<table>
<thead>
<tr>
<th>Title</th>
<th>On stochastic dynamical systems leaving fields of geometric objects invariant: revisited (Invariants of Dynamical Systems and Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Akiyama, Hiroshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1072: 1-8 (1998)</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1998-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62589">http://hdl.handle.net/2433/62589</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
On stochastic dynamical systems leaving
fields of geometric objects invariant: revisited

(Hiroshi AKIYAMA (明山 浩))
Faculty of Engineering, Shizuoka University

1 Introduction

Given a stochastic dynamical system on a manifold described by a stochastic differential equation (cf. [5]), we obtain a condition for the stochastic dynamical system to have a "field of geometric objects" (not necessary a tensor field) as an invariant. We also study a stochastic dynamical system leaving a \(G\)-structure of degree \(r\) invariant. For this end, we use a generalized Itô's formula applicable to fields of geometric objects ([1]).

In §2, we recall the notion of fields of geometric objects of order \(r\), as well as the important notion of Lie differentiation of a field of geometric objects with respect to a vector field in the sense of Salvioli ([10]). In §3, using a generalized Itô’s formula, we obtain a condition for the stochastic flow of diffeomorphisms generated by a stochastic differential equation on a manifold to leave a field of geometric objects invariant. In particular, we also obtain a condition for such a stochastic dynamical system to leave a \(G\)-structure of degree \(r\) invariant. Several examples are given in §4.

Some results in this note are based on [1] and [3].

2 Fields of geometric objects

Let \(M\) be a \(\sigma\)-compact, \(n\)-dimensional \(C^\infty\) manifold, and let \(P^r(M)\) be the bundle of frames of \(r\)-th order contact over \(M\) with structure group \(G^r(n)\) and projection \(\pi\). Here, each point of \(P^r(M)\) is the \(r\)-jet \(j_0^r(f)\) at the origin \(0 \in \mathbb{R}^n\) given by a diffeomorphism \(f\) of an open neighborhood of \(0 \in \mathbb{R}^n\) onto an open set of \(M\), and
$\pi(j_{0}(f)) := f(0)$. Also,
\[
G^{r}(n) = \{ j_{0}(\psi) : \psi \text{ is a diffeomorphism of an open neighborhood of } 0 \in \mathbb{R}^{n} \text{ onto an open neighborhood of } 0 \in \mathbb{R}^{n} \text{ such that } \psi(0) = 0 \}
\]
acts on $P^{r}(M)$ on the right by the usual composition law of jets ([6]).

By a field of geometric objects (of order $r$), we shall mean a $C^{\infty}$ section of a fiber bundle $E$ associated with $P^{r}(M)$. (For simplicity, we assume that $E$ admits a global $C^{\infty}$ section and that the domain of the definition of a field of geometric objects is $M$.) Note that the $(C^{\infty})$ vector fields, differential forms, (usual) tensor fields, pseudo-tensor fields, oriented tensor fields, and tensor densities are examples of fields of geometric objects of order one, and the affine connections without torsion and the projective structures (over $M$) are examples of fields of geometric objects of order two.

In particular, when $G^{r}(n)$ acts transitively on the standard fiber $E_{0}$ of $E$ on the left, that is, $E_{0}$ is a homogeneous space $G^{r}(n)/G$ with $G$ a closed subgroup of $G^{r}(n)$, then $E = P^{r}(M)/G = P^{r}(M) \times_{G^{r}(n)} (G^{r}(n)/G)$, and the $G$-structures of degree (or order) $r$ (that is, the $G$-subbundles of $P^{r}(M)$) are in one-to-one correspondence with the fields of geometric objects $\sigma : M \to P^{r}(M)/G ([6])$.

Now let $\pi_{T(M)} : T(M) \to M$ denote the tangent bundle over $M$ and $T_{x}(M)$ the tangent space of $M$ at $x \in M$. Let $\tilde{\varphi}$ be the transformation of $P^{r}(M)$ induced naturally from a $C^{\infty}$ transformation $\varphi$ of $M$, that is,
\[
\tilde{\varphi}(j_{0}(f)) = j_{0}(\varphi \circ f), \quad j_{0}(f) \in P^{r}(M).
\]
Then $\tilde{\varphi}$ induces naturally a transformation $\varphi$ of $E$ such that $\pi_{E} \circ \tilde{\varphi} = \varphi \circ \pi_{E}$, where $\pi_{E} : E \to M$ is the projection. We define a section $\varphi^{*}\sigma : M \to E$ by $\varphi^{*}\sigma = \tilde{\varphi}^{-1} \circ \pi_{E} \circ \varphi$.

Correspondingly, a $C^{\infty}$ vector field $X : M \ni x \mapsto X(x) \in T_{x}(M)$ induces a $C^{\infty}$ vector field $\tilde{X}$ on $P^{r}(M)$ and a $C^{\infty}$ vector field $\tilde{X}$ on $E$, respectively, in a natural manner. In other words, $X$ generates a local one-parameter group of local transformations $\varphi_{t}$ of $M$, and $\varphi_{t}$ induces a local one-parameter group of local transformations $\tilde{\varphi}_{t}$ (resp. $\tilde{\varphi}_{t}$) of $P^{r}(M)$ (resp. $E$). Then $\tilde{X}$ (resp. $\tilde{X}$) is the vector field generating $\tilde{\varphi}_{t}$ (resp. $\tilde{\varphi}_{t}$). We set
\[
\varphi_{t}^{*}\sigma = (\tilde{\varphi}_{t})^{-1} \circ \sigma \circ \varphi_{t}.
\]
The vector field $\tilde{X}$ (resp. $\tilde{X}$) is called the natural lift of $X$ to $P^{r}(M)$ (resp. $E$) (see [8] (for the case $r = 1$) and [6]).

Define the Lie derivative
\[
\hat{L}_{X}\sigma : M \to T(E)
\]
of \( \sigma \) with respect to \( X \) in the sense of Salvioli by ([10])

\[
(\hat{L}_X \sigma)(x) := \frac{d}{dt} (\varphi_t^\ast \sigma)(x) \bigg|_{t=0} = \sigma_\ast(X(x)) - \bar{X}(\sigma(X)) \in T_{\sigma(x)}(E), \quad x \in M,
\]
where \( \sigma_\ast \) denotes the differential of the map \( \sigma : M \to E \). Note that \( (\hat{L}_X \sigma)(x) \) is tangent to the fiber of \( E \) through \( \sigma(x) \).

Let \( G \) be a closed subgroup of \( G^r(n) \) and let \( P \) be a \( G \)-structure of degree \( r \) on \( M \). A \( C^\infty \) transformation \( \varphi \) of \( M \) is called an automorphism of \( P \) if the induced transformation \( \tilde{\varphi} \) of \( P^r(M) \) maps \( P \) onto \( P \). We prepare the following lemma (see, e.g., [3]).

**Lemma 2.1** Let \( G \) be a closed subgroup of \( G^r(n) \). Let \( P \) be a \( G \)-structure of degree \( r \) on \( M \), and \( \sigma : M \to P^r(M)/G \) the field of geometric objects corresponding to \( P \). Then:

1. For a \( C^\infty \) transformation \( \varphi \) of \( M \), the \( G \)-structure of degree \( r \) corresponding to \( \varphi^\ast \sigma \) is given by \( \tilde{\varphi}^{-1}(P) \).
2. A \( C^\infty \) transformation \( \varphi \) of \( M \) is an automorphism of \( P \) \iff \( \varphi^\ast \sigma = \sigma \).
3. \( X \) is an infinitesimal automorphism of \( P \) \iff \( \hat{L}_X \sigma = 0 \).

## 3 Stochastic dynamical systems leaving fields of geometric objects invariant

Let \( M \) and \( G \) be as in §2. Let \( X_0, X_1, \ldots, X_k \) be \( C^\infty \) vector fields on \( M \). For each \( \lambda = 0, 1, \ldots, k \), let \( \bar{X}_\lambda \) be the natural lift of \( X_\lambda \) to \( P^r(M) \), and consider the following stochastic differential equation in the Stratonovich form:

\[
dp_t = \sum_{\lambda=0}^{k} \bar{X}_\lambda(p_t) \circ dw^\lambda_t. \tag{3.1}
\]

Here, \( w^0_t \equiv t \), and \( w_t = (w^1_t, \ldots, w^k_t) \) is the \( k \)-dimensional Wiener process realized canonically on the \( k \)-dimensional standard Wiener space. The solution with the initial condition \( p_s = p \in P^r(M) \) is denoted by \( p_{s,t}(p) = (p_{s,t}(p,w)) \), \( 0 \leq s \leq t \). Then \( p_{s,t} \) is a (stochastic) map \( P^r(M) \to P^r(M) \). Assume that \( p_{s,t} \) defines a stochastic flow of \( \langle C^\infty \rangle \) diffeomorphisms of \( P^r(M) \). Then \( p_{s,t} \) induces a stochastic flow \( \xi_{s,t} \) of diffeomorphisms of \( M \). Note that \( \xi_{s,t} \) is also generated by the following stochastic differential equation:

\[
d\xi_t = \sum_{\lambda=0}^{k} X_\lambda(\xi_t) \circ dw^\lambda_t.
\]
Note also that, for almost all $w$,
\[ p_{s,t}(j_0^b(f)) = j_0^b(\xi_s \circ f) = \xi_s (j_0^b(f)), \quad j_0^b(f) \in P^r(M), \quad 0 \leq s \leq t. \]
Moreover, $\xi_{s,t}$ generates a stochastic flow $\eta_{s,t}(= \tilde{\xi}_{s,t})$ of diffeomorphisms of $E$; $\eta_{s,t}$ is also generated by the following stochastic differential equation:
\[ d\eta_t = \sum_{\lambda=0}^{k} \bar{X}_\lambda(\eta_t) \circ dw^\lambda_t, \]
where $\bar{X}_\lambda$ is the natural lift of $X_\lambda$ to $E$.

We define the stochastic deformation of $\sigma$ by
\[ \xi_{s,t}^\# \sigma = \sigma \circ \xi_{s,t}. \]
We shall say that a field of geometric objects $\sigma : M \to E$ is an invariant of $\xi_{s,t}$ if
\[ \xi_{s,t}^\# \sigma = \sigma \text{ (a.s.)}. \]

**Theorem 3.1** Let $\sigma : M \to E$ be a field of geometric objects. Suppose the equation (3.1) generates a stochastic flow $p_{s,t}$ of $(C^\infty)$ diffeomorphisms of $P^r(M)$ (with probability 1). Then for the stochastic flow $\xi_{s,t}$ of diffeomorphisms of $M$ induced from $p_{s,t}$, it holds that
\[ \sigma \text{ is an invariant of } \xi_{s,t} \iff \hat{L}_{X_\lambda} \sigma = 0 \quad (\lambda = 0, 1, \ldots, k). \]

**Theorem 3.2** Let $G$ be a closed subgroup of $G^r(n)$, and let $P$ be a $G$-structure of degree $r$ on $M$. Let $\sigma : M \to P^r(M)/G$ be the field of geometric objects corresponding to $P$. Assume that the equation (3.1) generates a stochastic flow $p_{s,t}$ of diffeomorphisms of $P^r(M)$ (with probability 1). Then for the stochastic flow $\xi_{s,t}$ of diffeomorphisms of $M$ induced from $p_{s,t}$, it holds that
\[ \xi_{s,t} \text{ is a stochastic flow of automorphisms of } P \iff \hat{L}_{X_\lambda} \sigma = 0 \quad (\lambda = 0, 1, \ldots, k). \]

These theorems are proved by using the following theorem ([1]). (For a tangent vector or a vector field $Y$ on $N$, we denote by $Y[H]$ the operation of $Y$ on a $C^\infty$ function $H : N \to \mathbb{R}$.)

**Theorem 3.3** (Generalized Itô’s formula for $\xi_{s,t}^\# \sigma$) For a $C^\infty$ function $F : E \to \mathbb{R}$, it holds that
\[ F(\xi_{s,t}^\# \sigma)(x) - F \circ \sigma(x) = \sum_{\lambda=0}^{k} \Phi_\lambda_{s,t}(x, F) + \frac{1}{2} \sum_{\alpha=1}^{k} \int_{s}^{t} \left( X_\alpha(\xi_{s,u}(x)) \cdot [((\hat{L}_{X_\alpha} \sigma)(\cdot))(F \circ \eta_{s,u}^{-1})] - ((\hat{L}_{X_\alpha} \sigma) \circ \xi_{s,u}(x)) \cdot [X_\alpha(F \circ \eta_{s,u}^{-1})] \right) \cdot du, \quad (x \in M), \]
where
\[ \Phi_{s,t}^\lambda(x,F) := \int_s^t (\eta_{s,u})^{-1}((\hat{L}_x\sigma) \circ \xi_{S,u})[F] \cdot dw_u^\lambda, \]
and \( \cdot dw_u^\lambda \) denotes the Itô stochastic differential.

In the case where \( E \) is a vector bundle, we have the following.

**Theorem 3.4** ([1]) Let \( E \) be a vector bundle associated with \( P^r(M) \). Then, for a field of geometric objects \( \sigma : M \to E \),
\[ \xi_{s,t} \sigma - \sigma = \sum_{\alpha=1}^k \int_s^t \xi_{s,u}^\alpha \mathcal{L}_{X_0} \sigma \cdot dw_u^\alpha + \int_s^t \xi_{s,u}^\alpha \left[ \mathcal{L}_{X_0} + \frac{1}{2} \sum_{\alpha=1}^k (\mathcal{L}_{X_0})^2 \right] \sigma \, du. \]
Here
\[ \mathcal{L}_X \sigma = \lim_{t \to 0} \frac{1}{t} (\varphi_t^{-1} \circ \sigma \circ \varphi_t - \sigma) \]
under the notations in §2.

**Remark 3.1** In particular, in the case where \( \sigma \) is a tensor field, the corresponding result is also given in, e.g., [4] and [9].

## 4 Examples

**Example 4.1 Stochastic flow of projective transformations.**

Let \( P \) be a projective structure on \( M \) with \( \dim M = n \geq 2 \). (Therefore, \( r = 2 \) and \( G = H^2(n) \) in the sense of [6] and [7].) Then \( \xi_{s,t} \) is a stochastic flow of projective transformations of \( M \) with respect to \( P \) if and only if each \( X_\lambda \) is an infinitesimal projective transformation ([3]).

**Example 4.2 Stochastic dynamical system having an \( \ell \)-dimensional \( C^\infty \) distribution on \( M \) as an invariant.**

Let \( L(M)(= P^1(M)) \) be the bundle of linear frames over \( M \) (\( \dim M \geq 2 \)). Let \( \ell \in \mathbb{N} \) be such that \( 1 \leq \ell < n = \dim M \), and let \( G(n,\ell) \) be the Grassmann manifold formed of \( \ell \)-dimensional subspaces of \( \mathbb{R}^n \) ([8]). The general linear group \( G(n,\mathbb{R}) \) acts on \( G(n,\ell) \) on the left. Then we have a fiber bundle \( E \) (with standard fiber \( G(n,\ell) \) and structure group \( GL(n,\mathbb{R}) \)) associated with \( L(M) \). This \( E \) is called the (unoriented) **Grassmann bundle** of \( \ell \)-planes over \( M \) in the literature. The \( C^\infty \) sections of \( E \) are in one-to-one correspondence with the \( \ell \)-dimensional \( C^\infty \) distributions on \( M \).

Let \( \mathcal{D} \) be an \( \ell \)-dimensional \( C^\infty \) distribution on \( M \), and let \( \sigma_{\mathcal{D}} : M \to E \) be the field of geometric objects corresponding to \( \mathcal{D} \). Then we can define an \( \ell \)-dimensional
stochastic distribution $D_{s,t}$ (the stochastic deformation of $D$) as the (stochastic) distribution corresponding to $\xi_{s,t}^\# \sigma_D$, and it holds that

$D$ is an invariant of $\xi_{s,t} \iff \xi_{s,t}^\# \sigma_D = \sigma_D$ (a.s.) $\iff \hat{L}_{X_\lambda} \sigma_D = 0$ ($\lambda = 0, 1, \ldots, k$).

**Example 4.3** *Stochastic dynamical system leaving a second order linear (partial) differential operator on $C^\infty$ functions on $M$ invariant.*

Let $\mathbb{F} = (\mathbb{R}^n \oplus \mathbb{R}^n) \oplus \mathbb{R}^n \oplus \mathbb{R}^n (\cong \mathbb{R}^m, m = n(n+1)/2 + n + 1)$, where the symbol $\oplus$ stands for the symmetric tensor product. Define the action of $G^2(n)$ on $\mathbb{F}$ as follows:

$$(s^i_j; s_{jm}^i)(a^{ij}; b^i; c) = \left( \sum_{m,\ell=1}^{n} s^i_m s^j_{\ell} a^{m\ell}; \sum_{j,m=1}^{n} s^i_j a^{jm} + \sum_{j=1}^{n} s^j b^i; c \right),$$

where $(s^i_j; s_{jm}^i)$ and $(a^{ij}; b^i; c)$ are natural coordinates of $G^2(n)$ and $\mathbb{F}$, respectively. Then we obtain a vector bundle $E(M, \mathbb{F}, G^2(n), P^2(M))$ with standard fiber $\mathbb{F}$ and structure group $G^2(n)$, associated with $P^2(M)$. Each $C^\infty$ section $\sigma$ of $E$ corresponds to a second order (possibly degenerate) linear (partial) differential operator $A_\sigma$ on $\mathbb{R}$-valued $C^\infty$ functions on $M$ (cf. [2]). Each field of geometric objects $\sigma : M \to E$ is expressed locally as

$$\left( x^i; \left( \frac{\partial^2}{\partial x^i \partial x^j}; \frac{\partial}{\partial x^i} \right), (a^{ij}(x); b^i(x); c(x)) \right), (i, j = 1, \ldots, n),$$

and $A_\sigma$ is expressed locally as

$$A_\sigma = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x^i} + c(x).$$

It holds that

$A_\sigma$ is an invariant of $\xi_{s,t} \iff \mathcal{L}_{X_\lambda} \sigma = 0$ ($\lambda = 0, 1, \ldots, k$).

**Example 4.4** *Mean invariants.*

Let $E$ be a vector bundle associated with $P^r(M)$. Then for a field of geometric objects $\sigma : M \to E$, we have, by Theorem 3.4,

$\sigma$ is a **mean invariant** of $\xi_{s,t} \iff (\text{the expectation}) \mathbb{E}[\xi_{s,t}^\# \sigma] = \sigma$

$$\iff \left( \frac{1}{2} \sum_{\alpha=1}^{k} (\mathcal{L}_{X_\alpha})^2 + \mathcal{L}_{X_0} \right) \sigma = 0.$$

We also give the following example, although the setting is slightly different from that of §3.
Example 4.5 Random acceleration on a Riemannian manifold.

Let $(M,g)$ be a $C^\infty$ Riemannian manifold, and let $\Phi$ be the geodesic spray. Let $\hat{X}_\lambda$ be the vertical lift of $X_\lambda$ to $T(M)$, $\lambda = 0, 1, \ldots, k$; that is,

$$
(\hat{X}_\lambda)_v = \frac{d}{dt} (v + t(X_\lambda)_{\pi_T(M)}(v)) \bigg|_{t=0} \in T_v(T(M)).
$$

Consider the following stochastic differential equation on $T(M)$ (an equation of random acceleration):

$$
dV_t = (\Phi + \hat{X}_0)(V_t)dt + \sum_{\alpha=1}^{k} \hat{X}_{\alpha}(V_t) \circ dw^\alpha_t.
$$

Let $\theta$ be a $C^\infty$ differential 1-form on $M$, and define a $C^\infty$ function $F_\theta$ on $T(M)$ by

$$
F_\theta(v) = \theta(v), \quad v \in T(M).
$$

Then

$$
d(\theta(V_t)) = dF_\theta(V_t) = (\Phi + \hat{X}_0)[F_\theta](V_t)dt + \sum_{\alpha=1}^{k} \hat{X}_{\alpha}[F_\theta](V_t) \circ dw^\alpha_t.
$$

For $v \in T(M)$, we have

$$
\hat{X}_\lambda[F_\theta](v) = \frac{d}{dt} (v + t(X_\lambda)_{\pi_T(M)}(v)[F_\theta] \bigg|_{t=0} = \frac{d}{dt} \theta(v + t(X_\lambda)_{\pi_T(M)}(v)) \bigg|_{t=0} = \theta(X_\lambda)(\pi_T(M)(v)).
$$

Also, $\Phi[F_\theta](v)$ is expressed locally as follows:

$$
\left( \sum_{i=1}^{n} \frac{\partial}{\partial x^i} - \sum_{i=1}^{n} \sum_{j,m=1}^{n} \Gamma_{jm}^i v^j v^m \frac{\partial}{\partial v^i} \right) \left[ \sum_{\ell=1}^{n} \theta_v v^\ell \right]
= \sum_{i,j=1}^{n} \left( \frac{\partial \theta_j}{\partial x^i} - \sum_{m=1}^{n} \Gamma_{jm}^i \theta_m \right) v^i v^j = \frac{1}{2} \sum_{i,j=1}^{n} (\nabla_i \theta_j + \nabla_j \theta_i) v^i v^j,
$$

where $\nabla$ stands for covariant differentiation with respect to the Levi-Civita connection and $\Gamma_{jm}^i$ the connection coefficients. Therefore, if

$$
\nabla_i \theta_j + \nabla_j \theta_i = 0 \quad \text{(Killing equation)} \quad \text{and} \quad \theta(X_\lambda) = 0
$$

for $i, j = 1, \ldots, n$ and $\lambda = 0, 1, \ldots, k$, that is, if the vector field $X_\theta$ corresponding to $\theta$ through $g$ (namely, $g(X_\theta, \cdot) = \theta$) is a Killing vector field and is orthogonal to each $X_\lambda$ in the sense that $g(X_\theta, X_\lambda) = 0$, then $F_\theta$ is an invariant of the solution of the above equation of random acceleration.
References


