g - g(y)1_X)^2 \otimes a)(y) = L_{\alpha}(g^2 \otimes a)(y) - T(g^2 \otimes a)(y) \]
\[ -2g(y)\{ L_{\alpha}(g \otimes a)(y) - T(g \otimes a)(y) \} + g^2(y)\{ L_{\alpha}(1_X \otimes a)(y) - T(1_X \otimes a)(y) \}, \]
we have
\[ \mu_{\alpha}(g, a) \leq \| L_{\alpha}(g^2 \otimes a) - T(g^2 \otimes a) \| \]
\[ + 2\| g \| \| L_{\alpha}(g \otimes a) - T(g \otimes a) \| + \| g^2 \| \| L_{\alpha}(1_X \otimes a) - T(1_X \otimes a) \|. \]
Therefore (a) implies (b). Next we suppose that (b) is valid. Let $v \in C(X), b \in E$ and $\epsilon > 0$ be given. Note that $b$ has the representation
\[ b = b^+ - b^-, \]
where $b^+$ and $b^-$ are the positive part and the negative part of $b$, respectively. Since $X$ is compact and $G$ separates the points of $X$, the original topology on $X$ is identical with the weak topology on $X$ induced by $G$. Therefore, there exists a finite subset $\{ g_1, g_2, \cdots, g_m \}$ of $G$ and a constant $K > 0$ such that
\[ |v(x) - v(y)| \leq \epsilon + K \sum_{i=1}^m (g_i(x) - g_i(y))^2 \]
for all $x, y \in X$. Hence it follows that
\[ \| L_{\alpha}((v - v(y)1_X) \otimes b^+)(y) \| \leq \epsilon \| L_{\alpha}(1_X \otimes b^+)(y) \| \]
\[ + K \sum_{i=1}^m \| L_{\alpha}((g_i - g_i(y)1_X)^2 \otimes b^+)(y) \| \]
for all $y \in X$, and so we have
\[ \| L_{\alpha}(v \otimes b^+) - T(v \otimes b^+) \| \]
\[ \leq \| L_\alpha(v \otimes b^+) - vL_\alpha(1_X \otimes b^+) \| + \| v \| \| L_\alpha(1_X \otimes b^+) - T(1_X \otimes b^+) \| \]
\[ \leq \epsilon \| L_\alpha(1_X \otimes b^+) \| + K \sum_{i=1}^{m} \mu_{\alpha}(g_i, b^+) + \| v \| \| L_\alpha(1_X \otimes b^+) - T(1_X \otimes b^+) \| , \]

which together with the assertion (b) yields \( \lim_\alpha \| L_\alpha(v \otimes b^+) - T(v \otimes b^+) \| = 0 \). Similarly, we have \( \lim_\alpha \| L_\alpha(v \otimes b^-) - T(v \otimes b^-) \| = 0 \). Now, we have

\[ \| L_\alpha(v \otimes b) - T(v \otimes b) \| \leq \| L_\alpha(v \otimes b^+) - T(v \otimes b^+) \| + \| L_\alpha(v \otimes b^-) - T(v \otimes b^-) \| , \]

and so

\[ \lim_\alpha \| L_\alpha(v \otimes b) - T(v \otimes b) \| = 0. \]

Hence, in view of (3), Lemma 2 and the theorem of Banach-Steinhaus establish the statement (c). It is obvious that (c) implies (a).

\square

Remark 1. Theorem 1 can be applied in the following situation: Let \( X \) be a compact subset of a real locally convex Hausdorff vector space \( F \) with its dual space \( F^* \) and \( G = \{ u|_X : u \in F^* \} \), where \( u|_X \) denotes the restriction of \( u \) to \( X \). If \( X \) is a compact convex subset of \( F \), then \( G \) can be taken as the space of all real-valued continuous affine functions on \( X \).

3. Bernstein-type operators

Let \( B[E] \) denote the normed algebra of all bounded linear operators of \( E \) into itself with the identity operator \( I \). Let \( X_1, X_2, \ldots, X_r \) be compact Hausdorff spaces and we here consider their product space

\[ X = \prod_{i=1}^{r} X_i = \{ x = (x_1, x_2, \ldots, x_r) : x_i \in X_i, i = 1, 2, \ldots, r \}. \]

Let \( \Phi = \{ (\Phi^{(i)}_{a,k})_{a,k \geq 0} : i = 1, 2, \ldots, r \} \) be a set of infinite lower triangular matrices of continuous functions from \( X_i \) into \( B[E] \) and let
$T = \{ T_{n,k_1,k_2,\ldots,k_r} : 0 \leq k_i \leq n, i = 1,2,\ldots,r \}$ be a set of bounded linear operators of $C(X, E)$ into $E$. Then we define

$$B_n(f)(x) = B_{n,T} \Phi(f)(x) = \sum_{k_1=0}^{n} \cdots \sum_{k_r=0}^{n} \prod_{i=1}^{r} \Phi_{n,k_i}^{(i)}(x_i)(T_{n,k_1,\ldots,k_r}(f))$$

for all $f \in C(X, E)$ and all $x \in X$. Notice that each $B_n$ is a bounded linear operator of $C(X, E)$ into itself. We call $B_n$ the $n$-th Bernstein-type operator with respect to $T$ and $\Phi$.

If we take

$$X_i = I_1 = [0,1] \quad (i = 1,2,\ldots,r) \tag{6}$$

and

$$\Phi_{n,k}^{(i)}(t) = \varphi_{n,k}^{(i)}(t)I \quad (t \in X_i, i = 1,2,\ldots,r),$$

where

$$\varphi_{n,k}^{(i)} \in C(X_i) \quad (i = 1,2,\ldots,r),$$

then (5) becomes

$$B_n(f)(x) = B_{n,T} \Phi(f)(x) = \sum_{k_1=0}^{n} \cdots \sum_{k_r=0}^{n} \prod_{i=1}^{r} \varphi_{n,k_i}^{(i)}(x_i)T_{n,k_1,\ldots,k_r}(f). \tag{7}$$

Furthermore, in particular, if we take

$$\varphi_{n,k}^{(i)}(t) = \binom{n}{k} t^k (1-t)^{n-k} \quad (t \in X_i, i = 1,2,\ldots,r)$$

and define

$$T_{n,k_1,k_2,\ldots,k_r}(f) = f(k_1/n, k_2/n, \ldots, k_r/n) \quad (f \in C(X, E)), \tag{8}$$

then (7) reduces to (1) in case of $E = \mathbb{R}$.

From now on let $X_i, i = 1,2,\ldots,r$, be as in (6) and each operator $T_{n,k_1,k_2,\ldots,k_r}$ is defined by (8).

**Lemma 3.** Suppose that for all $t \in X_i, i = 1,2,\ldots,r$,

$$\sum_{k=0}^{n} \Phi_{n,k}^{(i)}(t) = I, \quad \sum_{k=1}^{n} k \Phi_{n,k}^{(i)}(t) = ntI \tag{9}$$
and
$$
\sum_{k=2}^{n} k(k-1) \Phi_{n,k}^{(i)}(t) = n(n-1)t^2 I. \tag{10}
$$

Then we have

$$
B_n(1_X \otimes a) = 1_X \otimes a, \quad B_n(e_j \otimes a) = e_j \otimes a
$$

and

$$
B_n(e_j^2 \otimes a) = e_j^2 \otimes a + \frac{1}{n}(e_j - e_j^2) \otimes a
$$

for all $a \in E, n \geq 1$ and $j = 1, 2, \cdots, r$. Here, $e_j$ denotes the $j$-th coordinate function on $X$ defined by

$$
e_j(x) = x_j \quad (x = (x_1, x_2, \cdots, x_r) \in X).
$$

Proof. Let $x \in X$. Then we have

$$
B_n(1_X \otimes a)(x) = \sum_{k_1=0}^{n} \cdots \sum_{k_r=0}^{n} \prod_{i=1}^{r} \Phi_{n,k_i}^{(i)}(x_i)(a) = I(a) = a,
$$

$$
B_n(e_j \otimes a)(x) = \sum_{k_j=1}^{n} \Phi_{n,k_j}^{(j)}(x_j) \left( \frac{k_j}{n} a \right) = \frac{1}{n}(nx_j I)(a) = x_j a
$$

and

$$
B_n(e_j^2 \otimes a)(x) = \sum_{k_j=1}^{n} \Phi_{n,k_j}^{(j)}(x_j) \left( \frac{k_j^2}{n^2} a \right)
$$

$$
= \frac{1}{n^2} \left\{ \sum_{k_j=1}^{n} k_j \Phi_{n,k_j}^{(j)}(x_j)(a) + \sum_{k_j=2}^{n} k_j(k_j - 1) \Phi_{n,k_j}^{(j)}(x_j)(a) \right\}
$$

$$
= \frac{1}{n^2} \left\{ (nx_j I)(a) + (n(n-1)x_j^2 I)(a) \right\} = x_j^2 a + \frac{1}{n}(x_j - x_j^2)a,
$$

which implies desired result. \hfill \Box

Theorem 2. Suppose that for every $t \in X_i, i = 1, 2, \cdots, r$, each operator $\Phi_{n,k}^{(i)}(t)$ is positive, and (9) and (10) are fulfilled. Then we have

$$
\lim_{n \to \infty} \|B_n(f) - f\| = 0 \quad \text{for all } f \in C(X, E).
$$
Proof. We take $G = \{e_1, e_2, \cdots, e_r\}$, which clearly separates the points of $X$ (cf. Remark 1). Since each $B_n$ is positive, by Lemma 1, it is quasi-positive and $\|B_n\| = \|B_n(1_X \otimes e)\|$. Therefore, the desired result follows from Theorem 1 and Lemma 3.

Lemma 4. Let $\{\psi_{n,k}^{(i)} : i = 1, 2, \cdots, r\}$ be a set of infinite matrices of continuous mappings from $X_i$ into $B[E]$ such that for all $t \in X_i, i = 1, 2, \cdots, r$,

\[ \psi_{n,k+m}^{(i)}(t) = t^m \psi_{n,k}^{(i)}(t) \quad (n, k = 0, 1, 2, \cdots, m = 1, 2) \] \hfill (11)

and

\[ \sum_{k=0}^{n} \binom{n}{k} \psi_{n-k,k}^{(i)}(t) = I \quad (n = 0, 1, 2, \cdots). \] \hfill (12)

Then we have

\[ \sum_{k=1}^{n} k \binom{n}{k} \psi_{n-k,k}^{(i)}(t) = ntI \] \hfill (13)

and

\[ \sum_{k=2}^{n} k(k-1) \binom{n}{k} \psi_{n-k,k}^{(i)}(t) = n(n-1)t^2I \] \hfill (14)

for all $t \in X_i, i = 1, 2, \cdots, r$.

Proof. Since

\[ k \binom{n}{k} = n \binom{n-1}{k-1} \quad (1 \leq k \leq n) \]

and

\[ k(k-1) \binom{n}{k} = n(n-1) \binom{n-2}{k-2} \quad (2 \leq k \leq n), \]

it follow from (11) and (12) that

\[ \sum_{k=1}^{n} k \binom{n}{k} \psi_{n-k,k}^{(i)}(t) = n \sum_{k=1}^{n} \binom{n-1}{k-1} \psi_{n-k,k}^{(i)}(t) = n \sum_{j=0}^{n-1} \binom{n-1}{j} \psi_{n-j-1,j+1}^{(i)}(t) = nt \sum_{j=0}^{n-1} \binom{n-1}{j} \psi_{n-1-j,j}^{(i)}(t) = ntI \]

and

\[ \sum_{k=2}^{n} k(k-1) \binom{n}{k} \psi_{n-k,k}^{(i)}(t) = n(n-1) \sum_{k=2}^{n} \binom{n-2}{k-2} \psi_{n-k,k}^{(i)}(t) \]
\[ = n(n - 1) \sum_{j=0}^{n-2} \binom{n-2}{j} \Psi_{n-j-2,j+2}^{(i)}(t) \]
\[ = n(n - 1)t^2 \sum_{j=0}^{n-2} \binom{n-2}{j} \Psi_{n-2-j,j}^{(i)}(t) = n(n - 1)t^2 I. \]

Therefore, the equalities (13) and (14) hold. \[ \square \]

**Theorem 3.** Let \((\Psi_{n,k}^{(i)})_{n,k \geq 0}, i = 1, 2, \cdots, r\) be as in Lemma 4 with the additional assumption that all the operators \(\Psi_{n,k}^{(i)}(t)\) are positive for each \(t \in X_i, i = 1, 2, \cdots, r\), and define

\[
\phi_{n,k}^{(i)} = \begin{cases} 
\binom{n}{k} \Psi_{n-k,k}^{(i)} & (0 \leq k \leq n) \\
0 & (k \geq n).
\end{cases}
\]

Then we have \(\lim_{n \to \infty} \|B_n(f) - f\| = 0\) for all \(f \in C(X, E)\).

**Proof.** This follows from Lemma 4 and Theorem 2. \[ \square \]

Let \(\{\{\varphi_k^{(i)}\}_{k \geq 0} : i = 1, 2, \cdots, r\}\) be a set of sequences of continuous mappings from \(X_i\) into \(B[E]\), and we define

\[
\Delta^n \varphi_k^{(i)}(t) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \varphi_{n-k-j}^{(i)}(t) \quad (n, k = 0, 1, 2, \cdots). \quad (15)
\]

Suppose that for all \(t \in X_i, i = 1, 2, \cdots, r,\)

\[
\varphi_{k+m}^{(i)}(t) = t^m \varphi_k^{(i)}(t) \quad (k = 0, 1, 2, \cdots, m = 1, 2) \quad (16)
\]

and

\[
\sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} \varphi_k^{(i)}(t) = I \quad (n = 0, 1, 2, \cdots). \quad (17)
\]

**Corollary 1.** Assume that all the operator \(\Delta^n \varphi_k^{(i)}(t)\) given by (15) are positive for each \(t \in X_i, i = 1, 2, \cdots, r\), and define

\[
\phi_{n,k}^{(i)} = \begin{cases} 
\binom{n}{k} \Delta^{n-k} \varphi_k^{(i)} & (0 \leq k \leq n) \\
0 & (k \geq n)
\end{cases}
\]

Then we have \(\lim_{n \to \infty} \|B_n(f) - f\| = 0\) for all \(f \in C(X, E)\).
Indeed, setting

$$\Psi_{nk}(i) = \Delta n \varphi_{k}^{(i)} (n, k = 0, 1, 2, \ldots, i = 1, 2, \ldots r),$$

the conditions (16) and (17) imply the equalities (11) and (12), respectively. Thus, by Theorem 3, we have the claim of the corollary.

In particular, we take

$$\varphi_{k}^{(i)}(t) = t^k I \quad (t \in X_i, i = 1, 2, \ldots, r, k = 0, 1, 2, \ldots).$$

Then we have

$$\Delta n \varphi_{k}^{(i)}(t) = (1 - t)^n t^k I \quad (n, k = 0, 1, 2, \ldots, t \in X_i, i = 1, 2, \ldots r),$$

and the conditions (16) and (17) are also satisfied. Furthermore, we get again the Bernstein operators given by (1).

**Remark 2.** Suppose that $E$ is a Banach space. Let $r = 1$ and let $\Phi_{n,k}^{(1)}$ be as in Corollary 1. Then $B_n(f)$ becomes the $\Phi$-Bernstein approximation of $f$ of order $n$ due to Tucker [13]. Also, conversely if we have

$$\lim_{n \to \infty} \|B_n(f) - f\| = 0 \text{ for every } f \in C(X_1, E),$$

then

$$\varphi_{k}^{(1)}(t) = t^k I \quad (t \in X_1, k = 0, 1, 2, \ldots)$$

([13; Corollary]).

**References**


