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Coordinates on the moduli of curves induced from Mumford's uniformization

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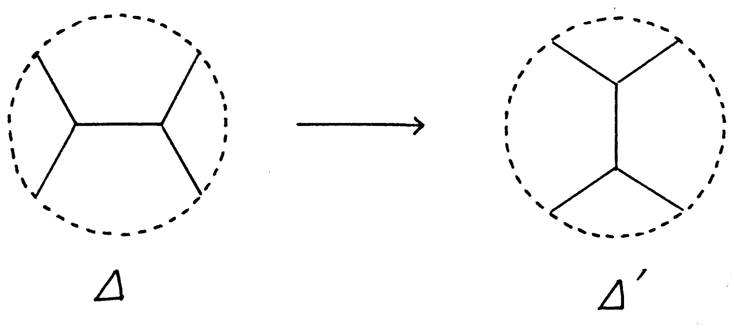
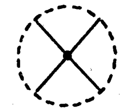
§0

In the celebrated paper [DM], Deligne and Mumford showed that the moduli space of (proper) smooth curves has a smooth compactification over $\mathbf{Z}$ as the moduli stack of stable curves by proving that the universal deformation of each stable curve has a smooth parameter space. Ihara and Nakamura [IhN] gave an explicit description of the universal deformation with parameter space over $\mathbf{Z}$ when the stable curve is a maximally degenerate curve consisting of smooth projective lines, and further, they used this deformation to study the Galois action on the algebraic fundamental group of the moduli of smooth curves. In [I], by extending the uniformization theory of curves by Mumford [M] to the case when base rings are nonlocal, for each stable curve consisting of smooth and singular projective lines, we construct its universal deformation which is Mumford uniformized.

In this note, we apply the result of [I] to construct formal neighborhoods over $\mathbf{Z}$ in the moduli of stable curves around certain paths which connect "points at infinity" corresponding to maximally degenerate curves. Let Δ be a (connected and finite) trivalent graph, and let Δ' be the graph obtained by changing a part of Δ as



Then the moduli space of degenerate curves with dual graph  becomes

$$\mathbf{P}^1 - \{3 \text{ points}\} \cong \mathbf{P}^1 - \{0, 1, \infty\}$$

by changing the coordinate. Let C_x denote the curve corresponding to $x \in \mathbf{P}^1 - \{0, 1, \infty\}$. Then a deformation of $\{C_x\}_x$ in the category of complex geometry is called a "duality" in conformal field theory. Here we will report that by using the Mumford uniformization theory over nonlocal base rings, we can construct a deformation of $\{C_x\}_x$ which is defined over a formal power series ring of several variables with $\mathbf{Z}[x, 1/x, 1/(1-x)]$-coefficients. Consequently, we have a formal neighborhood defined over $\mathbf{Z}$ around the x-line in the moduli space of stable curves.

We note that by sending certain deformation parameters to 0, our result can be easily generalized to the case when the base stable curve consists of smooth and singular projective lines with marked points. Its application to arithmetic geometry of the moduli space of marked curves (cf. [K]) will be given elsewhere.

§1

First, we consider a deformation of the maximally degenerate curve with dual graph Δ over a formal power series ring of several variables with $\mathbf{Z}$-coefficients, which had been constructed by Ihara and Nakamura when Δ has no loops (cf. [IhN], §2).

Theorem 1. *Let E be the set of edges of Δ, t_e variables parametrized by $e \in E$, and put*

$$A_\Delta = \mathbf{Z}[[t_e \ (e \in E)]], \quad B_\Delta = A_\Delta \left[\prod_{e \in E} \frac{1}{t_e} \right].$$

Then there exists a stable curve C_Δ over A_Δ of (arithmetic) genus $\mathrm{rank}_{\mathbf{Z}} H_1(\Delta, \mathbf{Z})$ *satisfies the following:*

(1) *The special fiber $C_\Delta \otimes_{A_\Delta} \mathbf{Z}$ of C_Δ obtained by sending $t_e \mapsto 0$ $(e \in E)$ is the maximally degenerate curve over $\mathbf{Z}$ with dual graph Δ.*

(2) *$C_\Delta \otimes_{A_\Delta} B_\Delta$ is smooth over B_Δ and is Mumford uniformized.*

(3) *If τ_e $(e \in E)$ are nonzero complex numbers with sufficiently small $|\tau_e|$, then $C_\Delta|_{t_e = \tau_e}$ becomes a Schottky uniformized Riemann surface.*

Remark. The Schottky group uniformizing C_Δ, which we denote by Γ_Δ, is described as follows. Fix an orientation of Δ, and let v_e (resp. v_{-e}) be the terminal (resp. initial) vertex of e (Picture: $v_{-e} \xrightarrow{e} v_e$). Then for a map

$$\lambda : \{\pm e \mid e \in E\} \rightarrow \{0, 1, \infty\}$$

satisfying that $\lambda(e) \neq \lambda(-e)$ and that $h \neq h', v_h = v_{h'} \Rightarrow \lambda(h) \neq \lambda(h')$, Γ_Δ is obtained as the image of

$$\iota : \pi_1(\Delta, v) \to PGL_2(B_\Delta); \quad \iota(h_1 \circ h_2 \circ \cdots \circ h_n) := \varphi_{h_1}\varphi_{h_2}\cdots\varphi_{h_n},$$

where for each $h \in \{\pm e \mid e \in E\}$,

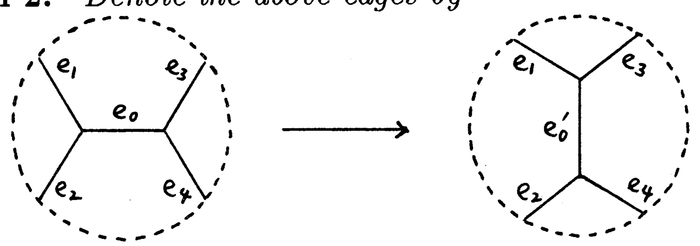
$$\varphi_h = \frac{1}{\lambda(h) - \lambda(-h)} \begin{pmatrix} \lambda(h) - \lambda(-h)t_h & -\lambda(h)\lambda(-h)(1-t_h) \\ 1 - t_h & -\lambda(-h) + \lambda(h)t_h \end{pmatrix} \bmod (B_\Delta)^\times$$

is the element of $PGL_2(B_\Delta)$ with fixed points $\lambda(\pm h)$ and multiplier t_h. In [I], under a more general setting, we show that 3 fixed points in $\mathbf{P}^1(B_\Delta)$ of $\Gamma_\Delta - \{1\}$ form naturally a tree T_Δ which is the universal cover of Δ with covering group Γ_Δ. Then C_Δ is obtained as the quotient by Γ_Δ of the glued scheme of $\mathbf{P}^1_{A_\Delta}$ associated with T_Δ especially using Grothendieck's formal existence theorem.

§2

Second, we consider a family of stable curves connecting C_Δ and $C_{\Delta'}$.

Theorem 2. *Denote the above edges by*



Δ $\quad\quad$ Δ'

and put $t_i = t_{e_i}$ $(0 \leq i \leq 4)$, $t'_0 = t_{e'_0}$. *Let* C_Δ/A_Δ *and* $C_{\Delta'}/A_{\Delta'}$ *be as in Theorem 1. Then there exists a Mumford uniformized stable curve* C *over the ring*

$$\mathbf{Z}\left[x, \frac{1}{x}, \frac{1}{1-x}\right][[s_i \ (1 \leq i \leq 4), \ s_e \ (e \in E'')]]; \ E'' := E - \{e_i \mid 0 \leq i \leq 4\}$$

satisfies the following:

(1) *Over* $\mathbf{Z}((x))[[s_i \ (1 \leq i \leq 4), \ s_e \ (e \in E'')]]$, C *is isomorphic to* C_Δ, *where the variables of the base rings are related as*

$$\frac{x}{t_0}, \ \frac{s_i}{t_0 t_i} \ (i = 1, 2), \ \frac{s_i}{t_i} \ (i = 3, 4), \ \frac{s_e}{t_e} \ (e \in E'') \ \in \ (A_\Delta)^\times.$$

(2) *Over* $\mathbf{Z}((1-x))[[s_i \ (1 \leq i \leq 4), \ s_e \ (e \in E'')]]$, C *is isomorphic to* $C_{\Delta'}$, *where the variables of the base rings are related as*

$$\frac{1-x}{t'_0}, \ \frac{s_i}{t'_0 t_i} \ (i = 1, 3), \ \frac{s_i}{t_i} \ (i = 2, 4), \ \frac{s_e}{t_e} \ (e \in E'') \ \in \ (A_{\Delta'})^\times.$$

Remark. The construction of $\mathcal{C}$ in Theorem 2 can be done by the method described in Remark of Theorem 1. The comparison between C_Δ, $C_{\Delta'}$ and $\mathcal{C}$ is obtained by caluculating cross ratios of fixed points of the Schottky groups uniformizing these deformations.

References

[DM] P. Deligne, D. Mumford: The irreducibility of the space of curves of given genus, Publ. Math. I.H.E.S. **36** (1969) 75-109.

[I] T. Ichikawa: Generalized Tate curve and integral Teichmüller modular forms, Preprint (1998).

[IhN] Y.Ihara, H.Nakamura: On deformation of maximally degenerate stable marked curves and Oda's problem, J. Reine Angew. Math. **487** (1997) 125-151.

[K] F. Knudsen: The projectivity of the moduli space of stable curves II, Math. Scand. **52** (1983) 161-199.

[M] D. Mumford: An analytic construction of degenerating curves over complete local rings, Compositio Math. **24** (1972) 129-174.