Some Two-Person Zero-Sum Dynamic Game (Nonlinear Analysis and Convex Analysis)

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by

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1 A Two-person Zero-Sum Dynamic Game with a Parameter

We give a two-person zero-sum dynamic game with a parameter \((DPG_\theta)\) by a sequence of the following objects

\[(S_n, A_n, B_n, t_{n+1}, u_n, v_n, \theta; n \in N)\]  \hspace{1cm} (1.1)

where

1. \(S_n\) is the state space at time \(n \in N\) and is assumed to be a Borel space, that is, a nonempty Borel subset of a complete separable metric space.

2. \(A_n\) and \(B_n\) are the action spaces at time \(n \in N\) of players I and II, respectively. It is assumed that \(A_n\) and \(B_n\) are Borel spaces.

3. \(\{t_{n+1}\}\) is the law of motion of the system; \(t_{n+1}\) is a Borel measurable transition probability from \(H_n A_n B_n\) to \(S_{n+1}\), \(n \in N\). Here, \(H_1 = S_1, H_n = S_1 A_1 B_1 \cdots S_{n-1} A_{n-1} B_{n-1} S_n, H_\infty = S_1 A_1 B_1 S_2 B_2 S_3 \cdots\). Then, \(H_n\) is the set of histories of the game for horizon \(n \in N\), while \(H_\infty\) is the set of all infinite histories of the game.

4. \(u_n : H_n A_n B_n \to \mathbb{R}\) is a Borel measurable function and \(v_n : H_n A_n B_n \to \mathbb{R}_+\), is a nonnegative bounded Borel measurable function, where \(\mathbb{R}_+ = (0, \infty)\). Of course, \(u_n\) and \(v_n\) may be recognized as functions on \(H_\infty\). Doing so, we assume that

\[\lim_{n \to \infty} u_n = u \in \mathbb{R}, \quad \lim_{n \to \infty} v_n = v \in \mathbb{R}_+.\]

5. \(\theta : S_1 \to \mathbb{R}\) is a real valued function, which is called a parameter function of the game.

6. \(T_\theta^n = u_n - \theta v_n : H_n A_n B_n \to \mathbb{R}\) is a loss function of player I at stage \(n \in N\) and \(-T_\theta^n\) is a loss function of player II.

Let \(F_n(G_n)\) be the set of all universally measurable transition probabilities from \(H_n(H_n)\) to \(A_n(B_n)\). A universally measurable strategy of player I(II) is a sequence \(f = \{f_n\}(g = \{g_n\})\) such that \(f_n \in F_n(g_n \in G_n)\) for each \(n \in N\). Denote by \(F(G)\) the set of all strategies for player I(II).

Let \(E_{f_n}, E_{g_n}, E_{t_{n+1}}\) denote the conditional expectation operator with respect to \(f_n \in F_n, g_n \in G_n, t_{n+1}\), respectively. Then, each pair of strategies \(f =
\{f_n\} (g = \{g_n\}), together with the law of motion \{t_{n+1}\}, defines uniquely a universally measurable transition probability \(P_{fg}(\cdot|\cdot)\) from \(S_1\) to \(A_1B_1S_2A_2B_2S_3\cdots\) such that, for two bounded Borel measurable functions \(u_n, v_n\) defined on \(H_nA_nB_n\) \((n \in \mathbb{N})\), we have for \(s_1 \in S_1\) and \(h \in H_\infty\),

\[
E(u_n, f, g)(s_1) = \int u_n(h) P_{fg}(dh|s_1) = E_{f_1}E_{g_1}E_{t_2} \cdots E_{f_{n-1}}E_{g_{n-1}}E_{t_n}E_{f_n}E_{g_n}u_n(s_1)
\]

and

\[
E(v_n, f, g)(s_1) = \int v_n(h) P_{fg}(dh|s_1) = E_{g_1}E_{t_2} \cdots E_{f_{n-1}}E_{g_{n-1}}E_{t_n}E_{f_n}E_{g_n}v_n(s_1)
\]

where \(u_n\) and \(v_n\) are also regarded as functions on \(H_\infty\).

Under our assumptions, we infer that, for each \(s_1 \in S_1\), \(f = \{f_n\} \in F\), \(g = \{g_n\} \in G\), from the dominated convergence theorem and Fubini’s theorem

\[
U(f, g)(s_1) = \lim_{n \to \infty} E(u_n, f, g)(s_1) = \lim_{n \to \infty} E_{f_1}E_{g_1}E_{t_2} \cdots E_{f_{n-1}}E_{g_{n-1}}E_{t_n}E_{f_n}E_{g_n}u_n(s_1)
\]

and

\[
V(f, g)(s_1) = \lim_{n \to \infty} E(v_n, f, g)(s_1) = \lim_{n \to \infty} E_{g_1}E_{f_1}E_{t_2} \cdots E_{g_{n-1}}E_{f_{n-1}}E_{t_n}E_{g_n}E_{f_n}v_n(s_1).
\]

For the loss function with the parameter function \(\theta\);

\[
T^\theta_n = u_n - \theta v_n,
\]

we have for each \(s_1 \in S_1\), \(f = \{f_n\} \in F\), \(g = \{g_n\} \in G\),

\[
T_\theta(f, g)(s_1) = \lim_{n \to \infty} E_{f_n}T^\theta_n(f, g)(s_1) = U(f, g)(s_1) - \theta(s_1)V(f, g)(s_1).
\]

We define for initial state \(s_1 \in S_1\),

\[
\overline{T}_\theta(s_1) = \inf_{f \in F} \sup_{g \in G} T_\theta(f, g)(s_1), \quad \underbar{T}_\theta(s_1) = \sup_{g \in G} \inf_{f \in F} T_\theta(f, g)(s_1).
\]

Then, \(\overline{T}_\theta(s_1)(\underbar{T}_\theta(s_1))\) is called the upper (the lower) value function of the parametric game. In general, it holds that \(\overline{T}_\theta(s_1) \geq \underbar{T}_\theta(s_1)\) for all \(s_1 \in S_1\). Further, we call the duality gap the interval \([\underbar{T}_\theta(s_1), \overline{T}_\theta(s_1)]\) for all \(s_1 \in S_1\).

**Definition 1.1** We shall say that the two-person zero-sum game \((DPG_\theta)\) has a saddle value function (in short, a value function), if

\[
\overline{T}_\theta(s_1) = \underbar{T}_\theta(s_1) = T^*_\theta(s_1)
\]

and this common function is called the value function of the game and is denoted by \(T^*_\theta(s_1)\).
Definition 1.2 A strategy $\bar{f} \in F$ is said to be a **mini-sup** of the game $(DPG_\theta)$ if 
$$\sup_{g \in G} T_\theta(\bar{f}, g)(s_1) = \underline{T}_\theta(s_1)$$
and a strategy $\bar{g} \in G$ is said to be a **max-inf** of the game $(DPG_\theta)$ if 
$$\inf_{f \in F} T_\theta(f, \bar{g})(s_1) = \overline{T}_\theta(s_1).$$

Definition 1.3 A pair strategies $(\bar{f}, \bar{g}) \in F \times G$ is said to be a **saddle point** of the game $(DPG_\theta)$ if 
$$\inf_{f \in F} T_\theta(f, \bar{g})(s_1) = T_\theta(\bar{f}, \bar{g})(s_1) = \sup_{g \in G} T_\theta(f, g)(s_1).$$

2 A Two-Person Zero-Sum Dynamic Fractional Game

We define a two-person zero-sum dynamic fractional game $(DFG)$ as follows:

$$(S_n, A_n, B_n, t_{n+1}, u_n, v_n, \overline{\theta}, \underline{\theta}; n \in N)$$

where $S_n$ is the state space and $A_n$ and $B_n$ are the action spaces at time $n \in N$ of players I and II, respectively. $\{t_{n+1}\}$ is the law of motion of the system. These terms are defined like as the game $(DPG_\theta)$. Further, $u_n : H_n A_n B_n \to \mathbb{R}$, is a bounded Borel measurable function and $v_n : H_n A_n B_n \to \mathbb{R}_+$, is a nonnegative bounded Borel measurable function, $\mathbb{R}_+ = (0, \infty)$. We assume that

$$\lim_{n \to \infty} u_n = u \in \mathbb{R}, \quad \lim_{n \to \infty} v_n = v \in \mathbb{R}_+.$$

Under our assumptions, we infer that, for each $s_1 \in S_1$, $f = \{f_n\} \in F$, $g = \{g_n\} \in G$,

$$U(f, g)(s_1) = \lim_{n \to \infty} E(u_n, f, g)(s_1), \quad V(f, g)(s_1) = \lim_{n \to \infty} E(v_n, f, g)(s_1) > 0.$$

Using the notations $U(f, g)(s_1)$ and $V(f, g)(s_1)$, we give

$$W(f, g)(s_1) = \frac{U(f, g)(s_1)}{V(f, g)(s_1)}$$

and we define for an initial state $s_1 \in S_1$,

$$\overline{\theta}(s_1) = \inf_{f \in F} \sup_{g \in G} W(f, g)(s_1), \quad \underline{\theta}(s_1) = \sup_{g \in G} \inf_{f \in F} W(f, g)(s_1).$$

Then, $\overline{\theta}(s_1)(\underline{\theta}(s_1))$ is called the upper (the lower) value function of the game $(DFG)$. In general, it holds that $\overline{\theta}(s_1) \geq \underline{\theta}(s_1)$ for all $s_1 \in S_1$ and the interval $[\underline{\theta}(s_1), \overline{\theta}(s_1)]$ is called the **duality gap** of the game $(DFG)$.

**Definition 2.1** The game $(DFG)$ is said to have a **value function** if the duality gap is equal to zero. We shall call the value function of the game $(DFG)$ the **common value function**

$$\bar{\theta}(s_1) = \bar{\theta}(s_1) = \theta^*(s_1).$$
Further, \( g^* \in G \) is said to be a \textbf{max-inf} of the game \((DFG)\) if
\[
\overline{\theta}(s_1) = \inf \sup_{f \in F, g \in G} W(f, g)(s_1) = \inf_{f \in F} W(f, g^*)(s_1).
\] (2.2)

Similarly, \( f^* \in F \) is said to be a \textbf{mini-sup} of the game \((DFG)\) if
\[
\underline{\theta}(s_1) = \sup \inf_{f \in F, g \in G} W(f, g)(s_1) = \sup_{g \in G} W(f^*, g)(s_1).
\] (2.3)

\textbf{Lemma 2.1} \( \overline{T}_\theta(s_1) \) has the following properties.

1. If two parameter functions \( \theta_1(s_1) \) and \( \theta_2(s_1) \) satisfy that \( \theta_1(s_1) > \theta_2(s_1) \geq 0 \), it follows that
\[
\overline{T}_{\theta_1}(s_1) \leq \overline{T}_{\theta_2}(s_1).
\] (1)

2. If \( \overline{T}_\theta(s_1) < 0 \), it holds that \( \theta(s_1) \geq \overline{\theta}(s_1) \).

3. If \( \overline{T}_\theta(s_1) > 0 \), it holds that \( \theta(s_1) \leq \overline{\theta}(s_1) \).

4. If \( \theta(s_1) > \overline{\theta}(s_1) \), it holds that \( \overline{T}_\theta(s_1) \leq 0 \).

5. If \( \theta(s_1) < \overline{\theta}(s_1) \), it holds that \( \overline{T}_\theta(s_1) \geq 0 \).

\textbf{Proof.} (1) If \( \theta_1(s_1) > \theta_2(s_1) \), then, we get \( \theta_1(s_1) U(f, g)(s_1) > \theta_2(s_1) U(f, g)(s_1) \), because \( U(f, g)(s_1) \) is positive for all \((f, g) \in F \times G\). Then, it follows that for all \((f, g) \in F \times G\),
\[
T_{\theta_1}(f, g)(s_1) < T_{\theta_2}(f, g)(s_1).
\]

Therefore, we get that
\[
\overline{T}_{\theta_1}(s_1) = \inf \sup_{f \in F, g \in G} T_{\theta_1}(f, g)(s_1)
\leq \inf \sup_{f \in F, g \in G} T_{\theta_2}(f, g)(s_1)
= \overline{T}_{\theta_2}(s_1).
\]

Thus, the proof of (1) in the lemma is complete.

(2) Since \( \overline{T}_\theta(s_1) < 0 \), from the definition of \( \overline{T}_\theta(s_1) \), there exists \( \overline{f} \in F \) such that \( \sup_{g \in G} T_\theta(\overline{f}, g)(s_1) < 0 \), that is, for all \( g \in G \),
\[
T_\theta(\overline{f}, g)(s_1) = U(\overline{f}, g)(s_1) - \theta(s_1)V(\overline{f}, g)(s_1) < 0.
\] (2.4)

From (2.4), this shows that for all \( g \in G \),
\[
W(\overline{f}, g)(s_1) = \frac{U(\overline{f}, g)(s_1)}{V(\overline{f}, g)(s_1)} < \theta(s_1)
\] (2.5)

that is,
\[
\sup_{g \in G} W(\overline{f}, g)(s_1) \leq \theta(s_1).
\] (2.6)

From the definition of \( \overline{\theta}(s_1) \) and (2.6), it follows that \( \theta(s_1) \geq \overline{\theta}(s_1) \).

(3) Since \( \overline{T}_\theta(s_1) > 0 \), that is, for all \( f \in F \), \( \sup_{g \in G} T_\theta(f, g)(s_1) > 0 \), there exists \( g_f \in G \), which depends on \( f \), such that
\[
T_\theta(f, g_f)(s_1) = U(f, g_f)(s_1) - \theta(s_1)V(f, g_f)(s_1) > 0.
\] (2.7)
From (2.7), it follows that for all $f \in F$, $W(f, g_f)(s_1) = U(f, g_f)(s_1)/V(f, g_f)(s_1) > \theta(s_1)$. This shows that $\overline{\theta}(s_1) \geq \theta(s_1)$.

(4) Since $\theta(s_1) > \overline{\theta}(s_1)$, from the definition of $\overline{\theta}(s_1)$, there exists $\overline{f} \in F$ such that for all $g \in G$,

$$\theta(s_1) > \sup_{g \in G} W(\overline{f}, g)(s_1).$$

This shows that for all $g \in G$, $T_\theta(\overline{f}, g)(s_1) < 0$. Hence, we get that

$$0 \geq \sup_{g \in G} T_\theta(\overline{f}, g)(s_1) \geq \inf_{f \in F} \sup_{g \in G} \tau_\theta(f, g)(s_1) = \overline{T}_\theta(s_1).$$

(5) Since $\overline{\theta}(s_1) > \theta(s_1)$, from the definition of $\overline{\theta}(s_1)$, it follows that for all $f \in F$,

$$\sup_{g \in G} W(f, g)(s_1) > \theta(s_1).$$

Thus, there exists $g_f \in G$, which depends on $f$, such that $W(f, g_f)(s_1) > \theta(s_1)$, that is, for all $f \in F$,

$$\sup_{g \in G} T_\theta(f, g)(s_1) \geq T_\theta(f, g_f)(s_1) > 0.$$

Hence, we get that

$$\overline{T}_\theta(s_1) = \inf_{f \in F} \sup_{g \in G} T_\theta(f, g)(s_1) \geq 0.$$

\[\square\]

\textbf{Lemma 2.2} $T_\theta(s_1)$ has the following properties.

(1) If two parameter functions $\theta_1(s_1)$ and $\theta_2(s_1)$ satisfy that $\theta_1(s_1) > \theta_2(s_1) \geq 0$, it follows that

$$T_{\theta_1}(s_1) \leq T_{\theta_2}(s_1).$$

(2) If $T_\theta(s_1) < 0$, it holds that $\theta(s_1) \geq \overline{\theta}(s_1)$.

(3) If $T_\theta(s_1) > 0$, it holds that $\theta(s_1) \leq \underline{\theta}(s_1)$.

(4) If $\theta(s_1) > \overline{\theta}(s_1)$, it holds that $T_\theta(s_1) \leq 0$.

(5) If $\theta(s_1) < \underline{\theta}(s_1)$, it holds that $T_\theta(s_1) \geq 0$.

Proof. Using $T_\theta(s_1)$ and $\theta(s_1)$ instead of $\overline{T}_\theta(s_1)$ and $\overline{\theta}(s_1)$, respectively. we can prove this lemma by similar arguments to the previous one. \[\square\]

We have the following relations between the game $(DFG)$ and $(DPG_\theta)$.

\textbf{Theorem 2.1} Suppose that $g^* \in G$ is a max-$\inf$ of the game $(DFG)$. Then, it holds that

(1) $\overline{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1)$.
(2) If \( \overline{T}_{\theta^*}(s_1) \leq 0 \), \( g^* \) is a max-inf of the game \( (DPG_{\theta^*}) \).

Proof. (1) From the definition of \( \overline{\theta}(s_1) \) and \( \underline{\theta}(s_1) \), in general it holds that \( \overline{\theta}(s_1) \geq \underline{\theta}(s_1) \).

On the other hand, since \( g^* \in G \) is a max-inf of the game \( (DFG) \), it follows that

\[
\overline{\theta}(s_1) = \inf_{f \in F} W(f, g^*)(s_1)
\leq \sup_{g \in G} \inf_{f \in F} W(f, g)(s_1)
= \underline{\theta}(s_1).
\]

Thus, the game \( (DFG) \) has a value function, that is, \( \overline{\theta} = \underline{\theta} \) on \( S_1 \).

(2) Since \( g^* \in G \) is a max-inf of the game \( (DFG) \), it holds that for all \( f \in F \),

\[
\theta^*(s_1) = \inf_{f \in F} W(f, g^*)(s_1) \leq W(f, g^*)(s_1)
\]

that is, for all \( f \in F \),

\[
0 \leq T_{\theta^*}(f, g^*)(s_1) \leq \sup_{g \in G} T_{\theta^*}(f, g)(s_1).
\]

Thus, from (2.8) and (2) of the theorem, we get the following:

\[
0 \leq \inf_{f \in F} T_{\theta^*}(f, g^*)(s_1)
\leq \inf_{f \in F} \sup_{g \in G} T_{\theta^*}(f, g)(s_1)
= \overline{T}_{\theta^*}(s_1) \leq 0.
\]

This shows that

\[
\inf_{f \in F} T_{\theta^*}(f, g^*)(s_1) = \inf_{f \in F} \sup_{g \in G} T_{\theta^*}(f, g)(s_1).
\]

That is, \( g^* \) is a max-inf of the game \( (DPG_{\theta^*}) \).

Corollary 2.1 Suppose that \( (f^*, g^*) \in F \times G \) is a saddle point of the game \( (DFG) \). Then, it holds that

(1) \( T_{\theta^*}(f^*, g^*)(s_1) = 0 \).

(2) \( (f^*, g^*) \) is a saddle point of the game \( (DPG_{\theta^*}) \).

The proof of the corollary is easily given by Theorem 2.1.

Theorem 2.2 Under \( \overline{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1) \), suppose that \( g^* \in G \) is a max-inf of the game \( (DPG_{\theta^*}) \) and

\[
\inf_{f \in F} T_{\theta^*}(f, g^*)(s_1) = \overline{T}_{\theta^*}(s_1) \geq 0.
\]

Then, \( g^* \) is a max-inf of the game \( (DFG) \).
Proof. Since $\overline{\tau}\theta^{*}(s_1) \geq 0$ and $g^*$ is a max-inf of the game $(DPG_{\theta^*})$, it follows that
\[
0 \leq \inf_{f \in F} \sup_{g \in G} T_{\theta^*}(f, g)(s_1)
= \inf_{f \in F} T_{\theta^*}(f, g^*)(s_1)
\leq T_{\theta^*}(f, g^*)(s_1) \text{ for all } f \in F,
\]
which implies that for all $f \in F$,
\[
\theta^*(s_1) \leq W(f, g^*)(s_1) \leq \sup_{g \in G} W(f, g)(s_1).
\]
Therefore, we get that
\[
\theta^*(s_1) \leq \inf_{f \in F} W(f, g^*)(s_1)
\leq \inf_{f \in F} \sup_{g \in G} W(f, g)(s_1)
= \theta^*(s_1).
\]
This shows that $g^*$ is a max-inf of the game $(DFG)$. \qed

Corollary 2.2 Under $\overline{\theta}(s_1) = \theta(s_1) = \theta^*(s_1)$, suppose that $(f^*, g^*) \in F \times G$ is a saddle point of the game $(DPG_{\theta^*})$ and $T_{\theta^*}(f^*, g^*)(s_1) = 0$ holds. Then, $(f^*, g^*)$ is a saddle point of the game $(DFG)$.

The proof of the corollary is easily given by Theorem 2.2.

References