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Kyoto University
Complex Ruelle Operator
and
Hyperbolic Complex Dynamical Systems

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1. Decomposition of Complex Ruelle operator

Let $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic rational mapping. We assume that all the attractive periodic points of $R$ are fixed points, all the critical points of $R$ are non-degenerate, and that the Julia set of $R$, $J_{R}$, is included in $\mathbb{C}$. Let $N$ denote the number of attractive fixed points and let $a_{1}, \cdots, a_{N}$ denote the attractive fixed points. Let $A_{k}$ denote the attractive basin of $a_{k}$. Let $C_{R}$ denote the set of critical points of $R$.

For $k = 1, \cdots, N$, let $\gamma_{k}$ denote an oriented multicurve in $A_{k}$, such that $\gamma_{k} = \partial \Omega_{k}$, where $\Omega_{k}$ is an open set satisfying $R^{-1}(\Omega_{k}) \subset \Omega_{k}$, $\Omega_{k} \cup A_{k} = \overline{\mathbb{C}}$, and $C_{R} \cap \Omega_{k} \cap A_{k} = \phi$. Let $\gamma = \bigcup_{k=1}^{N} \gamma_{k}$ and $\Omega = \bigcap_{k=1}^{N} \Omega_{k}$.

For open set $O \subset \overline{\mathbb{C}}$, let $\mathcal{O}_{0}(O)$ denote the space of functions $g : O \to \mathbb{C}$ holomorphic in $O$ and has an analytic extension to a neighbourhood of the closure of $O$, and satisfies $g(\infty) = 0$ if $\infty$ belongs to the closure of $O$. We have the following decomposition of holomorphic functions. The direct sum in the theorem means the uniqueness of the decomposition.

**Theorem 1.1**

$$\mathcal{O}_{0}(\Omega) = \bigoplus_{k=1}^{N} \mathcal{O}_{0}(\Omega_{k}).$$

**Proof** Let $g \in \mathcal{O}_{0}(\Omega)$. Then $g$ can be expressed as

$$g(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(\tau)}{\tau - x} d\tau, \quad x \in \Omega.$$
For $k = 1, \cdots, N$, let $\Gamma_k : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\Omega_k)$ be defined by

$$(\Gamma_k g)(x) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{g(\tau)}{\tau - x} d\tau, \quad x \in \Omega_k.$$ 

As $g(\tau)$ is bounded on $\gamma_k$, $\Gamma_k g$ is holomorphic in $\Omega_k$ and vanishes at the infinity. Hence $\Gamma_k g \in \mathcal{O}_0(\Omega_k)$. As $\gamma = \cup_{k=1}^N \gamma_k$, we have the decomposition

$$g = \sum_{k=1}^N \Gamma_k g.$$ 

To prove the uniqueness of the decomposition, assume $g_k \in \mathcal{O}_0(\Omega_k)$ for $k = 1, \cdots, N$, and

$$\sum_{k=1}^N g_k = 0.$$ 

Then $g_k$ is holomorphic in $\Omega_k$ and at the same time it can be analytically extended to $A_k$, since $-g_k = \Sigma_{j \neq k} g_j$ is holomorphic in $A_k$. This shows that $g_k$ is constant for $k = 1, \cdots, N$. However, $g_k$ takes value zero at the infinity if the infinity belongs to the domain of its definition. Therefore, $g_k = 0$ for all $k = 1, \cdots, N$, except one. But the exceptional one must be zero since $\sum_{k=1}^N g_k = 0$.

**Theorem 1.2**

$$\Gamma_k : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\Omega_k), \quad k = 1, \cdots, N$$

are projections.

**Proof** For all $g \in \mathcal{O}_0(\Omega)$, $\Gamma_k g$ is holomorphic in $\Omega_k$, hence we have $\Gamma_k^2 g = \Gamma_k g$. If $j \neq k$, then $\gamma_j \subset A_j \subset \Omega_k$. Therefore, $\Gamma_j \Gamma_k g = 0$ for all $g \in \mathcal{O}_0(\Omega)$. As we saw in the previous theorem, $\sum_{k=1}^N \Gamma_k = \text{id}$.

**Definition 1.3** We define complex Ruelle operator $L : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\Omega)$ by

$$(Lg)(x) = \sum_{y \in R^{-1}(x)} \frac{g(y)}{(R'(y))^2}, \quad g \in \mathcal{O}_0(\Omega), \quad x \in \Omega.$$ 

Note that $R^{-1}(x) \subset \Omega$ and $R'(y) \neq 0$ as we assumed $R$ is hyperbolic and $\Omega$ contains no critical points. As indicated by [1], the complex Ruelle
operator can be expressed as an integral operator of the form:

$$(Lg)(x) = \frac{1}{2\pi i} \int \gamma \frac{g(\tau)}{R'(\tau)(R(\tau) - x)} d\tau.$$ 

This formula is easily verified by applying the Cauchy's theorem about residues and it shows that $Lg \in \mathcal{O}_0(\Omega)$. Comparing $L$ with the Perron-Frobenius operator, we see that the spectral radius of $L$ is smaller than 1.

**Definition 1.4**

$L_{ij} : \mathcal{O}_0(\Omega_j) \to \mathcal{O}_0(\Omega_i)$ is defined by $L_{ij} = \Gamma_i \circ L|_{\mathcal{O}_0(\Omega_j)}$.

The Ruelle operator can be expressed as an $N \times N$ matrix of operators:

$L = (L_{ij})$

The components $L_{ij}$ are computed as follows.

**Proposition 1.5** If $i \neq j$, then for $g_j \in \mathcal{O}_0(\Omega_j)$ and $x \in \Omega_i$,

$$(L_{ij}g_j)(x) = - \sum_{c \in \mathcal{C}_i \cap A_i} \frac{g_j(c)}{R''(c)(R(c) - x)}.$$

**Proof** As $g_j \in \mathcal{O}_0(\Omega_j)$ and $L_{ij}g_j$ is defined by

$$(L_{ij}g_j)(x) = \frac{1}{2\pi i} \int \gamma i \frac{g_j(\tau)}{R'(\tau)(R(\tau) - x)} d\tau,$$

we can apply the residue theorem to the complement of $\Omega_i$. The residues at the critical points in $A_i$ give the formula.

**Proposition 1.6** For $g_j \in \mathcal{O}_0(\Omega_j)$ and for $x \in \Omega_j$,

$$(L_{jj})g_j(x) = \sum_{y \in R^{-1}(x)} \frac{g_j(y)}{(R'(y))^2} + \sum_{c \in \mathcal{C}_r \cap \Omega_j} \frac{g_j(c)}{R''(c)(R(c) - x)}.$$

**Proof** In this case, we can apply the residue theorem to $\Omega_j$.

2. Möbius transformation and complex Ruelle operator
In this section, we observe the behavior of the complex Ruelle operator under a coordinate change of the Riemann sphere by a Möbius transformation.

Let $M : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a Möbius transformation of the Riemann sphere. Let $\alpha = M^{-1}(\infty)$, $\beta = M(\infty)$, and $\tilde{R} = M \circ R \circ M^{-1}$. We set $\tilde{\Omega}_k = M(\Omega_k)$, $\tilde{\Omega} = M(\Omega)$, and assume $\alpha \not\in \Omega$. In order to avoid confusion, we denote the complex Ruelle operator defined in the previous section by $L_R$ associated to the rational mapping $R$. Now, we define a "complex Ruelle operator" associated to the Möbius transformation $M$.

**Definition 2.1**

$L_M : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\tilde{\Omega})$ is defined by $(L_M g)(\tilde{x}) = \frac{g \circ M^{-1}(\tilde{x})}{(M'(M^{-1}(\tilde{x})))^2}$, for $g \in \mathcal{O}_0(\Omega)$ and $\tilde{x} \in \tilde{\Omega}$.

**Proposition 2.2**

$L_{M^{-1}} = L_M^{-1}$, $L_{\tilde{R}} = L_M \circ L_R \circ L_{M^{-1}}$.

**Proof**  First equality is easily verified by computing $L_{M^{-1}} \circ L_M$ and $L_M \circ L_{M^{-1}}$ directly. Second equality is easily verified similarly by the definition of the complex Ruelle operator. However, we would like to give a proof for the operator defined as an integral operator. Let $\tilde{g} \in \mathcal{O}_0(\tilde{\Omega})$. Then we have, for $x \in \Omega$ and $\tilde{x} \in \tilde{\Omega}$,

$$(L_{M^{-1}} \tilde{g})(x) = \frac{\tilde{g} \circ M(x)}{((M^{-1})'(M(x)))^2},$$

$$(L_R L_{M^{-1}} \tilde{g})(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{(L_{M^{-1}} \tilde{g})(\tau)}{R'(\tau)(R(\tau) - x)} d\tau$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g} \circ M(\tau)}{R'(\tau)(R(\tau) - x)((M^{-1})' \circ M(\tau))^2} d\tau,$$

and

$$(L_M L_R L_{M^{-1}} \tilde{g})(\tilde{x})$$

$$= \frac{1}{(M' \circ M^{-1}(\tilde{x}))^2} \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g} \circ M(\tau) d\tau}{R'(\tau)(R(\tau) - M^{-1}(\tilde{x}))((M^{-1})' \circ M(\tau))^2}.$$
\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g} \circ M(\mathcal{T})(M'(\mathcal{T}))^2 d\mathcal{T}}{(M' \circ M^{-1}(\tilde{x}))^2 R(\tau)(R(\tau) - M^{-1}(\tilde{x}))}.
\]
On the other hand, by a change of variables \( \sigma = M(\tau) \), we have
\[
(L_{\tilde{R}}\tilde{g})(\tilde{x}) = \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g}(\sigma)d\sigma}{\tilde{R}'(\sigma)(\tilde{R}(\sigma) - \tilde{x})}
\]
\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g} \circ M(\mathcal{T})M'(\mathcal{T})d_{\mathcal{T}}}{M'(R(\mathcal{T}))R'(\mathcal{T})(M^{-1})' \circ M(\mathcal{T})(M \circ R(\tau) - \tilde{x})}.
\]
Hence we obtain
\[
(L_{M}L_{R}L_{M^{-1}}\tilde{g})(\tilde{x}) - (L_{\tilde{R}}\tilde{g})(\tilde{x})
\]
\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g} \circ M(\mathcal{T})(M'(\mathcal{T}))^2}{R'(\mathcal{T})} \times
\]
\[
\times \left( \frac{1}{(M' \circ M^{-1}(\tilde{x}))^2 (R(\tau) - M^{-1}(\tilde{x}))} - \frac{1}{M' \circ R(\tau)(M \circ R(\tau) - \tilde{x})} \right) d\tau.
\]
As \( R'(\tau) \neq 0 \) and \( M' \circ R(\tau) \neq 0 \) for \( \tau \in \Omega \), the integrand can have poles only at \( \tau \in R^{-1} \circ M^{-1}(\tilde{x}) \cap \Omega \). The residues at such points are, by setting \( x = M^{-1}(\tilde{x}) \) and \( y = R^{-1}(x) \), computed as
\[
\frac{\tilde{g} \circ M(y)(M'(y))^2}{R'(y)} \left( \frac{1}{(M'(x))^2 R'(y)} - \frac{1}{M' \circ R(y)(M \circ R(y)R'(y))} \right) = 0.
\]
Hence the proposition follows.

**DEFINITION 2.3** Components \( L_{M,ij} : O_0(\Omega_j) \to O_0(\tilde{\Omega_i}) \) is defined by \( L_{M,ij}g_j = \tilde{\Gamma}_iL_Mg_j \) for \( g_j \in O_0(\Omega_j) \), where \( \tilde{\Gamma}_i : O_0(\tilde{\Omega}) \to O_0(\tilde{\Omega_i}) \) denote the projection.

**PROPOSITION 2.4** If \( \infty \in \Omega_j \), then
\[
(L_{M,ij}g_j)(\tilde{x}) = (L_Mg_j)(\tilde{x}) + \operatorname{Res}_{\tau=\infty} \frac{g_j(\tau)}{M'(\tau)(M(\tau) - \tilde{x})}.
\]
If \( \infty \notin \Omega_j \), then
\[
(L_{M,ij}g_j)(\tilde{x}) = (L_Mg_j)(\tilde{x}).
\]
If \( i \neq j \) and \( \infty \in \Omega_i \), then \( L_{M,ij} = 0 \).
If \( i \neq j \) and \( \infty \notin \Omega_i \), then
\[
(L_{M,ij}g_j)(\tilde{x}) = -\operatorname{Res}_{\tau=\infty} \frac{g_j(\tau)}{M'(\tau)(M(\tau) - \tilde{x})}.
\]
PROOF These formulas are easily verified by a direct computation by applying the residue theorem.

3. Partial complex Ruelle operator

In this section, we examine a diagonal component of the complex Ruelle operator. The Fredholm determinant and the resolvent of the adjoint diagonal component $L_{ii} : \mathcal{O}_0(\Omega_i) \to \mathcal{O}_0(\Omega_i)$ of the complex Ruelle operator can be computed in a similar manner as is given by [1] and [2].

For the sake of simplicity, we assume $a_1 = \infty$, and $a_1$ is an attractive fixed point with eigenvalue $\sigma$ satisfying $0 < |\sigma| < 1$. We define the partial Ruelle operator as follows.

DEFINITION 3.1 Partial Ruelle operator $L_{11} : \mathcal{O}_0(\Omega_1) \to \mathcal{O}_0(\Omega_1)$ is defined by

\[(L_{11}g)(x) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{g(\tau)d\tau}{R'(\tau)(R(\tau) - x)}.\]

An explicit formula for the partial Ruelle operator is given by proposition 1.6. We shall consider the dual operator.

DEFINITION 3.2 The dual space $\mathcal{O}_0^*(\Omega_1)$ of $\mathcal{O}_0(\Omega_1)$ is the space of continuous, complex linear, and holomorphic functional $F : \mathcal{O}_0(\Omega_1) \to \mathbb{C}$. The topology of $\mathcal{O}_0(\Omega_1)$ is understood as the uniform convergence in a neighborhood of the closure of $\Omega_1$. A functional is said to be holomorphic if the value $F[g_\mu]$ is holomorphic with respect to the parameter $\mu$ for a holomorphic family of functions $g_\mu$.

PROPOSITION 3.3 For any $F \in \mathcal{O}_0^*(\Omega_1)$, there exists an $f \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1)$, such that

\[F[g] = \frac{1}{2\pi i} \int_{\gamma_1} f(\tau)g(\tau)d\tau, \quad \text{for} \quad g \in \mathcal{O}_0(\Omega_1).\]

PROOF In fact, the so called Cauchy transform

\[f(z) = F[\frac{1}{z-\zeta}]\]

gives such a function. As $\frac{1}{z-\zeta}$ is a holomorphic family of functions in $\mathcal{O}_0(\Omega_1)$ parametrized by $z \in \overline{\mathbb{C}} \setminus \Omega_1$. $f(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus \Omega_1$ and
\( f(\infty) = 0 \), hence \( f \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1) \). For \( g \in \mathcal{O}_0(\Omega_1) \),
\[
F[g] = F[\frac{1}{2\pi i} \int_{\gamma} \frac{g(z)}{z-\zeta} \, dz] \\
= \frac{1}{2\pi i} \int_{\gamma} g(z) F[\frac{1}{z-\zeta}] \, dz = \frac{1}{2\pi i} \int_{\gamma} g(z) f(z) \, dz.
\]
Note that such function \( f(z) \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1) \) is unique since
\[
\frac{1}{2\pi i} \int_{\gamma} f(\tau) \frac{1}{z-\tau} \, d\tau = f(z).
\]

**Proposition 3.4** The dual operator \( L_{11}^* : \mathcal{O}_0^*(\Omega_1) \rightarrow \mathcal{O}_0^*(\Omega_1) \) is represented by integral operator \( \mathcal{L}_{11}^* : \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1) \rightarrow \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1) \) defined, for \( f \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1) \) and \( z \in \overline{\mathbb{C}} \setminus \Omega_1 \), by
\[
(\mathcal{L}_{11}^* f)(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(R(\tau)) \, d\tau}{R'(\tau)(z-\tau)} = \frac{f(R(z))}{R'(z)} - \sum_{c \in C \cap A_1} \frac{f(R(c))}{R''(c)(z-c)}.
\]

**Proof** The proof is almost same as in [1]. By a direct computation, we have
\[
(\mathcal{L}_{11}^* f)(z) = (L_{11}^* F)[\frac{1}{z-\zeta}] = F[L_{11}[\frac{1}{z-\zeta}]] \\
= F[\frac{1}{2\pi i} \int_{\gamma} \frac{f(R(\tau)) \, d\tau}{R'(\tau)(z-\tau)}] \\
= \frac{1}{2\pi i} \int_{\gamma} f(\zeta) \, d\zeta \frac{1}{2\pi i} \int_{\gamma} \frac{f(R(\tau)) \, d\tau}{R'(\tau)(z-\tau)} \\
= \frac{1}{2\pi i} \int_{\gamma} \frac{f(R(z)) \, d\tau}{R'(z)(z-\tau)} - \sum_{c \in C \cap A_1} \frac{f(R(c))}{R''(c)(z-c)}.
\]

Note that \( \mathcal{L}_{11}^* f \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1) \) and the poles at the critical points in the last line of the above calculation cancel out.

**4. Fredholm determinant of the adjoint Ruelle operator**

In this section, we compute the Fredholm determinant and the resolvent of the adjoint operator \( \mathcal{L}_{11}^* \). The calculation is almost same as in [1].
Let $n_1$ denote the number of critical points in $A_1$. And let $\{c_1, \cdots, c_{n_1}\}$ be the critical points in $A_1$. Let

$$H(x, z; \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(R^{\circ n})'(z)(R^{\circ n}(z) - x)},$$

and let

$$M(\lambda) = \left( \delta_{ij} + \frac{\lambda}{R''(c_i)} H(c_j, R(c_i); \lambda) \right)_{i,j=1}^{n_1}$$

be an $n_1 \times n_1$ matrix.

**Theorem 4.1** The Fredholm determinant $D_{11}(\lambda)$ of $\mathcal{L}_{11}^*$ is given by

$$D_{11}(\lambda) = \prod_{n=1}^{\infty} (1 - \sigma^{n+1} \lambda) \det M(\lambda).$$

It is meromorphic in $\mathbb{C}$ and holomorphic for $|\lambda| < |\sigma|^{-2}$. $\mathcal{L}_{11}^*$ has no essential spectrum.

This theorem follows immediately from proposition 4.3 below. We assume that the backward orbits of critical points do not intersect with the curve $\gamma_1$. Let

$$\Omega_R = \overline{\mathbb{C}} \setminus (\Omega_1 \cup \bigcup_{n=0}^{\infty} \{ z \in \Omega_1 \mid R'(R^{\circ n}(z)) = 0 \}).$$

Then $\mathcal{O}_0(\Omega_1) \subset \mathcal{O}_0(\Omega_R)$. Define $\mathcal{L}^*_R : \mathcal{O}_0(\Omega_R) \to \mathcal{O}_0(\Omega_R)$ by

$$(\mathcal{L}^*_R f)(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(R(\tau)) d\tau}{R'(\tau)(z - \tau)}$$

for $f \in \mathcal{O}_0(\Omega_R)$. We see immediately that the image of $\mathcal{L}^*_R$ is included in $\mathcal{O}_0(\Omega_1)$, and $\mathcal{L}^*_R$ and $\mathcal{L}^*_{11}$ coincide on $\mathcal{O}_0(\Omega_1)$. Therefore, $\mathcal{L}^*_R$ and $\mathcal{L}^*_{11}$ has the same spectrum. Define an operator $\mathcal{K} : \mathcal{O}_0(\Omega_R) \to \mathcal{O}_0(\Omega_R)$ by

$$(\mathcal{K} f)(z) = \frac{f(R(z))}{R'(z)}, \quad f \in \mathcal{O}_0(\Omega_R), z \in \Omega_R.$$ 

**Proposition 4.2** The spectrum of $\mathcal{K}$ is $\{\sigma^{n+1}\}_{n=1}^{\infty}$, and the Fredholm determinant is given by

$$\det(I - \lambda \mathcal{K}) = \prod_{n=1}^{\infty} (1 - \sigma^{n+1} \lambda).$$
The eigenfunction $f_n(z)$ for eigenvalue $\lambda^{-1} = \sigma^{n+1}$ is given by
\[ f_n(z) = \frac{1}{\varphi'(z)(\varphi(z))^n}, \]
where holomorphic function $\varphi : \Omega_R \to \overline{\mathbb{C}}$ is the Schröder's function of the form $\varphi(z) = z + a_0 + \frac{a_2}{z} + \cdots \frac{a_n}{z^n} + \cdots$ near the infinity and satisfying the Schröder's equation $\varphi(R(z)) = \sigma^{-1}\varphi(z)$.

**PROOF** The fact that $f_n(z)$ is an eigenfunction for eigenvalue $\sigma^{n+1}$ is immediately verified by using the Shröder's equation. Eigenfunction $f_n(z)$ can be extended to $\Omega_R$ by using the function equations
\[ \mathcal{K}f_n = \sigma^{n+1}f_n \quad \text{and} \quad (\mathcal{K}f_n)(z) = \frac{f_n(R(z))}{R'(z)}. \]
As the eigenfunctions $\{f_n\}_{n=1}^\infty$ form a complete basis of $\mathcal{O}_0(\Omega_R^\infty)$, where $\Omega_R^\infty$ denotes the connected component of $\Omega_R$ containing the infinity, and as the eigenfunctions are determined from a germ at the infinity of the eigenfunction by the function equation above, the Fredholm determinant is given by the formula in the proposition.

Define linear maps $\mathcal{G} : \mathcal{O}_0(\Omega_R) \to \mathbb{C}^{n_1}$ and $\mathcal{F} : \mathbb{C}^{n_1} \to \mathcal{O}_0(\Omega_R)$ by
\[ \mathcal{G}f = \left( \frac{f(R(c_j))}{R''(c_j)} \right)_{j=1}^{n_1}, \quad f \in \mathcal{O}_0(\Omega_R), \]
\[ \mathcal{F}\alpha = \sum_{j=1}^{n_1} \frac{\alpha_j}{z - c_j}, \quad \alpha = (\alpha_j) \in \mathbb{C}^{n_1}. \]
We have
\[ \mathcal{L}_R^* = \mathcal{K} - \mathcal{F}\mathcal{G}. \]

The Fredholm determinant of the adjoint Ruelle operator $\mathcal{L}_R^*$ is computed as follows.

**PROPOSITION 4.3**
\[ D_{11}(\lambda) = \det(I - \lambda \mathcal{L}_R^*) = \det(I - \lambda \mathcal{K}) \det M(\lambda), \]
where $M(\lambda) = I_{n_1} + \lambda \mathcal{G}(I - \lambda \mathcal{K})^{-1}\mathcal{F}$. 

The $n_1 \times n_1$ matrix $M(\lambda)$ is computed as follows.

\[
M(\lambda) = I_{n_1} + \lambda G(I - \lambda \mathcal{K})^{-1} F
\]

\[
= I_{n_1} + \lambda G \left( \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n \right) F
\]

\[
= I_{n_1} + \lambda \left( \frac{1}{R''(c_i)} \sum_{n=0}^{\infty} \frac{\lambda^n}{(R^{(n)}(z)(R^{(n)}(z)-c_i))} \right)_{i,j=1}^{n_1}
\]

\[
= \left( \delta_{ij} + \frac{\lambda}{R''(c_i)} H(c_j, R(c_i); \lambda) \right)_{i,j=1}^{n_1}.
\]

As the spectrum of $\mathcal{K}$ is \( \{\sigma^{n+1}\}_{n=1}^{\infty} \), $M(\lambda)$ is meromorphic in $\mathbb{C}$ and holomorphic in $\{\lambda \mid |\lambda| < \sigma^{-2}\}$. This completes the proof of the proposition 4.3 and the Theorem 4.1.

5. The resolvent of the partial adjoint Ruelle operator

The resolvent of the partial adjoint Ruelle operator $\mathcal{L}_{11}^*$ can be computed in an analogous manner as in [1] and [2]. First, we compute the resolvent function of operator $\mathcal{K}$.

PROPOSITION 5.1  The function $H(x, z; \lambda)$ defined in the previous section is the resolvent function of $\mathcal{K}$, i.e.,

\[
H(x, z; \lambda) = (I - \lambda \mathcal{K})^{-1} \frac{1}{z - x}
\]

\[
= \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n \frac{1}{z - x} = \sum_{n=0}^{\infty} \frac{\lambda^n}{(R^{(n)}(z)(R^{(n)}(z)-x))},
\]

where $x \in \Omega_1$, $z \in \Omega_R$ and $\lambda \in \mathbb{C} \setminus \{\sigma^{-k}\}_{k=2}^{\infty}$. $H(x, z; \lambda)$ is holomorphic in $x, z$ and $\lambda$.

PROOF  As is easily observed, we have

\[
\mathcal{K}^n f(z) = \frac{f(R^{(n)}(z))}{(R^{(n)}(z))'},
\]
Let $f_n : \Omega_R \to \mathbb{C}, n = 1, 2, \cdots$, be the complete system of eigenfunctions of $\mathcal{K}$ given by proposition 4.2. For each $x \in \Omega_1$, we can expand the function $(z-x)^{-1} \in \mathcal{O}_0(\Omega_R)$ in the form

$$\frac{1}{z-x} = \sum_{n=1}^{\infty} b_n(x) f_n(z).$$

Observe that $b_n(x)$ is holomorphic in $\Omega_1$. With this expression, we have

$$(I - \lambda \mathcal{K})^{-1} \frac{1}{z-x} = \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n \frac{1}{z-x} = \sum_{n=0}^{\infty} b_n(x) \sum_{k=1}^{\infty} \lambda^k f_k(z) = \sum_{k=1}^{\infty} b_k(x) \lambda^{k+1} f_k(z).$$

This shows that $H(x, z; \lambda)$ has an analytic extension to the domain $\Omega_1 \times \Omega_R \times (\mathbb{C} \setminus \{\sigma^{-k}\}_{k=2}^{\infty})$.

The resolvent function $E(x, z; \lambda)$ is defined by

$$E(x, z; \lambda) = (I - \lambda \mathcal{L}_{11}^*)^{-1} \frac{1}{z-x}, \quad x \in \Omega_1, z \in \overline{\mathbb{C}} \setminus \Omega_1, \lambda \in \mathbb{C}, E(x, \infty; \lambda) = 0.$$

$E(x, z; \lambda)$ is holomorphic in $x$ and $z$, and meromorphic in $\lambda$.

**Proposition 5.2**

$$(I - \lambda \mathcal{L}_{11}^*)^{-1} = (I - \lambda \mathcal{K})^{-1} - \lambda (I - \lambda \mathcal{K})^{-1} \mathcal{F}(M(\lambda))^{-1} \mathcal{G}(I - \lambda \mathcal{K})^{-1},$$

where $M(\lambda) = I_{n_1} + \lambda \mathcal{G}(I - \lambda \mathcal{K})^{-1} \mathcal{F}$ is an $n_1 \times n_1$ matrice.

**Proof**  By a direct computation.

$$(I - \lambda \mathcal{L}_{11}^*)^{-1} = (I - \lambda \mathcal{K} + \lambda \mathcal{F} \mathcal{G})^{-1} = (I - \lambda \mathcal{K})^{-1} (I + \lambda \mathcal{F} \mathcal{G}(I - \lambda \mathcal{K})^{-1})^{-1} = (I - \lambda \mathcal{K})^{-1} (I + \sum_{k=1}^{\infty} (-\lambda)^k \mathcal{F} \mathcal{G}(I - \lambda \mathcal{K})^{-1})^{-1} = (I - \lambda \mathcal{K})^{-1} (I - \lambda \mathcal{F} \sum_{k=0}^{\infty} (-\lambda \mathcal{G}(I - \lambda \mathcal{K})^{-1} \mathcal{G}(I - \lambda \mathcal{K})^{-1})^{-1} = (I - \lambda \mathcal{K})^{-1} - \lambda (I - \lambda \mathcal{K})^{-1} \mathcal{F}(M(\lambda))^{-1} \mathcal{G}(I - \lambda \mathcal{K})^{-1}.$$

As is easily verified, $M(\lambda)$ in this proposition is same as $M(\lambda)$ given in the beginning of the previous section.

Let

$$H_1(z; \lambda) = ((I - \lambda \mathcal{K})^{-1} \frac{1}{z-c_i})_{i=1}^{n_1} = (H(c_i, z; \lambda))_{i=1}^{n_1}$$
be a row vector and let
\[ H_2(x; \lambda) = G(I - \lambda K)^{-1} \frac{1}{z-x} = GH(x, z; \lambda) = \left( \frac{H(x, R(c_i); \lambda)}{R''(c_i)} \right)_{i=1}^{n_1} \]
be a column vector. With all these things together, we find the explicit expression of the resolvent function \( E(x, z; \lambda) \).

**Theorem 5.3**
\[ E(x, z; \lambda) = H(x, z; \lambda) + \lambda H_1(z; \lambda)(M(\lambda))^{-1}H_2(x; \lambda). \]
The resolvent function is holomorphic in \( x \in \Omega_1 \), and in \( z \in \Omega_R \), and meromorphic in \( \lambda \in \mathbb{C} \). The poles are the zeros of the Fredholm determinant \( D_{11}(\lambda) \).

**References**

