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On Fricke groups corresponding to real lattices

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1 Introduction

We consider Teichmüller spaces of the closed torus and the once punctured torus. This is a report of our results obtained in a series of papers where we investigate explicit relations between these spaces. In [Ab1] and [Ab2] we found Fricke groups corresponding to real lattices and constructed holomorphic mappings between once punctured and closed tori determined by these groups and lattices, respectively.

We use throughout the convention that an element A in $\mathrm{PSL}(2,\mathbb{R})$ represents the Möbius transformation induced by A, *i.e.*,

$$\text{if } A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}) \text{ then } A(z)=\frac{az+b}{cz+d}.$$

We consider a Fuchsian group G consisting of Möbius transformations of $\mathrm{PSL}(2,\mathbb{R})$ and having the following properties: (i) G is discontinuous in the upper half-plane $\mathbb{H}$, (ii) Every real number is a limit point for G, (iii) G is finitely generated.

Definition 1.1 A Fuchsian group $\Gamma=\langle A,B\rangle$ for $A,B\in\mathrm{PSL}(2,\mathbb{R})$ is called *a Fricke group* if A,B are hyperbolic and $\mathrm{tr}\,[B^{-1},A^{-1}]=-2$.

In the definition above $\Gamma=\langle A,B\rangle$ is the free group generated by A,B and tr denotes the trace of a matrix. We consider a once punctured torus which is uniformized by a Fricke group Γ and take a normalized form for the presentation of Γ (see §4). By using the quantities $X=\mathrm{tr}\,A, Y=\mathrm{tr}\,B$ and $Z=\mathrm{tr}\,AB$, the above description of the Fricke group is characterized by $X^2+Y^2+Z^2=XYZ$ and $X,Y,Z>2$. Moreover, we obtain the following theorem (see [W]).

Theorem 1.1 *The Teichmüller space $\mathcal{T}_{1,1}$ of the once punctured torus is the sublocus of $X^2+Y^2+Z^2=XYZ$ with $X,Y,Z>2$.*

In this paper we denote a point in the Teichmüller space $\mathcal{T}_{1,1}$ of the once punctured torus by a triple (X, Y, Z) and we call it the (X, Y, Z) coordinates.

We describe a closed torus by $R_\tau = \mathbb{C}/\Gamma_\tau$, $\Gamma_\tau = \{m + n\tau \mid m, n \in \mathbb{Z}\}$, then the Teichmüller space $\mathcal{T}_{1,0}$ of the closed torus is the upper half-plane $\mathbb{H}$, *i.e.*, a point in the Teichmüller space $\mathcal{T}_{1,0}$ of the closed torus is denoted by $\tau \in \mathbb{H}$. (See [IT], etc.) We call it the τ coordinates.

It is well-known that theoretically we can identify Teichmüller spaces $\mathcal{T}_{1,0}$ and $\mathcal{T}_{1,1}$. For example in [W] the existence of a map from the (X, Y, Z) coordinates to the τ coordinates is described. But it does not give explicitly a holomorphic mapping between a once punctured torus determined by (X, Y, Z) and a closed torus determined by τ. Our problem is to construct such a holomorphic mapping.

Since the two Teichmüller spaces $\mathcal{T}_{1,0}$ and $\mathcal{T}_{1,1}$ are too large to find relations between them directly, we introduce the three subsets of $\mathcal{T}_{1,0}$: $L_1 = \{\tau \in \mathbb{H} \mid |\tau| \geq 1 \text{ and } \mathrm{Re}\,(\tau) = 0\}$, $L_2 = \{\tau \in \mathbb{H} \mid |\tau| = 1 \text{ and } -\frac{1}{2} \leq \mathrm{Re}\,(\tau) \leq 0\}$ and $L_3 = \{\tau \in \mathbb{H} \mid |\tau| \geq 1 \text{ and } \mathrm{Re}\,(\tau) = -\frac{1}{2}\}$. These sets are characterized by the fact that $\tau \in L_1 \cup L_2 \cup L_3$ if and only if τ is a closed torus associated to a real lattice (see §2). Define the three subsets of $\mathcal{T}_{1,1}$: $M_1 = \{(X, Y, Z) \in \mathcal{T}_{1,1} \mid 2 < X \leq Y \leq Z = XY/2\}$, $M_2 = \{(X, Y, Z) \in \mathcal{T}_{1,1} \mid 2 < X = Y \leq Z\}$ and $M_3 = \{(X, Y, Z) \in \mathcal{T}_{1,1} \mid 2 < X \leq Y = Z\}$. Then we obtained the following theorem in [Ab1]:

Theorem 1.2 *There exist correspondences of the sets:* $L_1 \Leftrightarrow M_1$, $L_2 \Leftrightarrow M_2$, $L_3 \Leftrightarrow M_3$.

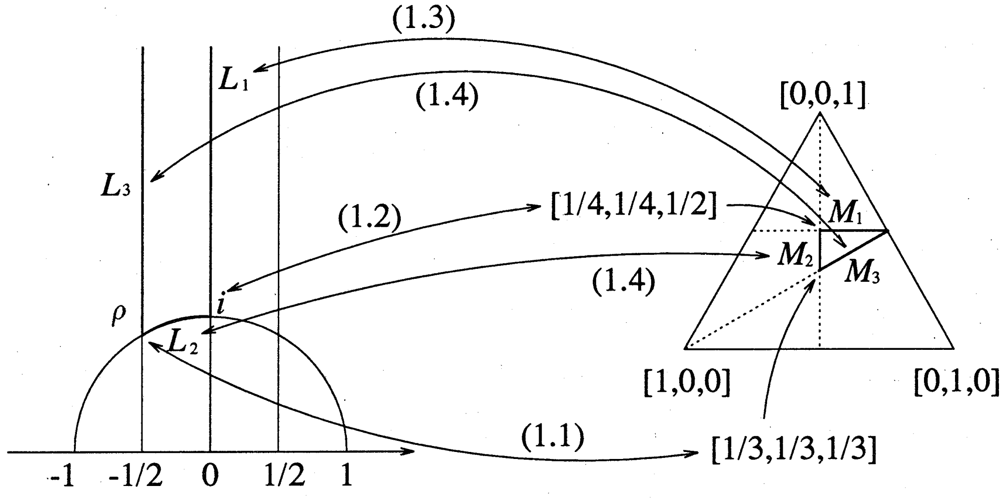
Our aim is to construct a holomorphic mapping between elements of L_l and M_l, $l = 1, 2, 3$ based on these correspondences.

We summarize our results. We represent a point in the upper half-plane $\mathbb{H}$ by z and a point in the complex plane $\mathbb{C}$ by u. We call $\mathbb{H}$ the z-plane and $\mathbb{C}$ the u-plane. Then a once punctured torus (X, Y, Z) and a closed torus τ can be identified with fundamental domains in the z-plane and the u-plane, respectively. For special two points $(3, 3, 3), (2\sqrt{2}, 2\sqrt{2}, 4)$ in $\mathcal{T}_{1,1}$ and $\rho = e^{\frac{2}{3}\pi i}, i$ in $\mathcal{T}_{1,0}$, holomorphic mappings between once punctured tori and closed tori can be constructed explicitly, which come from [C1], [C2]. A holomorphic mapping from $(3, 3, 3)$ to ρ is given by

$$1 - J(z) = \wp'(u)^2 = 4\wp(u)^3 + 1, \tag{1.1}$$

and a holomorphic mapping from $(2\sqrt{2}, 2\sqrt{2}, 4)$ to i is given by

$$J_i(z) = \wp(u)^2 \text{ and } \wp'(u)^2 = 4\wp(u)^3 - 4\wp(u), \tag{1.2}$$



In this figure we use another coordinate system $[a,b,c]$ defined by $a = X/YZ$, $b = Y/XZ$ and $c = Z/XY$ because the space $\mathcal{T}_{1,1}$ represented by $[a,b,c]$ is much easier to see than by (X,Y,Z). Two points $(3,3,3)$ and $(2\sqrt{2},2\sqrt{2},4)$ are identified with $[\frac{1}{3},\frac{1}{3},\frac{1}{3}]$ and $[\frac{1}{4},\frac{1}{4},\frac{1}{2}]$, respectively. It is easily checked that $\mathcal{T}_{1,1} = \{[a,b,c] \,|\, a+b+c=1 \text{ with } a,b,c>0\}$, $M_1 = \{[a,b,c] \in \mathcal{T}_{1,1} \,|\, 0 < a \le b, c = \frac{1}{2}\}$, $M_2 = \{[a,b,c] \in \mathcal{T}_{1,1} \,|\, 0 < a = b \le c \le \frac{1}{2}\}$ and $M_3 = \{[a,b,c] \in \mathcal{T}_{1,1} \,|\, 0 < a \le b = c\}$.

Fig. 1.1

where $\wp(u)$ are Weierstrass' pe-functions defined by the above equations, $J(z)$ is the modular function and $J_i(z)$ is a function having similar properties to $J(z)$ (see Proposition 3.1). We will show in §3 the reason why the relation (1.2) gives a holomorphic mapping between a once punctured torus and a closed torus. Generalizing the relations (1.1) and (1.2), we obtained the theorem for the case L_l and M_l for $l = 1,2,3$:

Theorem 1.3 *For any $(X,Y,Z) \in M_1$ there uniquely exists an element $\tau \in L_1$ satisfying the following conditions: let $x_1 < x_2 < x_3$ be three real roots of $p(x) = 4x^3 - g_2(\tau)x - g_3(\tau)$, then a holomorphic mapping between (X,Y,Z) and τ is given by a relation*

$$\wp(u) = (x_2 - x_1)J_{(X,Y,Z)}(z) + x_2, \tag{1.3}$$

where $\wp(u)$ is Weierstrass' pe-function defined by $\wp'(u)^2 = 4\wp(u)^3 - g_2(\tau)\wp(u) - g_3(\tau)$ and $J_{(X,Y,Z)}(z)$ is a function having similar properties to the modular function $J(z)$.

Theorem 1.4 *For any $(X,Y,Z) \in M_l$, $l = 2,3$ there uniquely exist an element $\tau \in L_l$*

and P such that a holomorphic mapping between (X, Y, Z) and τ is given by a relation

$$\wp(u) = J_{(X,Y,Z)}(z) - \frac{P}{J_{(X,Y,Z)}(z)} + \frac{1-P}{3}, \tag{1.4}$$

where $\wp(u)$ is Weierstrass' pe-function defined by

$$\wp'(u)^2 = 4\left(\wp(u) - \frac{2}{3}(P-1)\right)\left(\wp(u) - \left(\frac{1-P}{3} + 2\sqrt{P}i\right)\right)\left(\wp(u) - \left(\frac{1-P}{3} - 2\sqrt{P}i\right)\right),$$

$J_{(X,Y,Z)}(z)$ is a function having similar properties to the modular function $J(z)$, $7-4\sqrt{3} \leq P \leq 1$ if $\tau \in L_2$ and $P \geq 7+4\sqrt{3}$ if $\tau \in L_3$.

For proofs of Theorem 1.3 and 1.4 we refer the reader to [Ab1] and [Ab2], respectively. We will show in §5 precise definitions of $J_{(X,Y,Z)}$ in order to make their meaning clear. A construction of that mapping is one of important parts of our proof.

Finally, we note that the relations (1.1), (1.2), (1.3), (1.4) are closely related to symmetries of fundamental domains corresponding to a closed torus and a once punctured torus, that is, symmetries are reflected in the orders of $\wp(u)$, $J(z)$, $J_i(z)$ and $J_{(X,Y,Z)}(z)$ and that these relations give cusp forms of weight 1 for associated Fricke groups (see [Ab3]).

2 The functions $\wp$, J and real lattices

In this section we recall fundamental facts concerning Weierstrass' pe-function and the modular function. For detailed arguments and proofs we refer to chapters 3 and 6 of [JS].

For $\omega_1, \omega_2 \in \mathbb{C}$, we define a lattice Ω by

$$\Omega = \Omega(\omega_1, \omega_2) = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\},$$

and we call $\{\omega_1, \omega_2\}$ a basis for Ω and a point in Ω a lattice point in $\mathbb{C}$.

Definition 2.1 We call the following series *Weierstrass' pe-function*

$$\wp(u) = \frac{1}{u^2} + \sum_{\omega\in\Omega, \omega\neq 0}\left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2}\right) \quad \text{for all } u \in \mathbb{C}.$$

From the definition, it follows immediately that $\wp(u)$ depends on the lattice Ω and is an even function which is analytic on $\mathbb{C}\backslash\Omega$ and has a pole of order 2 at each $\omega \in \Omega$. Note that $\wp(u)$ is doubly periodic with respect to the lattice Ω. We define

$$g_2 = g_2(\Omega) = 60 \sum_{\omega\in\Omega,\omega\neq 0} \frac{1}{\omega^4} \quad \text{and} \quad g_3 = g_3(\Omega) = 140 \sum_{\omega\in\Omega,\omega\neq 0} \frac{1}{\omega^6}.$$

If $\Omega = \Omega(1,\tau)$ we can write

$$g_2 = g_2(\tau) = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^4} \text{ and } g_3 = g_3(\tau) = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^6}.$$

Note that $g_2(\rho) = g_3(i) = 0$, where $\rho = e^{\frac{2}{3}\pi i}$. Moreover, we can give characterizations of the function $\wp(u)$ and the lattice Ω when $g_2(\Omega)$ and $g_3(\Omega)$ are real. We represent by $\overline{z}$ the conjugate of $z \in \mathbb{C}$.

Proposition 2.1 *The following conditions are equivalent:*

(i) $g_2(\Omega), g_3(\Omega) \in \mathbb{R}$.

(ii) $\wp$ *defined by using the lattice* Ω *satisfies* $\wp(\overline{u}) = \overline{\wp(u)}$ *for all* $u \in \mathbb{C}$.

(iii) Ω *is a real lattice, i.e.,* $\overline{\Omega} = \{\overline{\omega} \mid \omega \in \Omega\} = \Omega$.

In order to characterize real lattices we introduce the following sets:

$$L_1 = \{\tau \in \mathbb{H} \mid |\tau| \geq 1 \text{ and } \mathrm{Re}\,(\tau) = 0\},$$
$$L_2 = \left\{\tau \in \mathbb{H} \,\middle|\, |\tau| = 1 \text{ and } -\frac{1}{2} \leq \mathrm{Re}\,(\tau) \leq 0\right\},$$
$$L_3 = \left\{\tau \in \mathbb{H} \,\middle|\, |\tau| \geq 1 \text{ and } \mathrm{Re}\,(\tau) = -\frac{1}{2}\right\}.$$

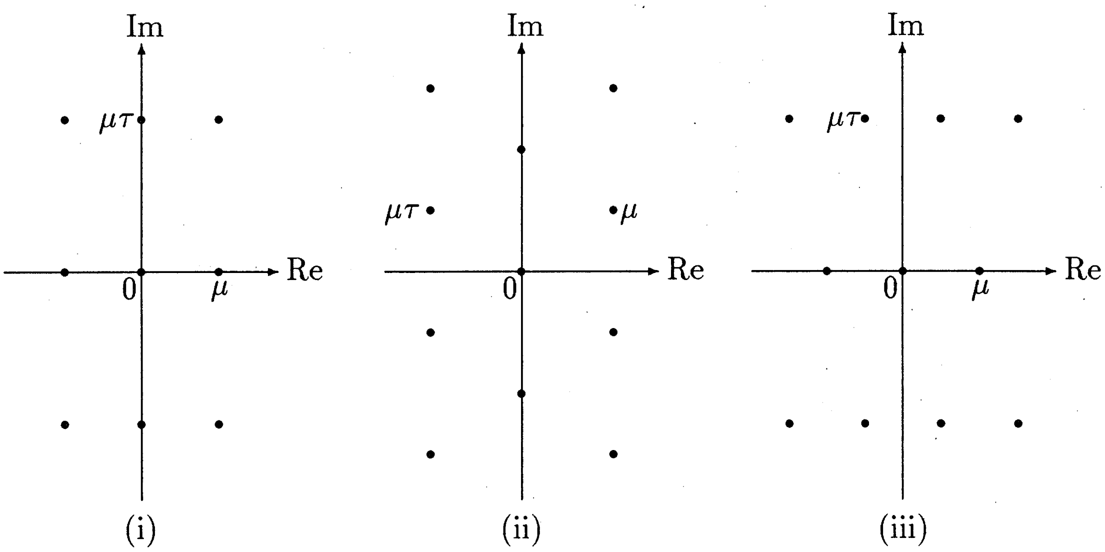
Let τ be an element in $L_1 \cup L_3$. If we take $\mu > 0$ or $\mu = ri$ with $r > 0$ then $\Omega = \Omega(\mu, \mu\tau)$ is a real lattice. Let $\tau = e^{i\theta}$ with $\pi/2 \leq \theta \leq 2\pi/3$ be an element in L_2. Then $\Omega = \Omega(\mu, \mu\tau)$ is a real lattice, where $\mu = re^{i\frac{\pi-\theta}{2}}$ or $\mu = re^{-\frac{\theta}{2}i}$ with $r > 0$. Conversely, any real lattice Ω can be represented as above, because real lattices are classified into two cases: one is a rectangular case, *i.e.*, $\tau \in L_1$ and the other is a rhombic case, *i.e.*, $\tau \in L_2 \cup L_3$. (See Fig. 2.1.) We call a lattice Ω defined by $\tau \in L_1$ as above a rectangular lattice and a lattice Ω defined by $\tau \in L_2 \cup L_3$ as above a rhombic lattice.

Definition 2.2 We define *the modular function* $J(\tau)$ by

$$J(\tau) = \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2} \quad \text{for all } \tau \in \mathbb{H}.$$

The properties of the modular function which we will use are the following:

Proposition 2.2 *(i)* J *is invariant under actions of the modular group* $\mathrm{PSL}(2,\mathbb{Z})$, *i.e.,* $J(T(\tau)) = J(\tau)$ *for all* $\tau \in \mathbb{H}$ *and* $T \in \mathrm{PSL}(2,\mathbb{Z})$, *where* $T(\tau)$ *is a Möbius transformation.*



Lattice points are represented by •. (i) The case $\tau \in L_1$ and $\mu > 0$. (ii) The case $\tau \in L_2$ and $\mu = re^{i\frac{\pi-\theta}{2}}$ with $r > 0$. (iii) The case $\tau \in L_3$ and $\mu > 0$.

Fig. 2.1

(ii) The mapping $J : \mathbb{H} \to \mathbb{C}$ *is holomorphic on* $\mathbb{H}$.

(iii) J *maps* L *onto* $\mathbb{R}$ *where* $L = L_1 \cup L_2 \cup L_3$. *Especially,* $J(i) = 1$ *and* $J(\rho) = 0$.

(iv) J *maps* F *onto* $\mathbb{C}$ *where* $F = \{\tau \in \mathbb{H} \mid |\tau| \geq 1 \ \text{and} \ |\mathrm{Re}\,(\tau)| \leq \frac{1}{2}\}$ *is a fundamental domain for the modular group.*

By using g_2 and g_3, we can introduce an important differential equation connecting $\wp(u)$ and $\wp'(u)$:

Proposition 2.3 $\wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3$.

Note that $\wp, g_2$ and g_3 depend on a lattice Ω.

Proposition 2.4 *Let* $\Omega = \Omega(\omega_1, \omega_2)$ *and* $\omega_3 = \omega_1 + \omega_2$. *Then we have* $\wp'(\frac{1}{2}\omega_1) = \wp'(\frac{1}{2}\omega_2) = \wp'(\frac{1}{2}\omega_3) = 0$.

Points $\frac{1}{2}\omega_l + (n_1\omega_1 + n_2\omega_2)$ for all $n_1, n_2 \in \mathbb{Z}$ and $l = 1, 2, 3$ are called ramification points of Ω.

Proposition 2.5 *We define* $e_l = \wp(\frac{1}{2}\omega_l)$ *for* $l = 1,2,3$. *Then* e_1, e_2, e_3 *are mutually distinct.*

That is to say, if we take a lattice Ω, the values $g_2(\Omega)$ and $g_3(\Omega)$ define a cubic polynomial $p(x) = 4x^3 - g_2x - g_3$, which has distinct roots. Conversely, we obtain the following assertion.

Proposition 2.6 *Let* $p(x) = 4x^3 - c_2x - c_3$ *for* $c_2, c_3 \in \mathbf{C}$ *be any cubic polynomial with distinct roots. Then there is a lattice* Ω *with* $c_2 = g_2(\Omega)$ *and* $c_3 = g_3(\Omega)$. *Precisely,*

(i) *if* $c_2 = 0,\ c_3 \neq 0$ *then* $\Omega = \Omega(\mu, \mu\rho)$, *where* $\mu \in \mathbf{C}$ *is determined by* $(1/\mu^6)g_3(\rho) = c_3$.

(ii) *if* $c_2 \neq 0,\ c_3 = 0$ *then* $\Omega = \Omega(\mu, \mu i)$, *where* $\mu \in \mathbf{C}$ *is determined by* $(1/\mu^4)g_2(i) = c_2$.

(iii) *if* $c_2 \neq 0,\ c_3 \neq 0$ *then* $\Omega = \Omega(\mu, \mu\tau)$, *where* μ *is any element of* $\mathbf{C}\backslash\{0\}$ *and* $\tau \in \mathbf{C}$ *is determined by* $J(\tau) = c_2^3/(c_2^3 - 27c_3^2)$.

We use this proposition in order to define real lattices.

3 An example

Recall that we represent a point in the upper half-plane $\mathbb{H}$ by z and a point in the complex plane $\mathbf{C}$ by u. We call $\mathbb{H}$ the z-plane and $\mathbf{C}$ the u-plane. Let Γ be a Fricke group associated with a once punctured torus $(X, Y, Z) \in \mathcal{T}_{1,1}$. Then Γ determines a fundamental domain in the z-plane, which can be identified with the once punctured torus (X, Y, Z). Let Γ_τ be a lattice in the u-plane for $\tau \in \mathcal{T}_{1,0}$. Then a fundamental domain for Γ_τ in the u-plane can be identified with the closed torus τ.

We consider the point $(2\sqrt{2}, 2\sqrt{2}, 4)$ in $\mathcal{T}_{1,1}$. Before we show a construction of a holomorphic mapping between a once punctured torus and a closed torus, we state fundamental domains in the z-plane for the Fricke group associated with this point.

As a representation of $(2\sqrt{2}, 2\sqrt{2}, 4)$ we take $\Gamma_i = \langle A_i, B_i \rangle$ with $A_i = \begin{pmatrix} 0 & -1 \\ 1 & 2\sqrt{2} \end{pmatrix}$ and $B_i = \begin{pmatrix} \sqrt{2} & -1 \\ -1 & \sqrt{2} \end{pmatrix}$ and define $C_i = B_i^{-1}A_i^{-1}$. Then $C_iB_iA_i = \begin{pmatrix} -1 & -4\sqrt{2} \\ 0 & -1 \end{pmatrix}$. One fundamental domain is a quadrilateral whose opposite sides are identified by actions of A_i and B_i (shown in normal outline in Fig. 3.1),

$$D_q(\Gamma_i) = \left\{ z \in \mathbb{H} \;\middle|\; \left| z + \frac{3\sqrt{2}}{4} \right| > \frac{\sqrt{2}}{4},\ \left| z + \frac{\sqrt{2}}{4} \right| > \frac{\sqrt{2}}{4},\ -\sqrt{2} \leq \mathrm{Re}\,(z) \leq 0 \right\},$$

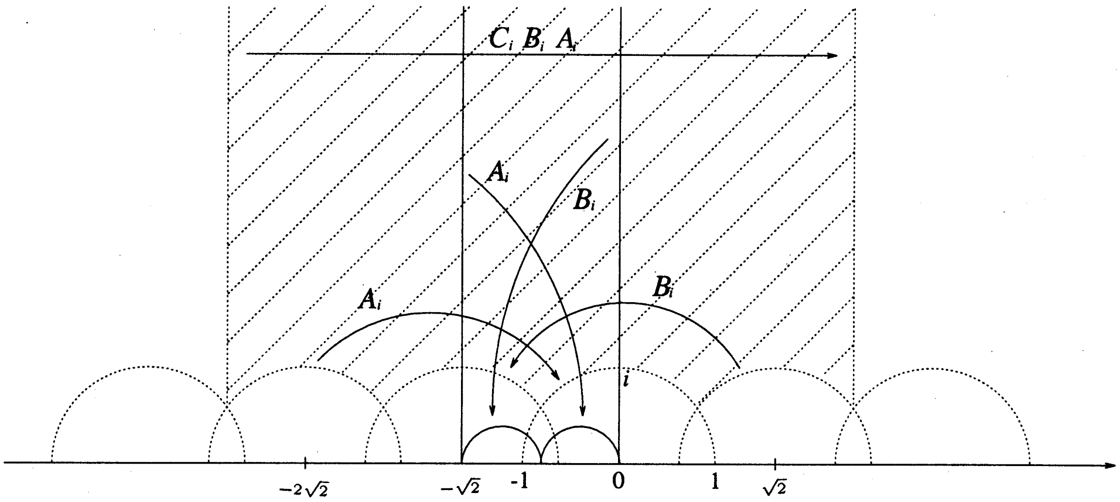


Fig. 3.1

another is a hexagon whose sides are identified by actions of A_i, B_i and $C_iB_iA_i$ (the shaded part in Fig. 3.1).

$$\begin{aligned} D_h(\Gamma_i) &= \Big\{ z \in \mathbb{H} \;\Big|\; |z+2\sqrt{2}| \geq 1\ |z+\sqrt{2}| \geq 1, \\ &\qquad |z| > 1,\ |z-\sqrt{2}| > 1,\ -\frac{5\sqrt{2}}{2} \leq \mathrm{Re}\,(z) < \frac{3\sqrt{2}}{2} \Big\}, \end{aligned}$$

We call them the quadrilateral fundamental domain and the hexagonal fundamental domain, respectively.

Theorem 3.1 *The once punctured torus* $(2\sqrt{2}, 2\sqrt{2}, 4) \in \mathcal{T}_{1,1}$ *is mapped to the closed torus* $i \in \mathcal{T}_{1,0}$ *holomorphically by using relations*

$$\wp'(u)^2 = 4\wp(u)^3 - 4\wp(u), \tag{3.1}$$

$$J_i(z) = \wp(u)^2. \tag{3.2}$$

The function J_i in (3.2) is defined as follows:

Proposition 3.1 *We can construct a function* J_i *which satisfies*

(i) J_i *maps* L_i *onto* $\mathbb{R}$ *where* $L_i = L_{i1} \cup L_{i2} \cup L_{i3}$ *and* $L_{i1} = L_1$*,*

$$L_{i2} = \left\{ \tau \in \mathbb{H} \,\middle|\, |\tau| = 1 \ \ \textit{and} \ \ -\frac{\sqrt{2}}{2} \leq \mathrm{Re}\,(\tau) \leq 0 \right\},$$

$$L_{i3} = \left\{ \tau \in \mathbb{H} \,\middle|\, |\tau| \geq 1 \ \textit{and} \ \mathrm{Re}\,(\tau) = -\frac{\sqrt{2}}{2} \right\}.$$

Especially, $J_i(i\infty) = \infty$, $J_i(i) = 1$ *and* $J_i(\rho_i) = 0$ *where* $\rho_i = e^{\frac{3}{4}\pi i}$.

(ii) J_i *maps* F_i *onto* $\mathbb{C}$ *where* $F_i = \{\tau \in \mathbb{H} \mid |\tau| \geq 1 \ \textit{and} \ \left|\mathrm{Re}\,(\tau)\right| \leq \frac{\sqrt{2}}{2}\}$.

(iii) The mapping $J_i : \mathbb{H} \to \mathbb{C}$ *is holomorphic on* $\mathbb{H}$.

For the proof of this proposition, we refer to Chapter III in [H].

We introduce two notations: one is $\mu\mathbb{R} = \{\mu u \mid u \in \mathbb{R}\}$ for $\mu \in \mathbb{C}$ and the other is $\wp(\mu_1\mathbb{R}) \subset \mu_2\mathbb{R} \cup \{\infty\}$ for $\mu_1, \mu_2 \in \mathbb{C}$ which means $\wp(\mu_1 u) \in \mu_2\mathbb{R} \cup \{\infty\}$ for all $u \in \mathbb{R}$.

Proposition 3.2 *(i) If* Ω *is a real lattice, then*

$$\wp(\mathbb{R}) \subset \mathbb{R} \cup \{\infty\} \ \ \textit{and} \ \ \wp(i\mathbb{R}) \subset \mathbb{R} \cup \{\infty\}.$$

(ii) If $\Omega = \Omega(\mu, \mu i)$ *for some* $\mu \in \mathbb{C}$ *is a real lattice, then*

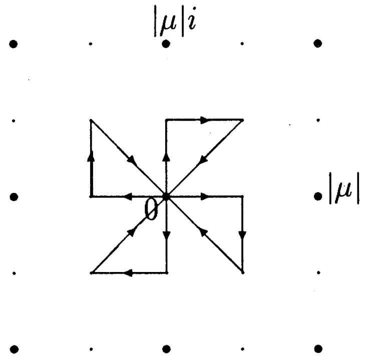
$$\wp(e^{\frac{\pi}{4}i}\mathbb{R}) \subset i\mathbb{R} \cup \{\infty\} \ \ \textit{and} \ \ \wp(e^{-\frac{\pi}{4}i}\mathbb{R}) \subset i\mathbb{R} \cup \{\infty\}.$$

(iii) For the lattice $\Omega = \Omega(\nu, \nu i)$ *with* $\nu > 0$, *we have*

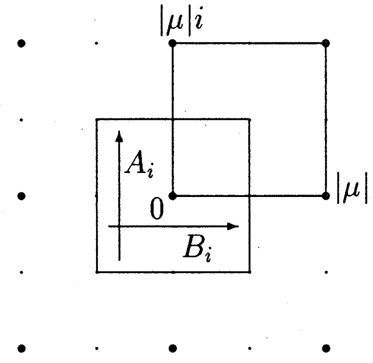
$$\begin{aligned} &\wp(\mathbb{R} + \tfrac{1}{2}\nu i) \subset \mathbb{R} \cup \{\infty\}, \quad \wp(\mathbb{R} - \tfrac{1}{2}\nu i) \subset \mathbb{R} \cup \{\infty\}, \\ &\wp(i\mathbb{R} + \tfrac{1}{2}\nu)) \subset \mathbb{R} \cup \{\infty\}, \quad \wp(i\mathbb{R} - \tfrac{1}{2}\nu)) \subset \mathbb{R} \cup \{\infty\}. \end{aligned}$$

These assertions are proved by using properties of real lattices, especially, Proposition 2.1 (ii).

Proof of Theorem 3.1. We take $\Gamma_i = \langle A_i, B_i \rangle$ with $A_i = \begin{pmatrix} 0 & -1 \\ 1 & 2\sqrt{2} \end{pmatrix}$ and $B_i = \begin{pmatrix} \sqrt{2} & -1 \\ -1 & \sqrt{2} \end{pmatrix}$ as a representation of $(2\sqrt{2}, 2\sqrt{2}, 4)$. Recall that two fundamental domains for Γ_i in the z-plane are as shown in Fig. 3.1. By Proposition 2.6 (ii), the relation (3.1) determines the lattice $\Omega = \Omega(\mu, \mu i)$ in the u-plane where μ satisfies $(1/\mu^4)g_2(i) = 4$. As $g_2(i) > 0$, we can write $\mu = |\mu|e^{\frac{j}{2}\pi i}$, $j = 0, 1, 2, 3$. Therefore we obtain $\Omega = \Omega(\mu, \mu i) = \Omega(|\mu|, |\mu| i)$ which is a real lattice. By the relation (3.2) the point $z = i\infty$ is mapped to a lattice point of the u-plane because of $J_i(i\infty) = \infty$ and $\wp(u_0) = \infty$ for $u_0 \in \Omega = \Omega(\mu, \mu i)$. We can assume without loss of generality that the image of $z = i\infty$ by (3.2) is the point $u = 0$. Suppose that J_i has a value $c_0 \in \mathbb{C}$ with $c_0 \neq 1$. Since for each $c \in \mathbb{C}$, $c \neq e_1, e_2, e_3, \infty$ the equation $\wp(u) = c$ has two simple solutions, for $J_i = c_0$ we obtain four points determined by the relation (3.2) in the u-plane. The hexagonal fundamental domain



(i)



(ii)

(i) The segments L_i $(i\infty \to i \to \rho_i \to i\infty)$, $L_i + \sqrt{2}$, $L_i - \sqrt{2}$ and $L_i - 2\sqrt{2}$ are mapped by the relations (3.1) and (3.2) to the segments $(0 \to \frac{1}{2}|\mu| \to \frac{1}{2}|\mu|(1-i) \to 0)$, $(0 \to -\frac{1}{2}|\mu|i \to -\frac{1}{2}|\mu|(1+i) \to 0)$, $(0 \to -\frac{1}{2}|\mu| \to -\frac{1}{2}|\mu|(1-i) \to 0)$ and $(0 \to \frac{1}{2}|\mu|i \to \frac{1}{2}|\mu|(1+i) \to 0)$.

(ii) The square determined by $\pm\frac{1}{2}|\mu|(1+i), \pm\frac{1}{2}|\mu|(1-i)$ is the image of the hexagonal fundamental domain by the relations (3.1) and (3.2).

Fig. 3.2

$D_h(\Gamma_i)$ is a 4-sheeted covering of the domain F_i, so we can take four points which attain $J_i = c_0$ in the hexagonal fundamental domain. Therefore we can obtain correspondences between these four points in the z-plane and the four points in the u-plane. Now we consider the case where $J_i(z)$ is real. From Proposition 3.1 (i) we decompose arguments into the following three cases.

(I) $J_i(z) \geq 1$, *i.e.*, $z \in L_{i1}$.

Set $J_i(z) = c$ then by using (3.2) we have $\wp(u) = \pm\sqrt{c}$. We can take a point u with $\wp(u) = \sqrt{c}$ on the line $\mathbb{R}$ because $\wp(\mathbb{R}) \subset \mathbb{R}\cup\{\infty\}$ and $\wp(u)$ is positively large if $u \in \mathbb{R}$ is small. In the same way we can take a point u with $\wp(u) = -\sqrt{c}$ on $i\mathbb{R}$. In the relation (3.2) if $J_i(i) = 1$ then $\wp^2(u) - 1 = 0$ which gives $\wp'(u)^2 = 4\wp(u)^3 - 4\wp(u) = 4\wp(u)(\wp(u)^2 - 1) = 0$, so the point u corresponding to $z = 1$ must be ramification points $\frac{1}{2}|\mu|e^{\frac{j}{2}\pi i}$, $j = 0, 1, 2, 3$ in the u-plane. Therefore we can take segments $\{t|\mu|e^{\frac{j}{2}\pi i} \mid 0 \leq t \leq \frac{1}{2}\}$, $j = 0, 1, 2, 3$ as images of L_{i1} by the mapping defined by the relations (3.1) and (3.2).

(II) $J_i(z) \leq 0$, *i.e.*, $z \in L_{i3}$.

Set $J_i(z) = c$ then by using (3.2) we have $\wp(u) = \pm\sqrt{-ci}$. By Proposition 3.2 (ii) we get $\wp(e^{-\frac{\pi}{4}i}\mathbb{R}) \subset i\mathbb{R} \cup \{\infty\}$. If $u = te^{-\frac{\pi}{4}i}$ and $t \in \mathbb{R}$ is small, $\wp(u) = i|\wp(u)|$ and $|\wp(u)|$ is large, so we can take a point u with $\wp(u) = \sqrt{-ci}$ on the line $e^{-\frac{\pi}{4}i}\mathbb{R}$. By using the same argument we can take a point u with $\wp(u) = -\sqrt{-ci}$ on $e^{\frac{\pi}{4}i}\mathbb{R}$. In the relation (3.2) if $J_i(\rho_i) = 0$ then $\wp(u) = 0$ which gives $\wp'(u)^2 = 4\wp(u)(\wp(u)^2 - 1) = 0$, so the point u corresponding to $z = \rho_i$ must be ramification points $\frac{\sqrt{2}}{2}|\mu|e^{(\frac{\pi}{4}+\frac{j}{2})\pi i}$, $j = 0, 1, 2, 3$ in the u-plane. Therefore we can take segments $\{t|\mu|e^{(\frac{\pi}{4}+\frac{j}{2})\pi i} \mid 0 \leq t \leq \frac{\sqrt{2}}{2}\}$, $j = 0, 1, 2, 3$ as images of L_{i3} by the mapping defined by the relations (3.1) and (3.2).

(III) $0 \leq J_i(z) \leq 1$, *i.e.*, $z \in L_{i2}$.

Set $J(z) = c$ then $\wp(u) = \pm\sqrt{c}$. Since the mapping defined by the relations (3.1) and (3.2) is conformal on $\mathbb{C}/\Omega$, we obtain Fig. 3.2 (i) by Proposition 3.2 (iii) and the arguments of (I) and (II).

From Proposition 3.1 and the symmetry of the domain F_i, the image of the hexagonal fundamental domain $D_h(\Gamma_i)$ by the mapping defined by the relations (3.1) and (3.2) can be represented by the square including 0 as in Fig. 3.2 (ii). Then we can show without loss of generality the actions of A_i and B_i as in Fig. 3.2 (ii) from the identification of the sides of $D_h(\Gamma_i)$. Moreover, the quadrilateral fundamental domain $D_q(\Gamma_i)$ is mapped to another square in Fig. 3.2 (ii) which is determined by the lattice $\Omega = \Omega(\mu, \mu i)$ in the u-plane. These facts imply that there exist the correspondences $\mu \Leftrightarrow A_i$ and $\mu i \Leftrightarrow B_i$ which we used in order to show the correspondence $i \Leftrightarrow (2\sqrt{2}, 2\sqrt{2}, 4)$ (see §3 in [Ab1]).

□

4 Introduction of fundamental domains

In order to construct holomorphic mappings between once punctured tori and closed tori in [Ab1] and [Ab2], it was very important to take proper fundamental domains for Fricke groups. In this section we will introduce fundamental domains for Fricke groups in M_l for $l = 1, 2, 3$.

We begin by recalling M, M_1, M_2 and M_3 defined in [Ab1]. M is a fundamental domain for the action of $\mathrm{PSL}(2, \mathbb{Z})$ on $\mathcal{T}_{1,1}$:

$$M = \left\{(X, Y, Z) \in \mathcal{T}_{1,1} \;\middle|\; 2 < X \leq Y \leq Z \leq \frac{1}{2}XY\right\},$$

and M_1, M_2, M_3 are the subsets of M defined by

$$M_1 = \left\{(X, Y, Z) \in M \;\middle|\; Z = \frac{1}{2}XY\right\}$$

$$= \left\{(X,Y,Z) \;\middle|\; (X,Y,Z) = \left(\frac{2\sqrt{1+\alpha^2}}{\alpha}, 2\sqrt{1+\alpha^2}, \frac{2(1+\alpha^2)}{\alpha}\right) \text{ for } \alpha \geq 1\right\},$$

$$\begin{aligned} M_2 &= \{(X,Y,Z) \in M \mid X = Y\} \\ &= \left\{(X,Y,Z) \;\middle|\; (X,Y,Z) = \left(\frac{2+\beta^2}{\beta}, \frac{2+\beta^2}{\beta}, 2+\beta^2\right) \text{ for } 1 \leq \beta \leq \sqrt{2}\right\}, \end{aligned}$$

$$\begin{aligned} M_3 &= \{(X,Y,Z) \in M \mid Y = Z\} \\ &= \left\{(X,Y,Z) \;\middle|\; (X,Y,Z) = \left(\frac{1+2\gamma^2}{\gamma^2}, \frac{1+2\gamma^2}{\gamma}, \frac{1+2\gamma^2}{\gamma}\right) \text{ for } \gamma \geq 1\right\}, \end{aligned}$$

where parameters α, β and γ are introduced by setting $Y = \alpha X, Z = \beta X$ and $Y = \gamma X$, respectively.

Definition 4.1 A Fricke group Γ is called *a special Fricke group* if the associated coordinate (X,Y,Z) of Γ is in M.

We recall some facts of special Fricke groups, which are due to [Sc]. Let (X,Y,Z) be an associated coordinate of a special Fricke group $\Gamma = \langle A, B\rangle$. Then A and B are given by

$$A = \begin{pmatrix} 0 & -\frac{k}{X} \\ \frac{X}{k} & X \end{pmatrix} \text{ and } B = \begin{pmatrix} Y - \frac{Z}{X} & -k\frac{Y}{X^2} \\ -\frac{Y}{k} & \frac{Z}{X} \end{pmatrix}$$

for some $k > 0$. In particular a fundamental domain for $\Gamma = \langle A, B\rangle$ is obtained by removing from the region

$$\left\{z = x + iy \;\middle|\; -k\left(\frac{3}{2} - \frac{X}{YZ}\right) \leq x < k\left(\frac{1}{2} + \frac{X}{YZ}\right) \text{ and } y > 0\right\}$$

the isometric circles for the Möbius transformations $A, A^{-1}, B, B^{-1}, C = B^{-1}A^{-1}$ and C^{-1}. We apply these facts to $(X,Y,Z) \in M_l$ for $l = 1,2,3$.

(I) the case $(X,Y,Z) \in M_1$.

We obtain a special Fricke group $\Gamma_\alpha = \langle A_\alpha, B_\alpha\rangle$ with

$$A_\alpha = \begin{pmatrix} 0 & -1 \\ 1 & \frac{2\sqrt{1+\alpha^2}}{\alpha} \end{pmatrix} \text{ and } B_\alpha = \begin{pmatrix} \sqrt{1+\alpha^2} & -\alpha \\ -\alpha & \sqrt{1+\alpha^2} \end{pmatrix} \text{ for } k = \frac{2\sqrt{1+\alpha^2}}{\alpha}.$$

Then we get

$$C_\alpha = B_\alpha^{-1}A_\alpha^{-1} = \begin{pmatrix} \frac{\frac{\alpha^2+2}{\alpha}}{\sqrt{1+\alpha^2}} & \sqrt{1+\alpha^2} \\ & \alpha \end{pmatrix} \text{ and } C_\alpha B_\alpha A_\alpha = \begin{pmatrix} -1 & -\frac{4\sqrt{1+\alpha^2}}{\alpha} \\ 0 & -1 \end{pmatrix}.$$

By using the basic facts cited above, we represent a fundamental domain for Γ_α:

$$\begin{aligned} D_h(\Gamma_\alpha) &= \Bigg\{ z \in \mathbb{H} \;\Bigg|\; \left| z + \frac{2\sqrt{1+\alpha^2}}{\alpha} \right| \geq 1, \left| z + \frac{\sqrt{1+\alpha^2}}{\alpha} \right| \geq \frac{1}{\alpha}, \\ & |z| > 1, \left| z - \frac{\sqrt{1+\alpha^2}}{\alpha} \right| > \frac{1}{\alpha}, -\frac{2+3\alpha^2}{\alpha\sqrt{1+\alpha^2}} \leq \mathrm{Re}\,(z) < \frac{2+\alpha^2}{\alpha\sqrt{1+\alpha^2}} \Bigg\}, \end{aligned}$$

which is a hexagon whose sides are identified by actions of $A_\alpha, B_\alpha, C_\alpha B_\alpha A_\alpha$ and is called the hexagonal fundamental domain. We can introduce another fundamental domain which is a quadrilateral whose opposite sides are identified by actions of A_α, B_α and is called the quadrilateral fundamental domain:

$$\begin{aligned} D_q(\Gamma_\alpha) &= \Bigg\{ z \in \mathbb{H} \;\Bigg|\; \left| z + \frac{1+2\alpha^2}{2\alpha\sqrt{1+\alpha^2}} \right| > \frac{1}{2\alpha\sqrt{1+\alpha^2}}, \\ & \left| z + \frac{\alpha}{2\sqrt{1+\alpha^2}} \right| > \frac{\alpha}{2\sqrt{1+\alpha^2}}, -\frac{\sqrt{1+\alpha^2}}{\alpha} \leq \mathrm{Re}\,(z) \leq 0 \Bigg\}. \end{aligned}$$

The quadrilateral fundamental domain (shown in normal outline in Fig. 5.1) can be identified with the once punctured torus (X, Y, Z).

(II) the case $(X, Y, Z) \in M_2$.

In this case a special Fricke group $\Gamma_\beta = \langle A_\beta, B_\beta \rangle$ is determined by

$$A_\beta = \begin{pmatrix} 0 & -1 \\ 1 & \frac{2+\beta^2}{\beta} \end{pmatrix} \text{ and } B_\beta = \begin{pmatrix} \frac{2}{\beta} & -1 \\ -1 & \beta \end{pmatrix} \text{ for } k = \frac{2+\beta^2}{\beta}.$$

Then we have

$$C_\beta = B_\beta^{-1} A_\beta^{-1} = \begin{pmatrix} \beta^2+1 & \beta \\ \beta & 1 \end{pmatrix} \text{ and } C_\beta B_\beta A_\beta = \begin{pmatrix} -1 & -\frac{4+2\beta^2}{\beta} \\ 0 & -1 \end{pmatrix}.$$

By using the basic facts cited above, we represent a fundamental domain for Γ_β:

$$\begin{aligned} D_o(\Gamma_\beta) &= \Bigg\{ z \in \mathbb{H} \;\Bigg|\; \left| z + \beta + \frac{2}{\beta} \right| \geq 1, \left| z + \frac{2}{\beta} \right| \geq 1, \left| z + \frac{1}{\beta} \right| \geq \frac{1}{\beta}, |z| > 1, \\ & |z - \beta| > 1, \left| z - \beta - \frac{1}{\beta} \right| > \frac{1}{\beta}, -\frac{3}{2}\beta - \frac{2}{\beta} \leq \mathrm{Re}\,(z) < \beta + \frac{1}{\beta} \Bigg\}, \end{aligned}$$

which is a octagon whose sides are identified by actions of $A_\beta, B_\beta, C_\beta, C_\beta B_\beta A_\beta$ and is called the octagonal fundamental domain. We can introduce another fundamental domain which is a quadrilateral whose opposite sides are identified by actions of A_β, B_β and is called the quadrilateral fundamental domain:

$$D_q(\Gamma_\beta) = \Bigg\{ z \in \mathbb{H} \;\Bigg|\; \left| z + \frac{3}{2\beta} \right| > \frac{1}{2\beta}, \left| z + \frac{1}{2\beta} \right| > \frac{1}{2\beta}, -\frac{2}{\beta} \leq \mathrm{Re}\,(z) \leq 0 \Bigg\}.$$

The quadrilateral fundamental domain (shown in normal outline in Fig. 5.2) can be identified with the once punctured torus (X, Y, Z).

(III) the case $(X, Y, Z) \in M_3$.

A special Fricke group $\Gamma_\gamma = \langle A_\gamma, B_\gamma \rangle$ is determined by the following A_γ and B_γ,

$$A_\gamma = \begin{pmatrix} 0 & -1 \\ 1 & \frac{1+2\gamma^2}{\gamma^2} \end{pmatrix} \text{ and } B_\gamma = \begin{pmatrix} \frac{1+\gamma^2}{\gamma} & -\gamma \\ -\gamma & \gamma \end{pmatrix} \text{ for } k = \frac{1+2\gamma^2}{\gamma^2}.$$

Then we get

$$C_\gamma = B_\gamma^{-1} A_\gamma^{-1} = \begin{pmatrix} \frac{1+\gamma^2}{\gamma} & \gamma \\ \gamma & \gamma \end{pmatrix} \text{ and } C_\gamma B_\gamma A_\gamma = \begin{pmatrix} -1 & -\frac{2+4\gamma^2}{\gamma^2} \\ 0 & -1 \end{pmatrix}.$$

In the same way as in (II) we obtain a octagon whose sides are identified by actions of $A_\gamma, B_\gamma, C_\gamma, C_\gamma B_\gamma A_\gamma$:

$$\begin{aligned} D_o(\Gamma_\gamma) &= \left\{ z \in \mathbb{H} \;\middle|\; \left| z + 2 + \frac{1}{\gamma^2} \right| \geq 1,\ \left| z + 1 + \frac{1}{\gamma^2} \right| \geq \frac{1}{\gamma},\ |z+1| \geq \frac{1}{\gamma},\ |z| > 1, \right. \\ & \left. |z-1| > \frac{1}{\gamma},\ \left| z - 1 - \frac{1}{\gamma^2} \right| > \frac{1}{\gamma},\ -3 - \frac{1}{2\gamma^2} \leq \mathrm{Re}\,(z) < 1 + \frac{3}{2\gamma^2} \right\}. \end{aligned}$$

We also have a quadrilateral whose opposite sides are identified by actions of A_γ, B_γ and which is identified with the once punctured torus:

$$D_q(\Gamma_\gamma) = \left\{ z \in \mathbb{H} \;\middle|\; \left| z + 1 + \frac{1}{2\gamma^2} \right| > \frac{1}{2\gamma^2},\ |z+1| > 1,\ -1 - \frac{1}{\gamma^2} \leq \mathrm{Re}\,(z) \leq 0 \right\}.$$

We call $D_o(\Gamma_\gamma)$ the octagonal fundamental domain for Γ_γ and $D_q(\Gamma_\gamma)$ the quadrilateral fundamental domain for Γ_γ (shown in normal outline in Fig. 5.3).

5 Investigation of fundamental domains

In this section we will introduce subregions of fundamental domains obtained in §4 and will define the mapping $J_{(X,Y,Z)}$ in Theorem 1.3 and 1.4 by using them.

5.1 The case M_1

In order to construct the holomorphic mapping we use the following modified fundamental domain $D(\Gamma_\alpha)$ (the shaded part in Fig. 5.1):

$$D(\Gamma_\alpha) = \left\{ z \in \mathbb{H} \;\middle|\; \left| z + \frac{3\sqrt{1+\alpha^2}}{\alpha} \right| \geq \frac{1}{\alpha},\ \left| z + \frac{2\sqrt{1+\alpha^2}}{\alpha} \right| \geq 1,\ \left| z + \frac{\sqrt{1+\alpha^2}}{\alpha} \right| \geq \frac{1}{\alpha}, \right.$$

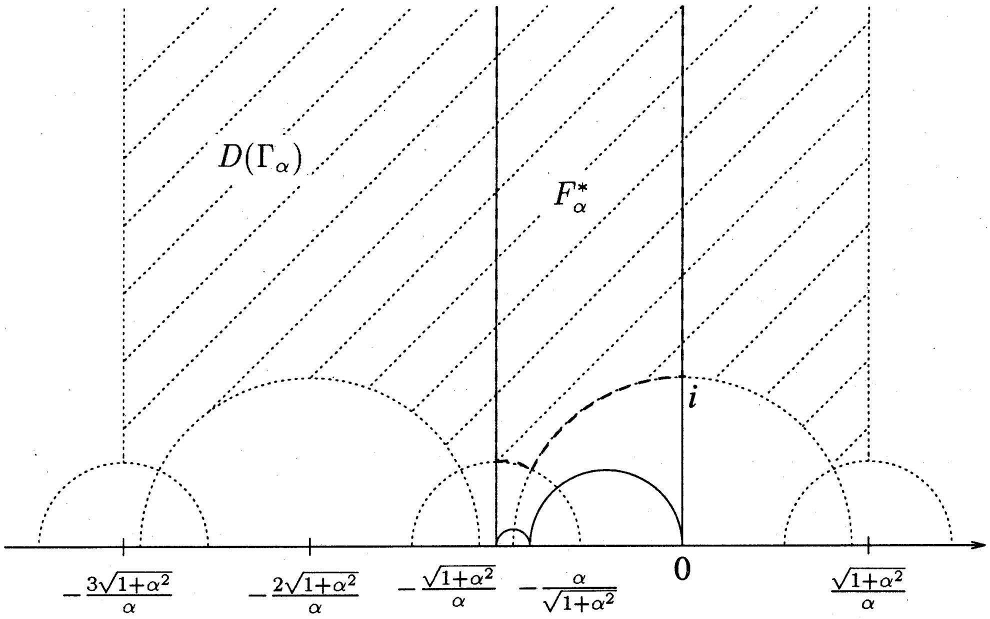


Fig. 5.1

$$|z| \geq 1, \left| z - \frac{\sqrt{1+\alpha^2}}{\alpha} \right| \geq \frac{1}{\alpha}, -\frac{3\sqrt{1+\alpha^2}}{\alpha} \leq \mathrm{Re}\,(z) \leq \frac{\sqrt{1+\alpha^2}}{\alpha} \Bigg\}.$$

We introduce the following notations:

$$F_\alpha^* = \left\{ z \in \mathbb{H} \;\middle|\; \left| z + \frac{\sqrt{1+\alpha^2}}{\alpha} \right| \geq \frac{1}{\alpha}, |z| \geq 1, -\frac{\sqrt{1+\alpha^2}}{\alpha} \leq \mathrm{Re}\,(z) \leq 0 \right\}$$

which is a subset of $D(\Gamma_\alpha)$,

$$\eta = \left(-\frac{\sqrt{1+\alpha^2}}{\alpha}, \frac{1}{\alpha} \right) \quad \text{and} \quad \zeta = \left(-\frac{\alpha}{\sqrt{1+\alpha^2}}, \frac{1}{\sqrt{1+\alpha^2}} \right).$$

The region F_α^* (shown in broken line in Fig. 5.1) is a quadrangle with angles $0, \pi/2, \pi/2, \pi/2$.

The holomorphic mapping used in Theorem 1.3 is constructed as follows:

Proposition 5.1 *We can construct a function J_α which satisfies*

(i) *J_α maps L_α onto $\mathbb{R}$ where $L_\alpha = L_{\alpha 1} \cup L_{\alpha 2} \cup L_{\alpha 3} \cup L_{\alpha 4}$ and*

$$L_{\alpha 1} = L_1, L_{\alpha 2} = \left\{ z \in \mathbb{H} \;\middle|\; |z| = 1 \;\; and \;\; -\frac{\alpha}{\sqrt{1+\alpha^2}} \leq \mathrm{Re}\,(z) \leq 0 \right\},$$

$$L_{\alpha 3} = \left\{ z \in \mathbb{H} \;\middle|\; \left| z + \frac{\sqrt{1+\alpha^2}}{\alpha} \right| = \frac{1}{\alpha} \;\; and \;\; -\frac{\sqrt{1+\alpha^2}}{\alpha} \leq \mathrm{Re}\,(z) \leq -\frac{\alpha}{\sqrt{1+\alpha^2}} \right\},$$

$$L_{\alpha 4} = \left\{ z \in \mathbb{H} \;\middle|\; |z| \geq 1 \;\; and \;\; \mathrm{Re}\,(z) = -\frac{\sqrt{1+\alpha^2}}{\alpha} \right\}.$$

Especially, $J_\alpha(i\infty) = \infty$, $J_\alpha(i) = P$ for some $P > 0$, $J_\alpha(\zeta) = 0$ and $J_\alpha(\eta) = -1$.

(ii) J_α *maps* F_α *onto* $\mathbb{C}$ *where*

$$F_\alpha = \left\{ z \in \mathbb{H} \;\middle|\; \left| z + \frac{\sqrt{1+\alpha^2}}{\alpha} \right| \geq \frac{1}{\alpha},\ |z| \geq 1,\ \left| z - \frac{\sqrt{1+\alpha^2}}{\alpha} \right| \geq \frac{1}{\alpha}, \right.$$
$$\left. -\frac{\sqrt{1+\alpha^2}}{\alpha} \leq \mathrm{Re}\,(z) \leq \frac{\sqrt{1+\alpha^2}}{\alpha} \right\}.$$

(iii) *The mapping* $J_\alpha : \mathbb{H} \to \mathbb{C}$ *is holomorphic on* $\mathbb{H}$.

5.2 The case M_2

We begin by introducing the following notations:

$$F_\beta = \left\{ z \in \mathbb{H} \;\middle|\; \left| z + \frac{1}{\beta} \right| \geq \frac{1}{\beta},\ \left| z - \frac{\beta}{2} \right| \geq \frac{\beta}{2},\ -\frac{1}{\beta} \leq \mathrm{Re}\,(z) \leq \frac{\beta}{2} \right\},$$

$$V_{\beta 1} = \left(-\frac{1}{\beta}, \frac{1}{\beta} \right), V_{\beta 2} = (0,0), V_{\beta 3} = \left(\frac{\beta}{2}, \frac{\beta}{2} \right).$$

The region F_β (shown in broken outline in Fig. 5.2) is a quadrilateral with angles $0, \pi/2, 0, \pi/2$.

Note that we change the octagonal fundamental $D_o(\Gamma_\beta)$ into

$$D(\Gamma_\beta) = \left\{ z \in \mathbb{H} \;\middle|\; \left| z + \beta + \frac{3}{\beta} \right| > \frac{1}{\beta},\ \left| z + \beta + \frac{2}{\beta} \right| \geq 1,\ \left| z + \frac{2}{\beta} \right| \geq 1,\ \left| z + \frac{1}{\beta} \right| \geq \frac{1}{\beta}, \right.$$
$$\left. |z| > 1,\ |z - \beta| > 1,\ \left| z - \beta - \frac{1}{\beta} \right| > \frac{1}{\beta},\ -\beta - \frac{3}{\beta} \leq \mathrm{Re}\,(z) < \beta + \frac{1}{\beta} \right\},$$

then $D(\Gamma_\beta)$ (the shaded part in Fig. 5.2) is also a fundamental domain for Γ_β.

Now we construct a holomorphic mapping from $\mathbb{H}$ to $\mathbb{C}$ which is used in Theorem 1.4.

Proposition 5.2 *We can construct a function* J_β *which satisfies the following conditions:*

(i) J_β *maps* L_β *onto* $\mathbb{R}$ *where* $L_\beta = L_{\beta 1} \cup L_{\beta 2} \cup L_{\beta 3} \cup L_{\beta 4}$ *and*

$$L_{\beta 1} = \left\{ z \in \mathbb{H} \;\middle|\; \left| z - \frac{\beta}{2} \right| \geq \frac{\beta}{2} \text{ and } \mathrm{Re}\,(z) = \frac{\beta}{2} \right\},$$

$$L_{\beta 2} = \left\{ z \in \mathbb{H} \;\middle|\; \left| z - \frac{\beta}{2} \right| = \frac{\beta}{2} \text{ and } 0 \leq \mathrm{Re}\,(z) \leq \frac{\beta}{2} \right\},$$

$$L_{\beta 3} = \left\{ z \in \mathbb{H} \;\middle|\; \left| z + \frac{1}{\beta} \right| = \frac{1}{\beta} \text{ and } -\frac{1}{\beta} \leq \mathrm{Re}\,(z) \leq 0 \right\},$$

$$L_{\beta 4} = \left\{ z \in \mathbb{H} \;\middle|\; \left| z + \frac{1}{\beta} \right| \geq \frac{1}{\beta} \text{ and } \mathrm{Re}\,(z) = -\frac{1}{\beta} \right\}.$$

Especially, $J_\beta(i\infty) = \infty$, $J_\beta(V_{\beta 3}) = P$ *for some* $P > 0$, $J_\beta(V_{\beta 2}) = 0$ *and* $J_\beta(V_{\beta 1}) = -1$.

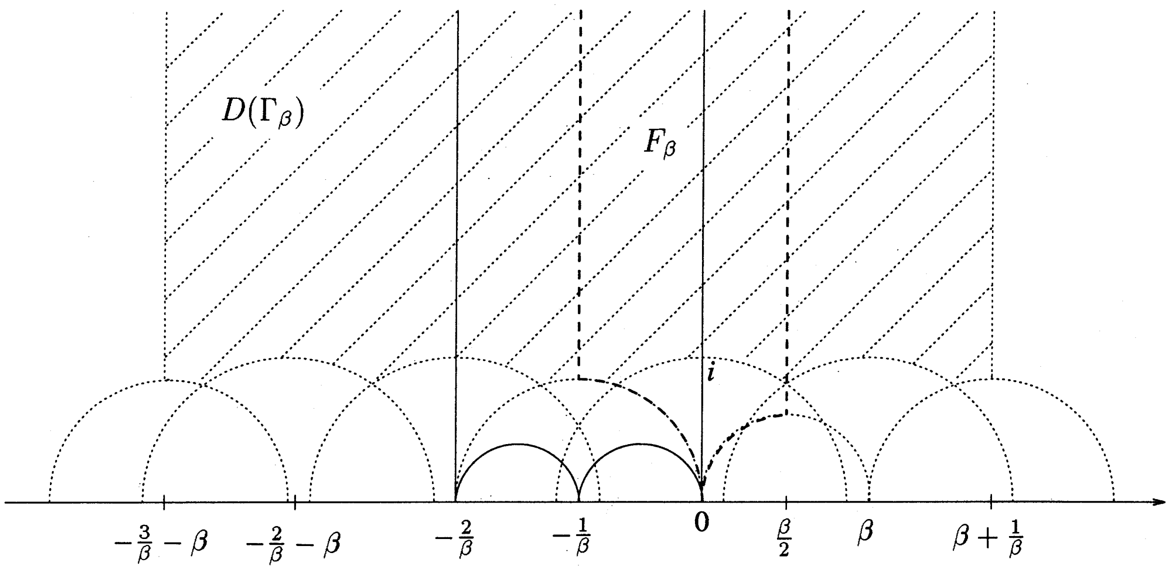


Fig. 5.2

(ii) *J_β maps $\hat{F}_\beta$ onto* $\mathbf{C}$ *where*

$$\hat{F}_\beta = \left\{ z \in \mathbb{H} \;\middle|\; \left|z+\frac{1}{\beta}\right| \geq \frac{1}{\beta},\ \left|z+\frac{\beta}{2}\right| \geq \frac{\beta}{2},\ \left|z-\beta-\frac{1}{\beta}\right| \geq \frac{1}{\beta}, \right.$$

$$\left. -\frac{1}{\beta} \leq \mathrm{Re}\,(z) \leq \beta + \frac{1}{\beta} \right\}.$$

(iii) *The mapping $J_\beta : \mathbb{H} \to \mathbf{C}$ is holomorphic on $\mathbb{H}$.*

5.3 The case M_3

We introduce the following notations:

$$F_\gamma = \left\{ z \in \mathbb{H} \;\middle|\; \left|z+1+\frac{1}{2\gamma^2}\right| \geq \frac{1}{2\gamma^2},\ |z| \geq 1,\ -1-\frac{1}{2\gamma^2} \leq \mathrm{Re}\,(z) \leq 0 \right\},$$

$$V_{\gamma 1} = \left(-1-\frac{1}{2\gamma^2}, \frac{1}{2\gamma^2}\right), V_{\gamma 2} = (-1,0), V_{\gamma 3} = (0,1).$$

The region F_γ (shown in broken outline in Fig. 5.3) is a quadrilateral with angles $0, \pi/2, 0, \pi/2$. Changing the octagonal fundamental domain $D_o(\Gamma_\gamma)$ into

$$\begin{aligned} D(\Gamma_\gamma) \;=\; & \left\{ z \in \mathbb{H} \;\middle|\; \left|z+3+\frac{1}{\gamma^2}\right| \geq \frac{1}{\gamma},\ \left|z+2+\frac{1}{\gamma^2}\right| \geq 1,\ \left|z+1+\frac{1}{\gamma^2}\right| \geq \frac{1}{\gamma}, \right. \\ & \left. |z+1| > \frac{1}{\gamma},\ |z| > 1,\ |z-1| > \frac{1}{\gamma},\ -3-\frac{3}{2\gamma^2} \leq \mathrm{Re}\,(z) < 1+\frac{1}{2\gamma^2} \right\}, \end{aligned}$$

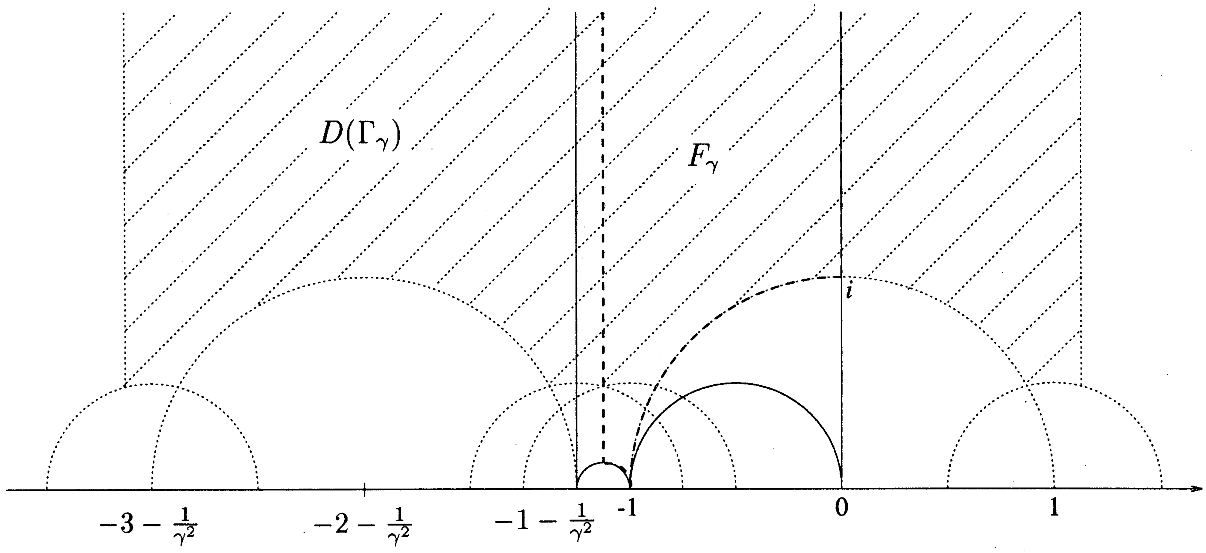


Fig. 5.3

we get that $D(\Gamma_\gamma)$ (the shaded part in Fig. 5.3) is also a fundamental domain for Γ_γ.

The mapping used in Theorem 1.4 is defined as follows:

Proposition 5.3 *We can construct a function J_γ which satisfies the following conditions:*

(i) *J_γ maps L_γ onto $\mathbb{R}$ where $L_\gamma = L_{\gamma 1} \cup L_{\gamma 2} \cup L_{\gamma 3} \cup L_{\gamma 4}$ and*

$$L_{\gamma 1} = \{z \in \mathbb{H} \mid |z| \geq 1 \ \ \textit{and} \ \operatorname{Re}(z) = 0\},$$

$$L_{\gamma 2} = \{z \in \mathbb{H} \mid |z| = 1 \ \ \textit{and} \ \ -1 \leq \operatorname{Re}(z) \leq 0\},$$

$$L_{\gamma 3} = \left\{z \in \mathbb{H} \;\middle|\; \left|z+1+\frac{1}{2\gamma^2}\right| = \frac{1}{2\gamma^2} \ \ \textit{and} \ \ -1-\frac{1}{2\gamma^2} \leq \operatorname{Re}(z) \leq -1\right\},$$

$$L_{\gamma 4} = \left\{z \in \mathbb{H} \;\middle|\; \left|z+1+\frac{1}{2\gamma^2}\right| \geq \frac{1}{2\gamma^2} \ \ \textit{and} \ \operatorname{Re}(z) = -1-\frac{1}{2\gamma^2}\right\}.$$

Especially, $J_\gamma(i\infty) = \infty$, $J_\gamma(V_{\gamma 3}) = P$ for some $P > 0$, $J_\gamma(V_{\gamma 2}) = 0$ and $J_\gamma(V_{\gamma 1}) = -1$.

(ii) *J_γ maps $\hat{F}_\gamma$ onto $\mathbb{C}$ where*

$$\hat{F}_\gamma = \left\{z \in \mathbb{H} \;\middle|\; \left|z+1+\frac{1}{2\gamma^2}\right| \geq \frac{1}{2\gamma^2},\ |z| \geq 1,\ \left|z-1-\frac{1}{2\gamma^2}\right| \geq \frac{1}{2\gamma^2},\right.$$
$$\left. -1-\frac{1}{2\gamma^2} \leq \operatorname{Re}(z) \leq 1+\frac{1}{2\gamma^2}\right\}.$$

(iii) *The mapping $J_\gamma : \mathbb{H} \to \mathbb{C}$ is holomorphic on $\mathbb{H}$.*

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