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EXISTENCE OF SOLUTIONS FOR CAUCHY PROBLEMS AND PERIODIC PROBLEMS WITH MULTIVALUED PSEUDO MONOTONE OPERATORS

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1. INTRODUCTION

Let V be a reflexive Banach space which is densely and continuously imbedded in a Hilbert space, let $T>0$ and let $\{A(t): t \in [0,T]\}$ be multivalued operators from V into its dual V'. We study the existence of solutions for Cauchy problems and periodic problems to a class of a nonlinear evolution equations of the form

$$u'(t)+A(t)u(t) \ni 0 \qquad \text{for } t \in [0,T].$$

In the case when $A(t)$ is single valued, Hirano [12], Ahmed and Xiang [1], Berkovits and Mustonen [3], and the author [15] studied the problems of this kind. Hirano and Ahmed and Xiang studied the Cauchy problems in the case when $A(t)$ is a monotone operator with nonmonotonic perturbations. Berkovits and Mustonen generalized their results to the case that $A(t)$ is pseudo monotone and they also studied the periodic problems. The author also studied the periodic problems in the case when $A(t)$ is pseudo monotone.

In this paper, we study the Cauchy problems and periodic problems in the case when $A(t)$ is a multivalued pseudo monotone operator. The conditions (A1)–(A4) in our theorems are more general than those in [1, 3, 12, 15] even in the case when $A(t)$ is single valued. To prove our results, we employ the method employed in [10]. We use a topological property of a solution set for a differential inclusion in Euclidean space [7], a topological fixed point theorem [9], Filippov's type implicit function theorem [16] and a minimax theorem [8].

The next section is devoted to some preliminaries and notations. In section 3, we state our main results and we prove them in section 4. In the final section, we study some applications.

2. PRELIMINARIES

Throughout this paper, all vector spaces are real, and we denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$, the set of positive integers, the set of real numbers and the set of nonnegative real numbers, respectively. Let X and Y be topological spaces. A multivalued mapping $F: X \to 2^Y$ is said to be *upper semicontinuous* if for every $x_0 \in X$ and open set $V \subset Y$ with $Fx_0 \subset V$, there exists an open neighborhood U of x_0 such that $Fx \subset V$ for every $x \in U$. Let E be a measure space. A multivalued mapping $G: E \to 2^Y$ is said to be *measurable* if for every closed set $C \subset Y$, the set $\{t \in E : G(t) \cap C \neq \emptyset\}$ is measurable. A multivalued mapping $H: E \times X \to 2^Y$ is said to be Carathéodory if for every $x \in X$, $H(\cdot, x): E \to 2^Y$ is measurable and for almost every $t \in E$, $H(t,\cdot): X \to 2^Y$ is upper semicontinuous. Let V be a reflexive Banach space. We denote by V', the topological dual of V. The value of $y \in V'$ at $x \in V$ will be denoted by (y,x). Let V be densely and continuously imbedded in a Hilbert space H. Since we identify H with its dual, we have $V \subset H \subset V'$. Let p, q and T be positive constants such that $1/p+1/q=1$. For every $u \in L^p(0,T;V)$ and $v \in L^q(0,T;V')$, $\langle v,u \rangle$ is defined by $\int_0^T (v(t),u(t))\,dt$. We denote by $W_p^1(0,T;V,H)$, the Banach space

$$W_p^1(0,T;V,H)=\{u \in L^p(0,T;V) : u' \in L^q(0,T;V')\}$$

with the norm $\|u\| + \|u'\|_*$, where u' is the generalized derivative [2, 17] of u and $\|\cdot\|$ and $\|\cdot\|_*$ are the norms of $L^p(0,T;V)$ and $L^q(0,T;V')$, respectively. We know from [17] that $W_p^1(0,T;V,H)$ is a reflexive Banach space and that $W_p^1(0,T;V,H)$ is continuously imbedded in $C(0,T;H)$.

Let V be a reflexive Banach space and let $A : V \to 2^{V'}$. A is said to be *monotone* if $(w-z, x-y) \geq 0$ for every $x, y \in V$, $w \in Ax$ and $z \in Ay$. A is said to be *pseudo monotone* if for every sequence $\{x_n\}$ in V and $\{w_n\}$ in V' such that $\{x_n\}$ converges weakly to $x \in V$, $w_n \in Ax_n$ for every $n \in \mathbb{N}$ and $\overline{\lim}(w_n, x_n - x) \leq 0$, for every $y \in V$, there exists $w_y \in Ax$ such that

$$(w_y, x-y) \leq \underline{\lim}_{n\to\infty} (w_n, x_n - y).$$

It is well known [4] that if A is monotone, for every $x \in V$, Ax is a nonempty, closed convex subset of V' and A is upper semicontinuous from every line segment in V to the weak topology of V', then A is pseudo monotone.

To prove our theorems in the next section, we use the following; see [7, Theorem and Lemma 5], [9, Example 1 in Section III.2 and Corollary V.3.8], [16, p. 864, Theorem 4.1 and Theorem 7.4] and [8, Theorem 2], respectively:

Proposition 1 (De Blasi and Myjak). Let $T > 0$ and let $g : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a Carathéodory mapping such that for almost every $t \in [0,T]$ and for every $y \in \mathbb{R}^n$, $g(t,y)$ is a nonempty, compact, convex subset of $\mathbb{R}^n$, and there exists $\mu \in L^1(0,T;\mathbb{R}_+)$ such that $\sup_{y\in\mathbb{R}^n} |g(t,y)| \leq \mu(t)$ for almost every $t \in [0,T]$. For every $y \in \mathbb{R}^n$, let

$$S_g(y) = \{x \in C(0,T;\mathbb{R}^n) : \text{ absolutely continuous, } x(0) = y,\ x'(t) \in g(t, x(t)) \text{ for a.e. } t \in [0,T]\}.$$

Then for every $y \in \mathbb{R}^n$, $S_g(y)$ is the intersection of a decreasing sequence of nonempty, compact, contractible subsets of $C(0,T;\mathbb{R}^n)$.

Proposition 2 (Górniewicz). Let X be a convex subset of a normed linear space and let $H : X \to 2^X$ be an upper semicontinuous mapping such that $\cup_{x\in X} Hx$ is contained in a compact subset of X and for every $x \in X$, Hx is a nonempty, acyclic, compact subset of X, where acyclic is in the sense of Čech homology with rational coefficients. Then H has a fixed point, i.e., there is an element x of X such that $x \in Hx$.

Proposition 3 (Wagner). Let $\mathcal{T}$ be a complete measure space, let V be a complete, separable metric space and let Y be a separable metric space. Let $F : \mathcal{T} \to 2^V$ be a measurable mapping such that for almost every $t \in \mathcal{T}$, $F(t)$ is nonempty, closed subset of V, let $g : \mathcal{T} \times V \to Y$ be a Carathéodory mapping and let $h : \mathcal{T} \to Y$ be a measurable mapping such that for almost every $t \in \mathcal{T}$, there exists $v_t \in F(t)$ with $h(t) = g(t, v_t)$. Then there exists a measurable mapping $f : \mathcal{T} \to V$ such that $f(t) \in F(t)$ and $h(t) = g(t, f(t))$ for almost every $t \in \mathcal{T}$.

Proposition 4 (Fan). Let U be a compact, convex subset of a Hausdorff topological vector space and let V be a convex subset of a vector space. Let $f : U \times V \to \mathbb{R}$ such that for every $v \in V$, $f(\cdot, v)$ is lower semicontinuous and convex, and for every $u \in U$, $f(u, \cdot)$ is concave. Then

$$\min_{u\in U} \sup_{v\in V} f(u,v) = \sup_{v\in V} \min_{u\in U} f(u,v).$$

3. Main results

Now we state our main results.

Theorem 1. Let T, p and q be positive constants such that $1/p + 1/q = 1$. Let $(V, \|\cdot\|)$ be a separable, reflexive Banach space which is densely and continuously imbedded in a Hilbert space $(H, |\cdot|)$ and let $\{A(t) : 0 \leq t \leq T\}$ be a family of multivalued mappings such that

(A1) for almost every $t \in [0,T]$, $A(t) : V \to 2^{V'}$ is pseudo monotone and $A(t)x$ is a nonempty, closed, convex subset of V' for every $x \in V$;
(A2) for every $u \in L^p(0,T;V)$, $t \mapsto A(t)u(t) : [0,T] \to 2^{V'}$ is measurable;
(A3) there exist $C_0, C_2 \in L^1(0,T;\mathbb{R}_+)$ and $C_1 > 0$ such that for almost every $t \in [0,T]$, for every $x \in V$ and for every $y \in A(t)x$,

$$(y,x) + C_0(t)|x|^2 \geq C_1\|x\|^p - C_2(t); \tag{3.1}$$

(A4) there exist an increasing function $C_3 : \mathbb{R}_+ \to \mathbb{R}_+$ and $C_4 : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $C_4(\cdot,a) \in L^q(0,T;\mathbb{R}_+)$ for every $a \in \mathbb{R}_+$, $C_4(t,\cdot)$ is increasing for almost every $t \in [0,T]$, and

$$\sup\{\|y\|_* : y \in A(t)x\} \leq C_3(|x|)\|x\|^{p-1} + C_4(t,|x|)$$

for almost every $t \in [0,T]$ and for every $x \in V$, where $\|\cdot\|_*$ is the norm of V'.

Then for every $u_0 \in H$, there exists $u \in W_p^1(0,T;V,H)$ such that

$$u(0) = u_0, \quad u'(t) + A(t)u(t) \ni 0 \quad \text{for almost every } t \in [0,T].$$

Theorem 2. Let T, p, q, V, H and $\{A(t) : 0 \leq t \leq T\}$ be as in Theorem 1. Assume $p > 2$ or $C_0(t) \equiv 0$. Then there exists $u \in W_p^1(0,T;V,H)$ such that

$$u(0) = u(T), \quad u'(t) + A(t)u(t) \ni 0 \quad \text{for almost every } t \in [0,T].$$

Remark 1. In both theorems, if V is compactly imbedded in H or $C_0(t) \equiv 0$, it is sufficient to assume the following (A5) and (A6) instead of (A3) and (A4):
(A5) there exist $C_0, C_2 \in L^1(0,T;\mathbb{R}_+)$ and $C_1 > 0$ such that for almost every $t \in [0,T]$ and for every $x \in V$, there exists $y \in A(t)x$ which satisfies (3.1).
(A6) there exist an increasing function $C_3 : \mathbb{R}_+ \to \mathbb{R}_+$ and $C_4 : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $C_4(\cdot,a) \in L^q(0,T;\mathbb{R}_+)$ for every $a \in \mathbb{R}_+$, $C_4(t,\cdot)$ is increasing for almost every $t \in [0,T]$, and

$$\sup\{\|y\|_* : \ y \in A(t)x \text{ satisfes (3.1) } \} \leq C_3(|x|)\|x\|^{p-1} + C_4(t,|x|)$$

for almost every $t \in [0,T]$ and for every $x \in V$.

For details, see Remark 2 in Section 4.

4. Proofs of theorems

First, we give the proof of Theorem 1. We assume that T, p, q, V, H, $\{A(t)\}$ and u_0 are as in Theorem 1.

Lemma 1. Let $t \in [0,T]$ such that $A(t)$ is pseudo monotone and $A(t)x$ is nonempty, closed, convex subset of V' for every $x \in V$. Let $x \in V$ and let $\{x_n\}$ be a sequence in V which converges weakly to x. Let $y \in V'$ and let $\{y_n\}$ be a sequence in V' such that it converges weakly to y, $y_n \in A(t)x_n$ for every $n \in \mathbb{N}$ and $\overline{\lim}_n(y_n, x_n - x) \leq 0$. Then $y \in A(t)x$. Especially, for almost every $t \in [0,T]$, $A(t)$ is upper semicontinuous from the strong topology of V to the weak topology of V'.

Proof. Since $A(t)$ is pseudo monotone, for every $z \in V$, there exists $y_z \in A(t)x$ such that $(y_z, x-z) \leq \underline{\lim}_n(y_n, x_n - x + x - z) = (y, x-z)$. So we have $\sup_{z\in V}\min_{w\in A(t)x}(w-y, x-z) \leq 0$. By Proposition 4, there exists $w \in A(t)x$ such that $(w-y, x-z) \leq 0$ for all $z \in V$, which implies $w = y$. Hence we obtain $y \in A(t)x$. □

We denote by $\mathcal{V}$ and $\mathcal{V}'$, the spaces $L^p(0,T;V)$ and $L^q(0,T;V')$, respectively and the norms of these spaces are also denoted by $\|\cdot\|$ and $\|\cdot\|_*$, respectively. By $\mathcal{W}$, we mean $W_p^1(0,T;V,H)$. For $u \in \mathcal{V}$ and $t \in [0,T]$, we write $\mathcal{A}u(t)$ instead of $A(t)u(t)$.

To prove the following, we use the method employed in [11, 12, 15].

Lemma 2. Let $\{u_n\}$ be a sequence in $\mathcal{W}$ and let $\{w_n\}$ be a sequence in $\mathcal{V}'$ such that $\{u_n\}$ converges weakly to u in $\mathcal{W}$, $w_n \in \mathcal{A}u_n$ for every $n \in \mathbb{N}$ and $\overline{\lim}_n \langle w_n, u_n - u\rangle \leq 0$. Then for every $v \in \mathcal{V}$, there exists $w_v \in \mathcal{A}u$ such that $\langle w_v, u - v\rangle \leq \underline{\lim}_n \langle w_n, u_n - v\rangle$. Further, if $\{w_n\}$ converges weakly to w in $\mathcal{V}'$, then $w \in \mathcal{A}u$.

Proof. Since $\{u_n\}$ converges weakly to u in $\mathcal{W}$, it also converges weakly to u in $C(0,T;H)$. So $\{u_n\}$ is bounded in $C(0,T;H)$. By (A3) and (A4), there exist $K_1 > 0$, $K_2 > 0$ and $K_3 \in L^1(0,T;\mathbb{R}_+)$ such that

$$(w_n(t), u_n(t) - v(t)) \geq K_1\|u_n(t)\|^p - K_2\|v(t)\|^p - K_3(t) \tag{4.1}$$

for almost every $t \in [0,T]$, for every $n \in \mathbb{N}$ and for every $v \in \mathcal{V}$. We will show that

$$\underline{\lim}_{n\to\infty} (w_n(t), u_n(t) - u(t)) \geq 0 \quad \text{for a.e. } t \in [0,T]. \tag{4.2}$$

Suppose not. Then the set

$$\begin{aligned}\{t \in [0,T] :\ &\underline{\lim}_{n\to\infty} (w_n(t), u_n(t) - u(t)) < 0,\\ &(w_n(t), u_n(t) - u(t)) \geq K_1\|u_n(t)\|^p - K_2\|u(t)\|^p - K_3(t) \quad \text{for all } n \in \mathbb{N}\}\end{aligned}$$

has a positive measure. Let t be an element of the set. Since $\{u_n(t)\}$ is bounded in V from (4.1), $\{u_n(t)\}$ converges weakly to $u(t)$ in V. By (A1), we have $\underline{\lim}_n (w_n(t), u_n(t) - u(t)) = 0$, which contradicts that t is an element of the above set. Hence we have (4.2). By (4.1) and Fatou's lemma, we have

$$0 \leq \int_0^T \underline{\lim}_{n\to\infty} (w_n(t), u_n(t) - u(t))\, dt \leq \underline{\lim}_{n\to\infty} \langle w_n, u_n - u\rangle \leq \overline{\lim}_{n\to\infty} \langle w_n, u_n - u\rangle \leq 0.$$

Hence we obtain $\lim_n \langle w_n, u_n - u\rangle = 0$. Next we will show that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\lim_{i\to\infty} (w_{n_i}(t), u_{n_i}(t) - u(t)) = 0 \quad \text{for a.e. } t \in [0,T]. \tag{4.3}$$

Put $h_n(t) = (w_n(t), u_n(t) - u(t))$ for $t \in [0,T]$. We know that $\underline{\lim}_n h_n(t) \geq 0$ for almost every $t \in [0,T]$ and $\lim_n \int_0^T h_n(t)\, dt = 0$. Set $h_n^-(t) = -\min\{h_n(t), 0\}$ for $t \in [0,T]$. Since $h_n^-(t) \leq K_2\|u(t)\|^2 + K_3(t)$, by Lebesgue's dominated convergence theorem, we get $\lim_n \int_0^T h_n^-(t)\, dt = 0$. So we obtain $\lim_n \int_0^T |h_n(t)|\, dt = 0$. Hence we can choose a subsequence $\{h_{n_i}\}$ of $\{h_n\}$ which satisfies (4.3).

Let $v \in \mathcal{V}$. By the preceding, there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_j \langle w_{n_j}, u_{n_j} - v\rangle = \underline{\lim}_n \langle w_n, u_n - v\rangle$ and $\lim_j (w_{n_j}(t), u_{n_j}(t) - u(t)) = 0$ for almost every $t \in [0,T]$. Since $\{u_{n_j}(t)\}$ is bounded in V by (4.1), $\{u_{n_j}(t)\}$ converges weakly to $u(t)$ in V for almost every $t \in [0,T]$. We know $\underline{\lim}_j (w_{n_j}(t), u_{n_j}(t) - v(t))$ is measurable and from Lemma 1, for almost every $t \in [0,T]$, there exists $y_t \in A(t)u(t)$ with $\underline{\lim}_j (w_{n_j}(t), u_{n_j}(t) - v(t)) = (y_t, u(t) - v(t))$. By Proposition 3, there exists $w_v \in \mathcal{V}'$ such that $w_v(t) \in A(t)u(t)$ and $\underline{\lim}_j (w_{n_j}(t), u_{n_j}(t) - v(t)) = (w_v(t), u(t) - v(t))$ for almost every $t \in [0,T]$. Hence by (4.1) and Fatou's lemma, we have

$$\langle w_v, u - v\rangle = \int_0^T \underline{\lim}_{j\to\infty} (w_{n_j}(t), u_{n_j}(t) - v(t))\, dt \leq \lim_{j\to\infty} \langle w_{n_j}, u_{n_j} - v\rangle = \underline{\lim}_{n\to\infty} \langle w_n, u_n - v\rangle.$$

Now, assume that $\{w_n\}$ converges weakly to w in $\mathcal{V}'$. Then for every $v \in \mathcal{V}$, there exists $w_v \in \mathcal{A}u$ such that $\langle w_v, u - v\rangle \leq \lim_n \langle w_n, u_n - u + u - v\rangle = \langle w, u - v\rangle$, which implies $\sup_{v\in\mathcal{V}} \min_{y\in\mathcal{A}u} \langle y - w, u - v\rangle \leq 0$. By Proposition 4, there exists $y \in \mathcal{A}u$ such that $\langle y - w, u - v\rangle \leq 0$ for all $v \in \mathcal{V}$. So we have $y = w$, and hence $w \in \mathcal{A}u$. $\square$

Let $\{e_1, e_2, \cdots\}$ be a subset of V such that the subspace spanned by $\{e_1, e_2, \cdots\}$ is dense in V and

$$(e_i, e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

i.e., $\{e_i\}$ is a complete orthonormal basis of H with $\{e_i\} \subset V$. For every $n \in \mathbb{N}$, we denote by F_n, the subspace of V spanned by $\{e_1, \cdots, e_n\}$.

We set $u_0^n = \sum_{i=1}^n (u_0, e_i)e_i$ for every $n \in \mathbb{N}$ and $K = \left(|u_0|^2 + 2\int_0^T C_2(s)\,ds\right)\exp\left(2\int_0^T C_0(s)\,ds\right)$.

Lemma 3. For every $n \in \mathbb{N}$, there exist an absolutely continuous function $u_n : [0,T] \to F_n$ and $w_n \in \mathcal{V}'$ such that $u_n' \in L^q(0,T;F_n)$, $w_n(t) \in A(t)u_n(t)$ for almost every $t \in [0,T]$, $|u_n(t)| \leq \sqrt{K}$ for every $t \in [0,T]$ and

(4.4) $\qquad u_n(0) = u_0^n, \qquad (u_n'(t) + w_n(t), v) = 0$ for a.e. $t \in [0,T]$ and for every $v \in F_n$.

Proof. Let $n \in \mathbb{N}$. Let $M > 0$ with $M|\cdot| \leq \|\cdot\|$. Since F_n is finite dimensional, there exists $L_n > 0$ such that $\|v\| \leq L_n|v|$ for all $v \in F_n$. Define $x_0 \in \mathbb{R}^n$ and $f : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ by

$$x_0 = \begin{pmatrix} (u_0^n, e_1) \\ \vdots \\ (u_0^n, e_n) \end{pmatrix}, \qquad f(t,x) = -\begin{pmatrix} \left(A(t)\left(\sum_{i=1}^n x_i e_i\right), e_1\right) \\ \vdots \\ \left(A(t)\left(\sum_{i=1}^n x_i e_i\right), e_n\right) \end{pmatrix}, \quad (t,x) \in [0,T] \times \mathbb{R}^n.$$

By (A2) and Lemma 1, f is a Carathéodory mapping. By (A4), we have

$$\begin{aligned} |f(t,x)| &\leq n\sum_{j=1}^n \left\| A(t)\left(\sum_{i=1}^n x_i e_i\right)\right\|_* \|e_j\| \\ &\leq n^2 L_n\left(C_3\left(\left|\sum_{i=1}^n x_i e_i\right|\right)\left\|\sum_{i=1}^n x_i e_i\right\|^{p-1} + C_4\left(t, \left|\sum_{i=1}^n x_i e_i\right|\right)\right) \qquad (4.5) \\ &\leq n^2 L_n\left(C_3(|x|)L_n^{p-1}|x|^{p-1} + C_4(t,|x|)\right) \end{aligned}$$

for almost every $t \in [0,T]$ and for every $x \in \mathbb{R}^n$. By (A3), we have

$$\begin{aligned} (x, f(t,x)) &= -\left(A(t)\left(\sum_{i=1}^n x_i e_i\right), \sum_{j=1}^n x_j e_j\right) \\ &\leq -C_1\left\|\sum_{i=1}^n x_i e_i\right\|^p + C_0(t)\left|\sum_{i=1}^n x_i e_i\right|^2 + C_2(t) \qquad (4.6) \\ &\leq -M^p C_1|x|^p + C_0(t)|x|^2 + C_2(t) \end{aligned}$$

for almost every $t \in [0,T]$ and for every $x \in \mathbb{R}^n$. We show an a priori estimate for an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ which satisfies $x(0) = x_0$ and $x'(t) \in f(t,x(t))$ for $t \in [0,T]$. Since from (4.6), $(|x(t)|^2)'/2 \leq C_0(t)|x(t)|^2 + C_2(t)$ for almost every $t \in [0,T]$, we have $|x(t)|^2 \leq |x_0|^2 + 2\int_0^T C_2(s)\,ds + 2\int_0^t C_0(s)|x(s)|^2\,ds$ for every $t \in [0,T]$. So by Gronwall's inequality, we get

$$|x(t)|^2 \leq \left(|x_0|^2 + 2\int_0^T C_2(s)\,ds\right)\exp\left(2\int_0^T C_0(s)\,ds\right) \quad \text{for every } t \in [0,T].$$

By the standard fixed point argument, there exists an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ such that $x'(t) \in f(t,x(t))$ for almost every $t \in [0,T]$ and $x(0) = x_0$. Let $u_n : [0,T] \to F_n$ be the absolutely continuous function defined by $u_n(t) = \sum_{i=1}^{n} x_i(t)e_i$ for every $t \in [0,T]$. It is easy to see that $u_n(0) = u_0^n$, $|u_n(t)| \leq \sqrt{K}$ for every $t \in [0,T]$ and $u_n' \in L^q(0,T;F_n)$. Since the mapping

$$t \mapsto \begin{pmatrix} (u_n'(t), e_1) \\ \vdots \\ (u_n'(t), e_n) \end{pmatrix} : [0,T] \to \mathbb{R}^n$$

is measurable, by Proposition 3, there exists a measurable mapping $w_n \in \mathcal{V}'$ such that $w_n(t) \in A(t)u_n(t)$ and $\big(u_n'(t) + w_n(t), e_j\big) = 0$ for almost every $t \in [0,T]$ and for every $j = 1, \dots, n$. □

Lemma 4. $\{u_n\}$ is bounded in $\mathcal{W}$.

Proof. Let $n \in \mathbb{N}$. From (4.4) and (A3), we have

$$\int_0^T \|u_n(t)\|^p \, dt \leq \frac{1}{C_1}\Big(K\int_0^T C_0(t)\,dt + \int_0^T C_2(t)\,dt + \frac{|u_0|^2}{2}\Big).$$

Since $u_n' \in L^q(0,T;F_n)$, there exists $v_n \in L^p(0,T;F_n)$ such that $\langle u_n', v_n\rangle = \|v_n\|^2 = \|u_n'\|_*^2$. So, by (4.4) and (A4), we get

$$\|u_n'\|_* \leq C_3\big(\sqrt{K}\big)\|u\|^{\frac{p}{q}} + \Big(\int_0^T |C_4(t,\sqrt{K})|\,dt\Big)^{\frac{1}{q}}.$$

Hence $\{u_n\}$ is bounded in $\mathcal{W}$. □

Since $\{u_n\}$ and $\{w_n\}$ are bounded in $\mathcal{W}$ and $\mathcal{V}'$, respectively, we may assume that $\{u_n\}$ converges weakly to u in $\mathcal{W}$ and $\{w_n\}$ converges weakly to w in $\mathcal{V}'$.

Lemma 5. $u' + w = 0$ and $u(0) = u_0$.

Proof. First, we will show $u' + w = 0$. Let $\varphi \in C_0^\infty(0,T)$ and let $n \in \mathbb{N}$. By (4.4), we have for every $m \geq n$,

$$\begin{aligned} 0 = (u_m(T), \varphi(T)e_n) - (u_m(0), \varphi(0)e_n) &= \int_0^T \big((u_m'(t), \varphi(t)e_n) + (\varphi'(t)e_n, u_m(t))\big)\,dt \\ &= \int_0^T \big((-w_m(t), \varphi(t)e_n) + (\varphi'(t)e_n, u_m(t))\big)\,dt = \langle -w_m, \varphi e_n\rangle + \langle \varphi' e_n, u_m\rangle. \end{aligned}$$

So we get $0 = \langle -\varphi w + \varphi' u, e_n\rangle$ for all $n \in \mathbb{N}$ and $\varphi \in C_0^\infty(0,T)$. Hence we obtain $u' + w = 0$. Since $\{u_n\}$ converges weakly to u in $\mathcal{W}$ and $\mathcal{W}$ is continuously imbedded in $C(0,T;H)$, $\{u_n(0)\}$ converges weakly to $u(0)$ in H. So we obtain $u(0) = u_0$. □

Proof of Theorem 1. By Lemma 3 and Lemma 5, we have $\langle w_n, u_n\rangle = 1/2|u_0^n|^2 - 1/2|u_n(T)|^2$ and $\langle w, u\rangle = 1/2|u_0|^2 - 1/2|u(T)|^2$. So we get

$$\varlimsup_{i\to\infty} \langle w_n, u_n - u\rangle = \frac{1}{2}\big(\lim_{n\to\infty} |u_0^n|^2 - |u_0|^2 + |u(T)|^2 - \varliminf_{i\to\infty} |u_n(T)|^2\big) \leq 0.$$

By Lemma 2, we have $w \in \mathcal{A}u$. Hence we obtain $u \in \mathcal{W}$ and $w \in \mathcal{A}u$ such that $u(0) = u_0$ and $u' + w = 0$. □

Next, we give the proof of Theorem 2. Till the end of this section, we assume $p > 2$ or $C_0(t) \equiv 0$. We fix $r > 0$ which satisfies

$$-C_1 M^p r^p T + \int_0^T C_2(s)\,ds + \big((r^2 + a)b + a\big)b\int_0^T C_0(s)\,ds < 0,$$

where $a = 2\int_0^T C_2(s)\,ds$ and $b = \exp\big(2\int_0^T C_0(s)\,ds\big)$. We set $\rho = \sqrt{(r^2+a)b}$ and $R = \sqrt{(\rho^2+a)b}$.

Lemma 6. For every $n \in \mathbb{N}$, there exist an absolutely continuous function $u_n : [0,T] \to F_n$ and $w_n \in \mathcal{V}'$ such that $u_n' \in L^q(0,T;F)$, $w_n(t) \in A(t)u_n(t)$ for almost every $t \in [0,T]$, $|u_n(t)| \leq R$ for every $t \in [0,T]$ and

$$u_n(0) = u_n(T), \quad (u_n'(t) + w_n(t), v) = 0 \quad \text{for a.e. } t \in [0,T] \text{ and for every } v \in F_n.$$

Proof. Let $n \in \mathbb{N}$. Let M, L_n and f be as in the proof of Lemma 3. Let $x : [0,T] \to \mathbb{R}^n$ be an absolutely continuous function which satisfies $x'(t) \in f(t, x(t))$ for almost every $t \in [0,T]$ and $|x(0)| \leq \rho$. We will show $|x(T)| \leq \rho$. By (4.6), it is easy to see that if there exists $t_0 \in [0,T]$ with $|x(t_0)| \leq r$ then $|x(T)| \leq \rho$. So we may assume that $|x(t)| > r$ for every $t \in [0,T]$. Since $|x(t)| \leq R$ for every $t \in [0,T]$ from (4.6), we get

$$\frac{1}{2}|x(T)|^2 - \frac{1}{2}|x(0)|^2 \leq -C_1 M^p r^p T + \int_0^T C_2(s)\,ds + R^2 \int_0^T C_0(s)\,ds < 0$$

from (4.6). So we obtain $|x(T)| \leq \rho$.

Let $g : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a Carathéodory mapping defined by

$$g(t,y) = \begin{cases} f(t,y) & \text{if } |y| \leq R, \\ f(t, Ry/|y|) & \text{if } |y| > R \end{cases} \qquad \text{for } (t,y) \in [0,T] \times \mathbb{R}^n.$$

We remark that $S_f(y) = S_g(y)$ for $y \in \mathbb{R}^n$ with $|y| \leq \rho$, where S_f and S_g are as in Proposition 1. From Proposition 1 and the continuity property of Čech homology, $S_g(y)$ is acyclic in the sense of Čech homology with rational coefficients for every $y \in \mathbb{R}^n$. We set $X = \{x \in C(0,T;\mathbb{R}^n) : |x(T)| \leq \rho\}$. For every $x \in X$, put $Hx = S_g(x(T)) = S_f(x(T))$. Then $H : X \to 2^X$ is an upper semicontinuous mapping such that $\cup_{x \in X} H(x)$ is contained in a compact subset of X and Hx is acyclic and compact for every $x \in X$. Hence, by Proposition 2, there exists a fixed point of H, i.e., there exists an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ such that $x'(t) \in f(t,x(t))$ for almost every $t \in [0,T]$, $x(0) = x(T)$ and $|x(t)| \leq R$ for every $t \in [0,T]$. Let $u_n : [0,T] \to F_n$ be the absolutely continuous function defined by $u_n(t) = \sum_{i=1}^n x_i(t) e_i$ for every $t \in [0,T]$. By the similar argument as in the proof of Lemma 3, we finish the proof. □

Lemma 7. $\{u_n\}$ is bounded in $\mathcal{W}$.

Proof. Let $n \in \mathbb{N}$. From Lemma 6 and (A3), we have

$$\begin{aligned} 0 &= \int_0^T (w_n(t), u_n(t))\,dt + \frac{1}{2}|u_n(T)|^2 - \frac{1}{2}|u_n(0)|^2 \\ &\geq C_1 \int_0^T \|u_n(t)\|^p\,dt - R^2 \int_0^T C_0(t)\,dt - \int_0^T C_2(t)\,dt. \end{aligned}$$

Hence by the same argument as in the proof of Lemma 4, $\{u_n\}$ is bounded in $\mathcal{W}$. □

Since $\{u_n\}$ and $\{w_n\}$ are bounded in $\mathcal{W}$ and $\mathcal{V}'$, respectively, we may assume that $\{u_n\}$ converges weakly to u in $\mathcal{W}$ and $\{w_n\}$ converges weakly to w in $\mathcal{V}'$.

By the same lines as those in the proof of Lemma 5, we obtain the following:

Lemma 8. $u' + w = 0$ and $u(0) = u(T)$.

Proof of Theorem 2. By Lemma 6 and Lemma 8, we have $\langle w_n, u_n \rangle = 0$ for every $n \in \mathbb{N}$ and $\langle w, u \rangle = 0$. So we get $\lim_n \langle w_n, u_n - u \rangle = 0$. By Lemma 2, we have $w \in \mathcal{A}u$. Hence we obtain $u \in \mathcal{W}$ and $w \in \mathcal{A}u$ such that $u(0) = u(T)$ and $u' + w = 0$. □

Remark 2. We give the proof of Remark 1. We set

$$\tilde{A}(t)x = \{y \in A(t)x : \ y \text{ satisfies } (3.1)\} \qquad \text{for } (t,x) \in [0,T] \times V.$$

Since it is easy to see that $\{\tilde{A}(t)\}$ satisfies (A2), (A3) and (A4), it is sufficient to show the following:

Lemma 9. $\tilde{A}(t)$ is pseudo monotone for almost every $t \in [0,T]$.

Proof. Let $t \in [0,T]$ such that $A(t)$ is pseudo monotone and $A(t)x$ is nonempty, closed, convex subset of V' for every $x \in V$. Let $x \in V$ and let $\{x_n\}$ be a sequence in V which converges weakly to x in V. Let $\{y_n\}$ be a sequence in V' such that $y_n \in \tilde{A}(t)x_n$ for every $n \in \mathbb{N}$ and $\overline{\lim}_n (y_n, x_n - x) \leq 0$. Fix $z \in V$. Then there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\{y_{n_i}\}$ converges weakly to y in V' and $\underline{\lim}_n (y_n, x_n - z) = \lim_i (y_{n_i}, x_{n_i} - z)$. Since $(y, x - z) = \underline{\lim}_n (y_n, x_n - z)$, it is sufficient to show that $y \in \tilde{A}(t)x$. From Lemma 1, we have $y \in A(t)x$. Since $\{y_{n_i}\}$ converges weakly to y, $y_{n_i} \in \tilde{A}(t)x_{n_i}$ and V is compactly imbedded in H or $C_0(t) \equiv 0$, we obtain $y \in \tilde{A}(t)x$ from (3.1). This completes the proof. □

5. Application

Throughout this section, $T > 0$, $p \geq 2$ with $1/p + 1/q = 1$, Ω is an bounded domain in $\mathbb{R}^N$ with smooth boundary and $Q = (0,T) \times \Omega$, m is a positive natural number. For real valued function u on Ω, we mean $\eta(u)$ and $\zeta(u)$ as follows:

$$\eta(u) = \{D^\alpha u : |\alpha| \leq m-1\} \in \mathbb{R}^{N_1}, \qquad \zeta(u) = \{D^\alpha u : |\alpha| = m\} \in \mathbb{R}^{N_2}.$$

Let $\{A_\alpha : |\alpha| = m\}$ be functions from $Q \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ into $\mathbb{R}$, let $\{A_\alpha : |\alpha| \leq m-1\}$ be multivalued functions $Q \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ into $2^{\mathbb{R}}$, and let h be a function from Q into $\mathbb{R}$.

We consider the following nonlinear differential inclusion

$$\text{(5.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \displaystyle\sum_{|\alpha|=m} (-1)^{|\alpha|} D^\alpha A_\alpha(t,x,\eta(u),\zeta(u)) \\ \qquad\qquad + \displaystyle\sum_{|\alpha|\leq m-1} (-1)^{|\alpha|} D^\alpha w_\alpha(t,x) = h(t,x) \quad \text{on } Q, \\ w_\alpha(t,x) \in A_\alpha(t,x,\eta(u),\zeta(u)) \quad \text{on } Q \text{ for } |\alpha| \leq m-1 \end{cases}$$

with Dirichlet boundary condition

$$\text{(5.2)} \qquad D^\alpha u = 0 \qquad \text{on } [0,T] \times \partial\Omega \text{ for } |\alpha| \leq m-1.$$

Theorem 3. Assume the following:

(i) for every α with $|\alpha| = m$, $A_\alpha(\cdot,\cdot,\eta,\zeta)$ is measurable for every $(\eta,\zeta) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $A_\alpha(t,x,\cdot,\cdot)$ is continuous for almost every $(t,x) \in Q$;

(ii) for every α with $|\alpha| \leq m-1$, $A_\alpha(t,x,\eta,\zeta)$ is nonempty, closed, convex subset of $\mathbb{R}$ for every $(t,x,\eta,\zeta) \in Q \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, $A_\alpha(\cdot,\cdot,\eta,\zeta)$ is measurable for every $(\eta,\zeta) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $A_\alpha(t,x,\cdot,\cdot)$ is upper semicontinuous for almost every $(t,x) \in Q$;

(iii) there exist $c_3 > 0$ and $c_4(t,x) \in L^q(Q;\mathbb{R}_+)$ such that

$$\sup |A_\alpha(t,x,\eta,\zeta)| \leq c_3\left(|\zeta|^{p-1} + |\eta|^{p-1}\right) + c_4(t,x)$$

for every $|\alpha| \leq m$ and $(t,x,\eta,\zeta) \in Q \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$;

(iv) for every $(t,x,\eta) \in Q \times \mathbb{R}^{N_1}$ and $\zeta, \zeta' \in \mathbb{R}^{N_2}$ with $\zeta \neq \zeta'$,

$$\sum_{|\alpha|=m} \left(A_\alpha(t,x,\eta,\zeta) - A_\alpha(t,x,\eta,\zeta')\right)(\zeta_\alpha - \zeta'_\alpha) > 0;$$

(v) there exist $c_1 > 0$, $c_0 \in L^1(0,T;\mathbb{R}_+)$ and $c_2 \in L^1(Q;R_+)$ such that

$$\sum_{|\alpha|=m} A_\alpha(t,x,\eta,\zeta)\zeta_\alpha + c_0(t)|\eta_0|^2 \geq c_1|\zeta|^p - c_2(t,x)$$

for every $(t,x,\eta,\zeta) \in Q \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, where $\eta_0 = \{\eta_\beta : |\beta| = 0\}$;

Then for every $h \in L^q(0,T;W^{-1,q}(\Omega))$ and $u_0 \in L^2(\Omega)$, there exists $u \in W^1_p(0,T;W_0^{1,p}(\Omega),L^2(\Omega))$ which satisfies (5.1) and (5.2) with $u(0)=u_0$, and if $p>2$ or $c_0(t) \in L^\infty(0,T;\mathbb{R}_+)$, then for every $h \in L^q(0,T;W^{-1,q}(\Omega))$, there exists $u \in W^1_p(0,T;W_0^{1,p}(\Omega),L^2(\Omega))$ which satisfies (5.1) and (5.2) with $u(0)=u(T)$.

Proof. For every $t \in [0,T]$, we define an operator $A(t): W_0^{m,p}(\Omega) \to 2^{W^{-m,q}(\Omega)}$ by $w \in A(t)u$ if for almost every $x \in \Omega$ and $|\alpha| \le m$, $w_\alpha(x) \in A_\alpha(t,x,\eta(u),\zeta(u))$ and for every $v \in W_0^{m,p}(\Omega)$, $(w,v) = \int_\Omega \sum_{|\alpha|\le m} w_\alpha(x) D^\alpha v\,dx$. First, we will show that $A(t)$ is pseudo monotone for almost every $t \in [0,T]$. We use the method employed in [5, 14]. Fix $t \in [0,T]$. Let $\{(u^n,w^n)\}$ be a sequence in $W_0^{m,p}(\Omega) \times W^{-m,q}(\Omega)$ such that $\{u^n\}$ converges weakly to u, $w^n \in Au^n$ and $\overline{\lim}_n (w^n, u^n-u) \le 0$. Fix $v \in W_0^{m,p}(\Omega)$. Taking a subsequence, if necessary, we may assume that $\underline{\lim}_n (w^n,u^n-v) = \lim_n (w^n,u^n-v)$, $\{D^\alpha u^n\}$ converges strongly and almost everywhere to $D^\alpha u$ for $|\alpha| \le m-1$ and $\{w^n_\alpha\}$ converges weakly to w_α in $L^q(\Omega)$ for $|\alpha| \le m$. We show $w_\alpha(x) \in A_\alpha(t,x,\eta(u),\zeta(u))$ for almost every $x \in \Omega$ and for every $|\alpha| \le m-1$. Fix α with $|\alpha| \le m-1$. We may assume that there exists $\omega^n_\alpha \in \mathrm{co}\{w^n_\alpha, w^{n+1}_\alpha, \cdots\}$ and $\omega^n_\alpha(x)$ converges to $w_\alpha(x)$ almost everywhere. Fix $x \in \Omega$. Let (β,γ) be an open interval with $A_\alpha(t,x,\eta(u),\zeta(u)) \subset (\beta,\gamma)$. From the upper semicontinuity of $A_\alpha(t,x,\cdot,\cdot)$, we have $w_\alpha(x) \in (\beta,\gamma)$. Since (β,γ) is an arbitrary open interval which contains $A_\alpha(t,x,\eta(u),\zeta(u))$ and $A_\alpha(t,x,\eta(u),\zeta(u))$ is closed and convex, we have $w_\alpha(x) \in A_\alpha(t,x,\eta(u),\zeta(u))$ almost everywhere. Set

$$
\begin{aligned}
p_n(x) &= \textstyle\sum_{|\alpha|=m} A_\alpha(t,x,\eta(u^n),\zeta(u^n))(D^\alpha u^n - D^\alpha u) + \sum_{|\alpha|\le m-1} w^n_\alpha(x)(D^\alpha u^n - D^\alpha u);\\
q_n(x) &= \textstyle\sum_{|\alpha|=m} (A_\alpha(t,x,\eta(u^n),\zeta(u^n)) - A_\alpha(t,x,\eta(u^n),\zeta(u)))(D^\alpha u^n - D^\alpha u);\\
r_n(x) &= \textstyle\sum_{|\alpha|=m} A_\alpha(t,x,\eta(u^n),\zeta(u))(D^\alpha u^n - D^\alpha u);\\
s_n(x) &= \textstyle\sum_{|\alpha|\le m-1} w^n_\alpha(x)(D^\alpha u^n - D^\alpha u).
\end{aligned}
$$

It is easy to see $q_n = p_n - r_n - s_n$ and $\overline{\lim}_n \int q_n \le 0$. From $q_n \ge 0$, $q_n(x) \to 0$ almost everywhere. By [13, Lemma 6], $D^\alpha u^n(x) \to D^\alpha u(x)$ almost everywhere for $|\alpha| = m$, and hence $p_n(x) \to 0$ almost everywhere. From the uniform integlability of $\{p_n^-\}$, we have $\lim_n \int p_n^- = 0$. So we obtain $\lim_n \int p_n^+ = 0$ and

$$\lim_{n\to\infty} (w^n, u^n - v) = \lim_{n\to\infty} \left(\int_\Omega p_n(x)\,dx + (w^n, u-v) \right) = (w, u-v).$$

Hence $A(t)$ is pseudo monotone, which implies (A1). By Sobolev's imbedding theorem, we have (A3). Hence by our theorems, we obtain the conclusion. □

As a direct consequence, we have the following, which improves [6, Theorem 4.1] and [15, Theorem 2].

Corollary. Let $b_i, a : \mathbb{R} \to \mathbb{R}$ be bounded and continuous functions and let $g : Q \times \mathbb{R} \times \mathbb{R}^n \to 2^{\mathbb{R}}$ be Carathéodory mapping such that there exist $\alpha, \beta > 0$ which satisfy $\sup |g(t,x,\eta,\zeta)| \le \alpha(|\eta|^{p-1} + |\zeta|^{p-1}) + \beta$ for every $(t,x,\eta,\zeta) \in Q \times \mathbb{R} \times \mathbb{R}^n$. Then

$$\frac{\partial u}{\partial t} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i} + a(x)|u(x)|^{p-2}u(x) + g(t,x,u(x),Du(x)) \ni 0$$

with Dirichlet boundary condition has a T-periodic, weak solution $u \in W^1_p(0,T;W_0^{1,p}(\Omega),L^2(\Omega))$, where $Du(x) = (\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n})$.

REFERENCES

[1] N. U. Ahmed and X. Xiang, *Existence of solutions for a class of nonlinear evolution equations with nonmonotone perturbations,* Nonlinear Anal. **13** (1994), 81-89.
[2] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces,* Noordhoff, Leyden, 1976.
[3] J. Berkovits and V. Mustonen, *Monotonicity methods for nonlinear evolution equations,* Nonlinear Anal. **27** (1996), 1397-1405.
[4] F. E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces,* Proc. Symp. Pure Math. **18**, part 2, 1976.
[5] F. E. Browder, *Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains,* Proc. Nat. Acad. Sci. USA **74** (1977), 2659-2661.
[6] J. P. Caşcaval and I. I. Vrabie, *Existence of periodic solutions for a class of nonlinear evolution equations,* Rev. Mate. Univ. Complutense Madrid **7** (1994), 325-338.
[7] F. S. De Blasi and J. Myjak, *On the solutions sets for differential inclusions,* Bull. Pol. Acad. Sci. Math. **33** (1985), 17-23.
[8] K. Fan, *Minimax theorems,* Proc. Nat. Acad. Sci. USA **39** (1953), 42-47.
[9] L. Górniewicz, *Homological methods in fixed point theory of multi-valued maps,* Dissertationes Math. **129** (1976), 1-66.
[10] L. Górniewicz and S. Plaskacz, *Periodic solutions of differential inclusions in R^n,* Boll. U. M. I. (7) **7-A** (1993), 409-420.
[11] N. Hirano, *Nonlinear Volterra equations with positive kernels,* in Nonlinear and convex analysis (Santa Barbara, Calif., 1985), Lecture Notes in Pure and Appl. Math. **107**, 83-98, 1987.
[12] N. Hirano, *Nonlinear evolution equations with nonmonotonic perturbations,* Nonlinear Anal. **13** (1989), 599-609.
[13] R. Landes, *On Galerkin's method in the existence theory of quasilinear elliptic equations,* J. Funct. Anal. **39** (1980), 123-148.
[14] R. Landes and V. Mustonen, *On pseudo-monotone operators and nonlinear noncoercive variational problems on unbounded domains,* Math. Ann. **248** (1980), 241-246.
[15] N. Shioji, *Existence of periodic solutions for nonlinear evolution equations with nonmonotonic perturbations,* Proc. Amer. Math. Soc. **125** (1997), 2921-2929.
[16] D. H. Wagner, *Survey of measurable selection theorems,* SIAM J. Control Optimization **15** (1977), 859-903.
[17] E. Zeidler, *Nonlinear functional analysis and its applications II/A,B, Linear monotone operators,* Springer-Verlag, New York, 1990.

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