Complex Ruelle Operator
and
Hyperbolic Complex Dynamical Systems

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1. Decomposition of Complex Ruelle operator

Let $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic rational mapping. We assume that all the attractive periodic points of $R$ are fixed points, all the critical points of $R$ are non-degenerate, and that the Julia set of $R$, $J_R$, is inculed in $\mathbb{C}$. Let $N$ denote the number of attractive fixed points and let $a_1, \cdots, a_N$ denote the attractive fixed points. Let $A_k$ denote the attractive basin of $a_k$. Let $C_R$ denote the set of critical points of $R$.

For $k = 1, \cdots, N$, let $\gamma_k$ denote an oriented multicurve in $A_k$, such that $\gamma_k = \partial \Omega_k$, where $\Omega_k$ is an open set satisfying $R^{-1}(\Omega_k) \subset \Omega_k$, $\Omega_k \cup A_k = \overline{\mathbb{C}}$, and $C_R \cap \Omega_k \cap A_k = \phi$. Let $\gamma = \bigcup_{k=1}^{N} \gamma_k$ and $\Omega = \bigcap_{k=1}^{N} \Omega_k$.

For open set $O \subset \overline{\mathbb{C}}$, let $\mathcal{O}_0(O)$ denote the space of functions $g : O \to \mathbb{C}$ holomorphic in $O$ and has an analytic extension to a neighbourhood of the closure of $O$, and satisfies $g(\infty) = 0$ if $\infty$ belongs to the closure of $O$. We have the following decomposition of holomorphic functions. The direct sum in the theorem means the uniqueness of the decomposition.

**Theorem 1.1**

$$\mathcal{O}_0(\Omega) = \bigoplus_{k=1}^{N} \mathcal{O}_0(\Omega_k).$$

**Proof** Let $g \in \mathcal{O}_0(\Omega)$. Then $g$ can be expressed as

$$g(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(\tau)}{\tau - x} d\tau, \quad x \in \Omega.$$
For \( k = 1, \cdots, N \), let \( \Gamma_k : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\Omega_k) \) be defined by

\[
(\Gamma_k g)(x) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{g(\tau)}{\tau - x} d\tau, \quad x \in \Omega_k.
\]

As \( g(\tau) \) is bounded on \( \gamma_k \), \( \Gamma_k g \) is holomorphic in \( \Omega_k \) and vanishes at the infinity. Hence \( \Gamma_k g \in \mathcal{O}_0(\Omega_k) \). As \( \gamma = \bigcup_{k=1}^{N} \gamma_k \), we have the decomposition

\[
g = \sum_{k=1}^{N} \Gamma_k g.
\]

To prove the uniqueness of the decomposition, assume \( g_k \in \mathcal{O}_0(\Omega_k) \) for \( k = 1, \cdots, N \), and

\[
\sum_{k=1}^{N} g_k = 0.
\]

Then \( g_k \) is holomorphic in \( \Omega_k \) and at the same time it can be analytically extended to \( A_k \), since \( -g_k = \sum_{j \neq k} g_j \) is holomorphic in \( A_k \). This shows that \( g_k \) is constant for \( k = 1, \cdots, N \). However, \( g_k \) takes value zero at the infinity if the infinity belongs to the domain of its definition. Therefore, \( g_k = 0 \) for all \( k = 1, \cdots, N \), except one. But the exceptional one must be zero since \( \sum_{k=1}^{N} g_k = 0 \).

**Theorem 1.2**

\[
\Gamma_k : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\Omega_k), \quad k = 1, \cdots, N
\]

are projections.

**Proof** For all \( g \in \mathcal{O}_0(\Omega) \), \( \Gamma_k g \) is holomorphic in \( \Omega_k \), hence we have \( \Gamma_k^2 g = \Gamma_k g \). If \( j \neq k \), then \( \gamma_j \subset A_j \subset \Omega_k \). Therefore, \( \Gamma_j \Gamma_k g = 0 \) for all \( g \in \mathcal{O}_0(\Omega) \). As we saw in the previous theorem, \( \sum_{k=1}^{N} \Gamma_k = \text{id} \).

**Definition 1.3** We define complex Ruelle operator \( L : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\Omega) \) by

\[
(Lg)(x) = \sum_{y \in R^{-1}(x)} \frac{g(y)}{(R'(y))^2}, \quad g \in \mathcal{O}_0(\Omega), \quad x \in \Omega.
\]

Note that \( R^{-1}(x) \subset \Omega \) and \( R'(y) \neq 0 \) as we assumed \( R \) is hyperbolic and \( \Omega \) contains no critical points. As indicated by [1], the complex Ruelle
operator can be expressed as an integral operator of the form:

$$(Lg)(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(\tau)}{R'(\tau)(R(\tau) - x)} d\tau.$$  

This formula is easily verified by applying the Cauchy’s theorem about residues and it shows that $Lg \in O_0(\Omega)$. Comparing $L$ with the Perron-Frobenius operator, we see that the spectral radius of $L$ is smaller than 1.

**Definition 1.4**

$L_{ij} : O_0(\Omega_j) \rightarrow O_0(\Omega_i)$ is defined by $L_{ij} = \Gamma_i \circ L |_{O_0(\Omega_j)}$.

The Ruelle operator can be expressed as an $N \times N$ matrice of operators:

$L = (L_{ij})$

The components $L_{ij}$ are computed as follows.

**Proposition 1.5** If $i \neq j$, then for $g_j \in O_0(\Omega_j)$ and $x \in \Omega_i$,

$$(L_{ij}g_j)(x) = - \sum_{c \in \mathcal{C}_R \cap A_i \cap \Omega_j} \frac{g_j(c)}{R''(c)(R(c) - x)}.$$  

**Proof** As $g_j \in O_0(\Omega_j)$ and $L_{ij}g_j$ is defined by

$$(L_{ij}g_j)(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{g_j(\tau)}{R'(\tau)(R(\tau) - x)} d\tau,$$

we can apply the residue theorem to the complement of $\Omega_i$. The residues at the critical points in $A_i$ give the formula.

**Proposition 1.6** For $g_j \in O_0(\Omega_j)$ and for $x \in \Omega_j$,

$$(L_{jj})g_j(x) = \sum_{y \in R^{-1}(x)} \frac{g_j(y)}{(R'(y))^2} + \sum_{c \in \mathcal{C}_R \cap \Omega_j} \frac{g_j(c)}{R''(c)(R(c) - x)}.$$  

**Proof** In this case, we can apply the residue theorem to $\Omega_j$.

2. Möbius transformation and complex Ruelle operator
In this section, we observe the behavior of the complex Ruelle operator under a coordinate change of the Riemann sphere by a Möbius transformation.

Let $M : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a Möbius transformation of the Riemann sphere. Let $\alpha = M^{-1}(\infty), \beta = M(\infty)$, and $\tilde{R} = M \circ R \circ M^{-1}$. We set $\tilde{\Omega}_k = M(\Omega_k), \tilde{\Omega} = M(\Omega)$, and assume $\alpha \not\in \Omega$. In order to avoid confusion, we denote the complex Ruelle operator defined in the previous section by $L_R$ associated to the rational mapping $R$. Now, we define a "complex Ruelle operator" associated to the Möbius transformation $M$.

**Definition 2.1**

$L_M : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\tilde{\Omega})$ is defined by $(L_M g)(\tilde{x}) = \frac{g \circ M^{-1}(\tilde{x})}{(M'(M^{-1}(\tilde{x})))^2}$, for $g \in \mathcal{O}_0(\Omega)$ and $\tilde{x} \in \tilde{\Omega}$.

**Proposition 2.2**

$L_{M^{-1}} = L_M^{-1}$, $L_{\tilde{R}} = L_M \circ L_R \circ L_{M^{-1}}$.

**Proof** First equality is easily verified by computing $L_{M^{-1}} \circ L_M$ and $L_M \circ L_{M^{-1}}$ directly. Second equality is easily verified similarly by the definition of the complex Ruelle operator. However, we would like to give a proof for the operator defined as an integral operator. Let $\tilde{g} \in \mathcal{O}_0(\tilde{\Omega})$. Then we have, for $x \in \Omega$ and $\tilde{x} \in \tilde{\Omega},$

$$(L_{M^{-1} \tilde{g}})(x) = \frac{\tilde{g} \circ M(x)}{((M^{-1})'(M(x)))^2},$$

$$(L_R L_{M^{-1} \tilde{g}})(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{(L_{M^{-1} \tilde{g}})(\tau)}{R'(\tau)(R(\tau) - x)} d\tau = \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g} \circ M(\tau)}{R'(\tau)(R(\tau) - x)((M^{-1})' \circ M(\tau))^2} d\tau,$$

and

$$(L_M L_R L_{M^{-1} \tilde{g}})(\tilde{x}) = \frac{1}{(M' \circ M^{-1}(\tilde{x}))^2} \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g} \circ M(\tau) d\tau}{R'(\tau)(R(\tau) - M^{-1}(\tilde{x}))((M^{-1})' \circ M(\tau))^2}.$$
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g} \circ M(\mathcal{T})(M'(\mathcal{T}))^2 d\mathcal{T}}{(M' \circ M^{-1}(\tilde{x}))^2 R(\tau)(R(\tau) - M^{-1}(\tilde{x}))}
\]

On the other hand, by a change of variables $\sigma = M(\tau)$, we have

\[
(L_{\tilde{R}} \tilde{g})(\tilde{x}) = \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g}(\sigma) d\sigma}{\tilde{R}'(\sigma)(\tilde{R}(\sigma) - \tilde{x})}
\]

\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g} \circ M(\mathcal{T})M'(\mathcal{T}) d\mathcal{T}}{M'(R(\mathcal{T}))(R(\mathcal{T}) - M^{-1}(\tilde{x}))}
\]

Hence we obtain

\[
(L_M L_R L_{M^{-1}} \tilde{g})(\tilde{x}) - (L_{\tilde{R}} \tilde{g})(\tilde{x})
\]

\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g} \circ M(\mathcal{T})(M'(\mathcal{T}))^2}{R(\tau)} \times 
\]

\[
\left( \frac{1}{(M' \circ M^{-1}(\tilde{x}))^2 (R(\tau) - M^{-1}(\tilde{x}))} - \frac{1}{M' \circ R(\tau)(M \circ R(\tau) - \tilde{x})} \right) d\tau.
\]

As $R'(\tau) \neq 0$ and $M' \circ R(\tau) \neq 0$ for $\tau \in \Omega$, the integrand can have poles only at $\tau \in R^{-1} \circ M^{-1}(\tilde{x}) \cap \Omega$. The residues at such points are, by setting $x = M^{-1}(\tilde{x})$ and $y \in R^{-1}(x)$, computed as

\[
\frac{\tilde{g} \circ M(y)(M'(y))^2}{R'(y)} \left( \frac{1}{(M'(x))^2 R'(y)} - \frac{1}{M' \circ R(y)(M' \circ R(y) R'(y))} \right) = 0.
\]

Hence the proposition follows.

**Definition 2.3** Components $L_{M,ij} : \mathcal{O}_0(\Omega_j) \to \mathcal{O}_0(\tilde{\Omega}_i)$ is defined by $L_{M,ij} g_j = \tilde{\Gamma}_i L_M g_j$ for $g_j \in \mathcal{O}_0(\Omega_j)$, where $\tilde{\Gamma}_i : \mathcal{O}_0(\tilde{\Omega}) \to \mathcal{O}_0(\tilde{\Omega}_i)$ denote the projection.

**Proposition 2.4** If $\infty \in \Omega_j$, then

\[
(L_{M,ij} g_j)(\tilde{x}) = (L_M g_j)(\tilde{x}) + \text{Res}_{\tau=\infty} \frac{g_j(\tau)}{M'(\tau)(M(\tau) - \tilde{x})}.
\]

If $\infty \notin \Omega_j$, then

\[
(L_{M,ij} g_j)(\tilde{x}) = (L_M g_j)(\tilde{x}).
\]

If $i \neq j$ and $\infty \in \Omega_i$, then

\[
L_{M,ij} = 0.
\]

If $i \neq j$ and $\infty \notin \Omega_i$, then

\[
(L_{M,ij} g_j)(\tilde{x}) = -\text{Res}_{\tau=\infty} \frac{g_j(\tau)}{M'(\tau)(M(\tau) - \tilde{x})}.
\]
These formulas are easily verified by a direct computation by applying the residue theorem.

3. Partial complex Ruelle operator

In this section, we examine a diagonal component of the complex Ruelle operator. The Fredholm determinant and the resolvent of the adjoint diagonal component $L_{ii} : \mathcal{O}_0(\Omega_i) \to \mathcal{O}_0(\Omega_i)$ of the complex Ruelle operator can be computed in a similar manner as is given by [1] and [2].

For the sake of simplicity, we assume $a_1 = \infty$, and $a_1$ is an attractive fixed point with eigenvalue $\sigma$ satisfying $0 < |\sigma| < 1$. We define the partial Ruelle operator as follows.

**Definition 3.1** Partial Ruelle operator $L_{11} : \mathcal{O}_0(\Omega_1) \to \mathcal{O}_0(\Omega_1)$ is defined by

$$(L_{11} g)(x) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{g(\tau)d\tau}{R'(\tau)(R(\tau) - x)}.$$ 

An explicit formula for the partial Ruelle operator is given by proposition 1.6. We shall consider the dual operator.

**Definition 3.2** The dual space $\mathcal{O}_0^*(\Omega_1)$ of $\mathcal{O}_0(\Omega_1)$ is the space of continuous, complex linear, and holomorphic functional $F : \mathcal{O}_0(\Omega_1) \to \mathbb{C}$. The topology of $\mathcal{O}_0(\Omega_1)$ is understood as the uniform convergence in a neighborhood of the closure of $\Omega_1$. A functional is said to be holomorphic if the value $F[g_\mu]$ is holomorphic with respect to the parameter $\mu$ for a holomorphic family of functions $g_\mu$.

**Proposition 3.3** For any $F \in \mathcal{O}_0^*(\Omega_1)$, there exists an $f \in \mathcal{O}_0(\mathbb{C} \setminus \Omega_1)$, such that

$$F[g] = \frac{1}{2\pi i} \int_{\gamma_1} f(\tau)g(\tau)d\tau, \quad \text{for} \quad g \in \mathcal{O}_0(\Omega_1).$$

**Proof** In fact, the so called Cauchy transform

$$f(z) = F[\frac{1}{z - \zeta}]$$

gives such a function. As $\frac{1}{z - \zeta}$ is a holomorphic family of functions in $\mathcal{O}_0(\Omega_1)$ parametrized by $z \in \mathbb{C} \setminus \Omega_1$. $f(z)$ is holomorphic in $\mathbb{C} \setminus \Omega_1$ and
$f(\infty) = 0$, hence $f \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1)$. For $g \in \mathcal{O}_0(\Omega_1)$,

$$F[g] = F\left[ \frac{1}{2\pi i} \int_{\gamma} \frac{g(z)}{z - \zeta} dz \right]$$

$$= \frac{1}{2\pi i} \int_{\gamma} g(z) F \left[ \frac{1}{z - \zeta} \right] dz = \frac{1}{2\pi i} \int_{\gamma} g(z) f(z) dz.$$ 

Note that such function $f(z) \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1)$ is unique since

$$\frac{1}{2\pi i} \int_{\gamma} f(\tau) \frac{1}{z - \tau} d\tau = f(z).$$

**Proposition 3.4** The dual operator $L_{11}^* : \mathcal{O}_0^*(\Omega_1) \rightarrow \mathcal{O}_0^*(\Omega_1)$ is represented by integral operator $\mathcal{L}_{11}^* : \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1) \rightarrow \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1)$ defined, for $f \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1)$ and $z \in \overline{\mathbb{C}} \setminus \Omega_1$, by

$$(\mathcal{L}_{11}^* f)(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(R(\tau))}{R'(\tau)(R(\tau) - z)} d\tau$$

$$= \frac{f(R(z))}{R'(z)} - \sum_{c \in C_R \cap A} \frac{f(R(c))}{R''(c)(z - c)}.$$ 

**Proof** The proof is almost same as in [1]. By a direct computation, we have

$$(\mathcal{L}_{11}^* f)(z) = (L_{11}^* F)[\frac{1}{z - \zeta}] = F[L_{11} \left[ \frac{1}{z - \zeta} \right]]$$

$$= F\left[ \frac{1}{2\pi i} \int_{\gamma} R'(\tau)(R(\tau) - \zeta)(z - \tau) d\tau \right]$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{R'(\tau)(R(\tau) - \zeta)(z - \tau)} d\tau$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\tau}{R'(\tau)(R(\tau) - \zeta)} - \sum_{c \in C_R \cap A} \frac{f(R(c))}{R''(c)(z - c)}.$$ 

Note that $\mathcal{L}_{11}^* f \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1)$ and the poles at the critical points in the last line of the above calculation cancel out.

**4. Fredholm determinant of the adjoint Ruelle operator**

In this section, we compute the Fredholm determinant and the resolvent of the adjoint operator $\mathcal{L}_{11}^*$. The calculation is almost same as in [1].
Let $n_1$ denote the number of critical points in $A_1$. And let $\{c_1, \cdots , c_{n_1}\}$ be the critical points in $A_1$. Let

$$H(x, z; \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(R^{\text{on}})'(z)(R^{\text{on}}(z) - x)},$$

and let

$$M(\lambda) = \left( \delta_{ij} + \frac{\lambda}{R''(c_i)} H(c_j, R(c_i); \lambda) \right)_{i,j=1}^{n_1}$$

be an $n_1 \times n_1$ matrice.

**Theorem 4.1** The Fredholm determinant $D_{11}(\lambda)$ of $\mathcal{L}_{11}^*$ is given by

$$D_{11}(\lambda) = \prod_{n=1}^{\infty} (1 - \sigma^{n+1} \lambda) \det M(\lambda).$$

It is meromorphic in $\mathbb{C}$ and holomorphic for $|\lambda| < |\sigma|^{-2}$. $\mathcal{L}_{11}^*$ has no essential spectrum.

This theorem follows immediately from proposition 4.3 below. We assume that the backward orbits of critical points do not intersect with the curve $\gamma_1$. Let

$$\Omega_R = \overline{\mathbb{C}} \setminus (\Omega_1 \cup \bigcup_{n=0}^{\infty} \{z \in \Omega_1 \mid R'(R^{on}(z)) = 0\}).$$

Then $\mathcal{O}_0(\Omega_1) \subset \mathcal{O}_0(\Omega_R)$. Define $\mathcal{L}_R^* : \mathcal{O}_0(\Omega_R) \rightarrow \mathcal{O}_0(\Omega_R)$ by

$$(\mathcal{L}_R^* f)(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(R(\tau))}{R'(\tau)(z - \tau)} d\tau$$

for $f \in \mathcal{O}_0(\Omega_R)$. We see immediately that the image of $\mathcal{L}_R^*$ is included in $\mathcal{O}_0(\Omega_1)$, and $\mathcal{L}_R^*$ and $\mathcal{L}_{11}^*$ coincide on $\mathcal{O}_0(\Omega_1)$. Therefore, $\mathcal{L}_R^*$ and $\mathcal{L}_{11}^*$ has the same spectrum. Define a operator $\mathcal{K} : \mathcal{O}_0(\Omega_R) \rightarrow \mathcal{O}_0(\Omega_R)$ by

$$(\mathcal{K} f)(z) = \frac{f(R(z))}{R'(z)}, \quad f \in \mathcal{O}_0(\Omega_R), z \in \Omega_R.$$

**Proposition 4.2** The spectrum of $\mathcal{K}$ is $\{\sigma^{n+1}\}_{n=1}^{\infty}$, and the Fredholm determinant is given by

$$\det(I - \lambda \mathcal{K}) = \prod_{n=1}^{\infty} (1 - \sigma^{n+1} \lambda).$$
The eigenfunction $f_n(z)$ for eigenvalue $\lambda^{-1} = \sigma^{n+1}$ is given by

$$f_n(z) = \frac{1}{\varphi'(z)(\varphi(z))^n},$$

where holomorphic function $\varphi : \Omega_R \rightarrow \overline{\mathbb{C}}$ is the Schröder's function of the form $\varphi(z) = z + a_0 + \frac{a_2}{z} + \cdots + \frac{a_n}{z^n} + \cdots$ near the infinity and satisfying the Schröder's equation $\varphi(R(z)) = \sigma^{-1}\varphi(z)$.

**Proof** The fact that $f_n(z)$ is an eigenfunction for eigenvalue $\sigma^{n+1}$ is immediately verified by using the Shröder's equation. Eigenfunction $f_n(z)$ can be extended to $\Omega_R$ by using the function equations

$$\mathcal{K}f_n = \sigma^{n+1}f_n \quad \text{and} \quad (\mathcal{K}f_n)(z) = \frac{f_n(R(z))}{R'(z)}.$$

As the eigenfunctions $\{f_n\}_{n=1}^{\infty}$ form a complete basis of $O_0(\Omega_R^\infty)$, where $\Omega_R^\infty$ denotes the connected component of $\Omega_R$ containing the infinity, and as the eigenfunctions are determined from a germ at the infinity of the eigenfunction by the function equation above, the Fredholm determinant is given by the formula in the proposition.

Define linear maps $\mathcal{G} : O_0(\Omega_R) \rightarrow \mathbb{C}^{n_1}$ and $\mathcal{F} : \mathbb{C}^{n_1} \rightarrow O_0(\Omega_R)$ by

$$\mathcal{G}f = \left(\frac{f(R(c_j))}{R'(c_j)}\right)_{j=1}^{n_1}, \quad f \in O_0(\Omega_R),$$

$$\mathcal{F}\alpha = \sum_{j=1}^{n_1} \frac{\alpha_j}{z-c_j}, \quad \alpha = (\alpha_j) \in \mathbb{C}^{n_1}.$$

We have

$$\mathcal{L}_R^* = \mathcal{K} - \mathcal{F}\mathcal{G}.$$

The Fredholm determinant of the adjoint Ruelle operator $\mathcal{L}_R^*$ is computed as follows.

**Proposition 4.3**

$$D_{11}(\lambda) = \det(I - \lambda \mathcal{L}_R^*) = \det(I - \lambda \mathcal{K}) \det M(\lambda),$$

where $M(\lambda) = I_{n_1} + \lambda \mathcal{G}(I - \lambda \mathcal{K})^{-1}\mathcal{F}$. 
\[ \det(I - \lambda \mathcal{L}^*_R) = \det(I - \lambda \mathcal{K} + \lambda \mathcal{F}\mathcal{G}) \]
\[ = \det(I - \lambda \mathcal{K}) \det(I + \lambda(I - \lambda \mathcal{K})^{-1} \mathcal{F}\mathcal{G}) \]
\[ = \det(I - \lambda \mathcal{K}) \det(I_{n_1} + \lambda \mathcal{G}(I - \lambda \mathcal{K})^{-1} \mathcal{F}) \]
\[ = \det(I - \lambda \mathcal{K}) \det M(\lambda). \]

The \( n_1 \times n_1 \) matrice \( M(\lambda) \) is computed as follows.

\[ M(\lambda) = I_{n_1} + \lambda \mathcal{G}(I - \lambda \mathcal{K})^{-1} \mathcal{F} \]
\[ = I_{n_1} + \lambda \mathcal{G}(\sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n) \mathcal{F} \]
\[ = I_{n_1} + \lambda \left( \sum_{n=0}^{\infty} \frac{\lambda^n}{(R^{\circ n})'(z)(R^{\circ n}(z) - c_i)} \right)_{j=1}^{n_1} \]
\[ = I_{n_1} + \lambda \left( \sum_{n=0}^{\infty} \frac{\lambda^n}{(R^{\circ n})'(c_i)(R^{\circ n}(c_i) - c_j)} \right)_{i,j=1}^{n_1} \]
\[ = \left( \delta_{ij} + \frac{\lambda}{R''(c_i)} H(c_j, R(c_i); \lambda) \right)_{i,j=1}^{n_1}. \]

As the spectrum of \( \mathcal{K} \) is \( \{\sigma^{n+1}\}_{n=1}^{\infty} \), \( M(\lambda) \) is meromorphic in \( \mathbb{C} \) and holomorphic in \( \{\lambda \mid |\lambda| < \sigma^{-2}\} \). This completes the proof of the proposition 4.3 and the Theorem 4.1.

5. The resolvent of the partial adjoint Ruelle operator

The resolvent of the partial adjoint Ruelle operator \( \mathcal{L}_{11}^* \) can be computed in an analogous manner as in [1] and [2]. First, we compute the resolvent function of operator \( \mathcal{K} \).

**Proposition 5.1** The function \( H(x, z; \lambda) \) defined in the previous section is the resolvent function of \( \mathcal{K} \), i.e.,

\[ H(x, z; \lambda) = (I - \lambda \mathcal{K})^{-1} \frac{1}{z - x} \]
\[ = \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n \frac{1}{z - x} = \sum_{n=0}^{\infty} \lambda^n \frac{1}{(R^{\circ n})'(z)(R^{\circ n}(z) - x)}, \]

where \( x \in \Omega_1, z \in \Omega_R \) and \( \lambda \in \mathbb{C} \setminus \{\sigma^{-k}\}_{k=2}^{\infty} \). \( H(x, z; \lambda) \) is holomorphic in \( x, z \) and \( \lambda \).

**Proof** As is easily observed, we have

\[ \mathcal{K}^n f(z) = \frac{f(R^{\circ n}(z))}{(R^{\circ n})'(z)}. \]
Let \( f_n : \Omega_R \to \mathbb{C}, n = 1, 2, \ldots \), be the complete system of eigenfunctions of \( \mathcal{K} \) given by proposition 4.2. For each \( x \in \Omega_1 \), we can expand the function \((z - x)^{-1} \in \mathcal{O}_0(\Omega_R)\) in the form

\[
\frac{1}{z - x} = \sum_{n=1}^{\infty} b_n(x) f_n(z).
\]

Observe that \( b_n(x) \) is holomorphic in \( \Omega_1 \). With this expression, we have

\[
(I - \lambda \mathcal{K})^{-1} \frac{1}{z - x} = \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n \frac{1}{z - x}
\]

\[
= \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n \sum_{k=1}^{\infty} b_k(x) f_k(z) = \sum_{k=1}^{\infty} b_k(x) \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n f_k(z)
\]

\[
= \sum_{k=1}^{\infty} b_k(x) \sum_{n=0}^{\infty} (\lambda \sigma^{k+1})^n f_k(z) = \sum_{k=1}^{\infty} b_k(x) \frac{1}{1 - \lambda \sigma^{k+1}} f_k(z).
\]

This shows that \( H(x, z; \lambda) \) has an analytic extension to the domain \( \Omega_1 \times \Omega_R \times (\mathbb{C} \setminus \{\sigma^{-k}\}_{k=2}^{\infty}) \).

The resolvent function \( E(x, z; \lambda) \) is defined by

\[
E(x, z; \lambda) = (I - \lambda \mathcal{L}_{11}^{*})^{-1} \frac{1}{z - x}, \quad x \in \Omega_1, z \in \overline{\mathbb{C}} \setminus \Omega_1, \lambda \in \mathbb{C}, E(x, \infty; \lambda) = 0.
\]

\( E(x, z; \lambda) \) is holomorphic in \( x \) and \( z \), and meromorphic in \( \lambda \).

**Proposition 5.2**

\[
(I - \lambda \mathcal{L}_{11}^{*})^{-1} = (I - \lambda \mathcal{K})^{-1} - \lambda (I - \lambda \mathcal{K})^{-1} \mathcal{F}(M(\lambda))^{-1} \mathcal{G}(I - \lambda \mathcal{K})^{-1},
\]

where \( M(\lambda) = I_{n_1} + \lambda \mathcal{G}(I - \lambda \mathcal{K})^{-1} \mathcal{F} \) is an \( n_1 \times n_1 \) matrice.

**Proof**

By a direct computation.

\[
(I - \lambda \mathcal{L}_{11}^{*})^{-1} = (I - \lambda \mathcal{K} + \lambda \mathcal{F} \mathcal{G})^{-1}
\]

\[
= (I - \lambda \mathcal{K})^{-1} (I + \lambda \mathcal{F} \mathcal{G}(I - \lambda \mathcal{K})^{-1})^{-1}
\]

\[
= (I - \lambda \mathcal{K})^{-1} (I + \sum_{k=1}^{\infty} (-\lambda)^k (\mathcal{F} \mathcal{G}(I - \lambda \mathcal{K})^{-1})^k)
\]

\[
= (I - \lambda \mathcal{K})^{-1} (I - \lambda \mathcal{F} \sum_{k=1}^{\infty} (-\lambda \mathcal{G}(I - \lambda \mathcal{K})^{-1})^k \mathcal{G}(I - \lambda \mathcal{K})^{-1})
\]

\[
= (I - \lambda \mathcal{K})^{-1} - \lambda (I - \lambda \mathcal{K})^{-1} \mathcal{F}(M(\lambda))^{-1} \mathcal{G}(I - \lambda \mathcal{K})^{-1}.
\]

As is easily verified, \( M(\lambda) \) in this proposition is same as \( M(\lambda) \) given in the beginning of the previous section.

Let

\[
H_1(z; \lambda) = ((I - \lambda \mathcal{K})^{-1} \frac{1}{z - c_i})_{i=1}^{n_1} = (H(c_i, z; \lambda))_{i=1}^{n_1}
\]
be a row vector and let

\[ H_2(x; \lambda) = \mathcal{G}(I - \lambda \mathcal{K})^{-1} \frac{1}{z - x} = \mathcal{G}H(x, z; \lambda) = \left( \frac{H(x, R(c_i); \lambda)}{R''(c_i)} \right)_{i=1}^{n_1} \]

be a column vector. With all these things together, we find the explicit expression of the resolvent function \( E(x, z; \lambda) \).

**Theorem 5.3**

\[ E(x, z; \lambda) = H(x, z; \lambda) + \lambda H_1(z; \lambda)(M(\lambda))^{-1}H_2(x; \lambda). \]

The resolvent function is holomorphic in \( x \in \Omega_1 \), and in \( z \in \Omega_R \), and meromorphic in \( \lambda \in \mathbb{C} \). The poles are the zeros of the Fredholm determinant \( D_{11}(\lambda) \).

**References**

