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# ON THE EXTENSIONS OF $\mathcal{W}_{n,A}$ BY $\mathbb{G}_{m,A}$

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## 1. MOTIVATION

Let $(A, \mathfrak{M})$ be a DVR with $K = Q(A)$ of characteristic 0 and $k = A/\mathfrak{M}$ of characteristic $p > 0$.

The so called Artin-Schreier-Witt exact sequence

$$(1) \qquad 0 \to \mathbb{Z}/p^n \to W_{n,k} \xrightarrow{F^n - \mathrm{id}} W_{n,k} \to 0$$

describes any étale $p^n$-cyclic coverings, where $W_{n,k}$ is the group scheme over $k$ of Witt vectors of length $n$ and $F$ is the Frobenius endomorphism.

On the other hand, when $K$ contains $\mu_{p^n}$, any étale $p^n$-cyclic coverings are descrived by the Kummer sequence

$$(2) \qquad 0 \to \boldsymbol{\mu}_{p^n,K} \to \mathbb{G}_{m.K} \xrightarrow{\theta_{p^n}} \mathbb{G}_{m,K} \to 0.$$

But we do not like to have two Gods (1) and (2) in the world.

In fact, we can constract a Kummer-Artin-Schreier-Witt exact sequence over DVR $A = \mathbb{Z}_{(p)}[\mu_{p^n}]$:

$$(3) \qquad 0 \to (\mathbb{Z}/p^n)_A \to \mathcal{W}_n \to \mathcal{W}_n/\,(\mathbb{Z}/p^n)_A \to 0$$

with an exact sequence of Kummer type as the generic fibre:

$$(4) \qquad 0 \to \boldsymbol{\mu}_{p^n,K} \to (\mathbb{G}_{m.K})^n \to (\mathbb{G}_{m,K})^n \to 0$$

and with (1) as the special fibre (cf. [12, 15]).

In $n = 1$ case, the exact sequence (3) is given explicitly as follows:

Let $\zeta$ be a primitive $p$-th root of unity, $\lambda = \zeta - 1$ and $A = \mathbb{Z}_{(p)}[\zeta]$. We define $\mathcal{W}_1$ by the group scheme

$$\mathcal{G}^{(\lambda)} = \mathrm{Spec}A[x, \frac{1}{\lambda x + 1}]$$

with group law $x \cdot y = x + y + \lambda xy$. Then (3) is given by

$$(5) \qquad 0 \to (\mathbb{Z}/p)_A \to \mathcal{G}^{(\lambda)} \xrightarrow{\Psi} \mathcal{G}^{(\lambda)}/(\mathbb{Z}/p) \cong \mathcal{G}(\lambda^p) \to 0,$$

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where $\Phi(x) = \frac{1}{\lambda^p}\{(\lambda x+1)^p - 1\}$. The exact sequence (3) for general $n$ is given by taking suitable extensions step by step starting from the exact sequence (5); that is to say, if we constract $\mathcal{W}_n$ for an $n$, then the next $\mathcal{W}_{n+1}$ is given by an extension of $\mathcal{W}_n$ by $\mathcal{G}^{(\lambda)}$:

$$0 \longrightarrow \mathcal{G}^{(\lambda)} \longrightarrow \mathcal{W}_{n+1} \longrightarrow \mathcal{W}_n \longrightarrow 0 \in \mathrm{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)}). \tag{6}$$

On the other hand, some matters concerning of $\mathcal{G}^{(\lambda)}$ can be calculated by using the exact sequence

$$0 \longrightarrow \mathcal{G}^{(\lambda)} \xrightarrow{\alpha^{(\lambda)}} \mathbb{G}_{m,A} \xrightarrow{r^{(\lambda)}} i_*\mathbb{G}_{A/\lambda} \longrightarrow 0, \tag{7}$$

where $i : \mathrm{Spec}A/\lambda \hookrightarrow \mathrm{Spec}A$ is the canonical inclusion, $\alpha^{(\lambda)}(x) = \lambda x + 1$ and $r^{(\lambda)}t \equiv t \mod \lambda$.

In fact, using this exact sequence (7), we can obtain a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}(\mathcal{W}_n, \mathcal{G}^{(\lambda)}) \longrightarrow \mathrm{Hom}(\mathcal{W}_n, \mathbb{G}_{m,A}) \longrightarrow \mathrm{Hom}(\mathcal{W}_n, i_*\mathbb{G}_{m,A/\lambda}) \\ &\xrightarrow{\partial} \mathrm{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)}) \longrightarrow \mathrm{Ext}^1(\mathcal{W}_n, \mathbb{G}_{m,A}). \end{aligned} \tag{8}$$

Here we have $\mathrm{Ext}^1(\mathcal{W}_n, \mathbb{G}_{m,A}) = 0$ by Hilbert theorem 90. Therefore for our purpose to search $\mathcal{W}_{n+1}$, to calculate

$$\mathrm{Hom}(\mathcal{W}_n, i_*\mathbb{G}_{m,A/\lambda}) \cong \mathrm{Hom}(\mathcal{W}_{n,A/\lambda}, \mathbb{G}_{m,A/\lambda})$$

is important.

Moreover to determine explicitly the quotient $\mathcal{W}_n/(\mathbb{Z}/p^n)_A$ is crucial when we apply our theory to the lifting problems of $p^n$-cyclic coverings of curves as was expanded by B. Green and M. Matignon [4]. When once we construct the quotient $\mathcal{W}_n/(\mathbb{Z}/p^n)_A$, the next one $\mathcal{W}_{n+1}/(\mathbb{Z}/p^{n+1})_A$ is given in $\mathrm{Ext}^1(\mathcal{W}_n/(\mathbb{Z}/p^n)_A, \mathcal{G}^{(\lambda^p)})$, and it is fixed explicitly by calculating $\mathrm{Hom}(\mathcal{W}_{n,A/\lambda^p}, \mathbb{G}_{m,A/\lambda^p})$.

Our aim of this report is to determine explicitly the groups $\mathrm{Hom}(\mathcal{W}_{n,A/\lambda}, \mathbb{G}_{m,A/\lambda})$ and $\mathrm{Ext}^1(\mathcal{W}_{n,A/\lambda}, \mathbb{G}_{m,A/\lambda})$.

## 2. The structure of $\mathcal{W}_n$

By using the exact sequence (8), for $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathfrak{M} \setminus \{0\}$, $\mathcal{W}_n$ can be written in the form:

(9)

$$\mathcal{W}_n = \mathrm{Spec}A[X_0, X_1, \dots, X_{n-1}, \frac{1}{\lambda_1 X_0 + 1}, \\ \frac{1}{\lambda_2 X_1 + D_1(X_0)}, \frac{1}{\lambda_3 X_2 + D_2(X_0, X_1)}, \dots \frac{1}{\lambda_n X_{n-1} + D_{n-1}(X_0, \dots, X_{n-2})}]$$

where for each $i = 1, 2, \ldots, n-1$,

$$D_i : \mathrm{Spec}A/\lambda_{i+1}[X_0, \ldots, X_{i-1}, \frac{1}{\lambda_1 X_0 + 1}, \ldots, \frac{1}{\lambda_i X_{i-1} + D_{i-1}}] \to \mathbb{G}_{m,A/\lambda_{i+1}}$$

is a homomorphism and we understand that $D_0 = 1$. The group law of $\mathcal{W}_n$ is that which makes the map

$$\begin{array}{rcl} \mathcal{W}_n & \to & (\mathbb{G}_{m,A})^n \\ \boldsymbol{x} = (x_0, \ldots, x_{n-1}) & \mapsto & (\lambda_1 x_1 + 1, \lambda_2 x_1 + D_1(\boldsymbol{x}), \ldots, \lambda_n x_{n-1} + D_{n-1}(\boldsymbol{x})) \end{array}$$

a homomorphism. One can refer to [15] for the details.

## 3. Deformed Artin-Hasse exponential series

Let $W_n$ (resp. $\widehat{W}_n$) be the group scheme (resp. the formal group scheme) over $\mathbb{Z}$ of Witt vectors of length $n$, and $W$ (resp. $\widehat{W}$) the group scheme (resp. the formal group scheme) of Witt vectors over $\mathbb{Z}$, and let $\mathbb{G}_m$ (resp. $\widehat{\mathbb{G}}_m$) be the multiplicative group scheme (resp. the multiplicative formal group scheme) over $\mathbb{Z}$.

We denote the Witt polynomials by

$$\begin{aligned} \Phi_0(\boldsymbol{X}) &= X_0 \\ \Phi_1(\boldsymbol{X}) &= X_0^p + pX_1 \\ &\vdots \\ \Phi_n(\boldsymbol{X}) &= X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n \\ &\vdots \end{aligned}$$

Let $F$ be the Frobenius endomorphism defined by

$$\begin{array}{rcccccc} F := \Phi^{-1} \circ \Phi^{(1)} : & W & \xrightarrow{\Phi^{(1)}} & \mathbb{G}_a^\infty & \xleftarrow{\Phi} & W \\ & \boldsymbol{x} & \longmapsto & (\Phi_1(\boldsymbol{x}), \Phi_2(\boldsymbol{x}), \ldots) & & \\ & & & (\Phi_0(\boldsymbol{y}), \Phi_1(\boldsymbol{y}), \ldots) & \longleftarrow & \boldsymbol{y} \end{array}$$

We note that $F$ is also an endomorphism of $\widehat{W}$.

For later use, we define a morphism $[p] : W \to W$ by

$$[p]\boldsymbol{b} := (0, b_0^p, b_1^p, \ldots)$$

for a vector $\boldsymbol{b} = (b_0, b_1, \ldots) \in W(A)$. Note that if $A$ is an $\mathbb{F}_p$-algebra, $[p]\boldsymbol{b}$ is nothing but $p\boldsymbol{b}$. Moreover, for a vector $\boldsymbol{a} = (a_0, a_1, \ldots) \in W(A)$, we define a map $T_{\boldsymbol{a}} : W(A) \to W(A)$ by

$$\Phi_n(T_{\boldsymbol{a}}\boldsymbol{b}) = a_0^{p^n}\Phi_n(\boldsymbol{b}) + pa_1^{p^{n-1}}\Phi_{n-1}(\boldsymbol{b}) + \cdots + p^n a_n \Phi_0(\boldsymbol{b})$$

for $\boldsymbol{b} \in W(A)$. Then we can easily see the following.

**Lemma 3.1.** *Actually, $T_{\boldsymbol{a}} : W(A) \longrightarrow W(A)$ is a well-defined endomorphism.*

Let $\overline{\Lambda}$ denote the Witt vector $(\Lambda, 0, 0, \dots)$ with coefficients in $\mathbb{Z}[\Lambda]$ and $F^{(\Lambda)}$ the endomorphism $F - \overline{\Lambda^{p-1}}$ of the group scheme $W_{\mathbb{Z}[\Lambda]}$.

The so called Artin-Hasse exponential series is given by

$$\begin{aligned} E_p(X) &:= \exp\left(X + \frac{X^p}{p} + \frac{X^{p^2}}{p^2} + \cdots\right) \\ &= e^X e^{\frac{X^p}{p}} e^{\frac{X^{p^2}}{p^2}} \cdots \in \mathbb{Z}_{(p)}[[X]]. \end{aligned}$$

Now we define a formal power series $E_p(U, \Lambda; X)$ in $\mathbb{Q}[U, \Lambda][[X]]$ by

$$E_p(U, \Lambda; X) := (1 + \Lambda X)^{\frac{U}{\Lambda}} \prod_{k=1}^{\infty} \left(1 + \Lambda^{p^k} X^{p^k}\right)^{\frac{1}{p^k}\left(\left(\frac{U}{\Lambda}\right)^{p^k} - \left(\frac{U}{\Lambda}\right)^{p^{k-1}}\right)}.$$

In our argument, one of the crucial points is to descide the integrality of this kind of series. For checking the integrality, Hazewinkel's lemma (cf. [2, §2]) is almost almighty in our case.

**Lemma 3.2** ([2, (2.3.3)]). *Let $A$ be an integral domain containing $\mathbb{Z}_{(p)}$, and $\sigma : K = Q(A) \longrightarrow K = Q(A)$ be a $\mathbb{Z}_{(p)}$-algebra homomorphism such that $\sigma(f) \equiv f \bmod pA$ for any $f \in A$. Let $d(X) = d_0 X + d_1 X^{p^1} + \cdots \in A[\frac{1}{p}][[X]]$. Then*

$$\exp\left(d(X)\right) = 1 + d(X) + \frac{1}{2!} d(X)^2 + \cdots \in A[[X]]$$

*if and only if there exist $b_i \in A$ $(i = 0, 1, \dots)$ such that $d_0 = b_0$, and $d_n = b_n + \frac{1}{p}\sigma(d_{n-1}) \in A$ for $n \geq 1$.*

By using this lemma, we can see that $E_p(U, \Lambda; X) \in \mathbb{Z}_{(p)}[U, \Lambda][[X]]$. Easily we can see that $E_p(1, 0; X) = E_p(X)$, that is to say, $E_p(U, \Lambda; X)$ gives a deformation of the Artin-Hasse exponential series $E_p(X)$.

Let $A$ be a $\mathbb{Z}_{(p)}$-algebra, $\lambda \in A$ and $\boldsymbol{a} = (a_0, a_1, \dots) \in W(A)$. We define a formal power series $E_p(\boldsymbol{a}, \lambda; X)$ in $A[[X]]$ by

$$\begin{aligned} (10) \qquad E_p(\boldsymbol{a}, \lambda; X) &:= \prod_{k=0}^{\infty} E_p(a_k, \lambda^{p^k}; X^{p^k}) \\ &= (1 + \lambda X)^{\frac{a_0}{\lambda}} \prod_{k=1}^{\infty} \left(1 + \lambda^{p^k} X^{p^k}\right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1} F^{(\lambda)} \boldsymbol{a}} \end{aligned}$$

Then the boundary of this power series $E_p(\boldsymbol{a},\lambda;X)$ is given by the following.

(11)

$$\begin{aligned}(\partial E_p(\boldsymbol{a},\lambda;\cdot))(X,Y) &= \frac{E_p(\boldsymbol{a},\lambda;X)E_p(\boldsymbol{a},\lambda;Y)}{E_p(\boldsymbol{a},\lambda;X+Y+\lambda XY)} \\ &= \prod_{k=1}^{\infty}\left(\frac{(1+\lambda^{p^k}X^{p^k})(1+\lambda^{p^k}Y^{p^k})}{1+\lambda^{p^k}(X+Y+\lambda XY)^{p^k}}\right)^{\frac{1}{p^k\lambda^{p^k}}\Phi_{k-1}F^{(\lambda)}\boldsymbol{a}}\end{aligned}$$

Now replacing $F^{(\lambda)}\boldsymbol{a}$ with a Witt vector $\boldsymbol{b}=(b_0,b_1,\dots)$ in the right hand side of (11), we define a cocyle as follows.

$$F_p(\boldsymbol{b},\lambda;X,Y) := \prod_{k=1}^{\infty}\left(\frac{(1+\lambda^{p^k}X^{p^k})(1+\lambda^{p^k}Y^{p^k})}{1+\lambda^{p^k}(X+Y+\lambda XY)^{p^k}}\right)^{\frac{1}{p^k\lambda^{p^k}}\Phi_{k-1}\boldsymbol{b}} \tag{12}$$

Again using the integrality lemma, we can see that

$$F_p(\boldsymbol{b},\lambda;X,Y)\in\mathbb{Z}_{(p)}[\boldsymbol{b},\lambda][[X,Y]].$$

## 4. Determination of $\mathrm{Hom}(\mathcal{W}_n,\mathbb{G}_{m,A})$ and $\mathrm{H}^2_0(\mathcal{W}_n,\mathbb{G}_{m,A})$

Let $A$ be a $\mathbb{Z}_{(p)}$-algebra and $\lambda\in A$. By (10) and (11), we can define homomorphisms

$$\xi_0^1 : \mathrm{Ker}(W(A)\xrightarrow{F^{(\lambda)}}W(A))\longrightarrow \mathrm{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}^{(\lambda)},\widehat{\mathbb{G}}_{m,A});\ \boldsymbol{a}\mapsto E_p(\boldsymbol{a},\lambda;X)$$

and, when $\lambda$ is nilpotent,

$$\xi_0^1 : \mathrm{Ker}(\widehat{W}(A)\xrightarrow{F^{(\lambda)}}\widehat{W}(A))\longrightarrow \mathrm{Hom}_{A\text{-gr}}(\mathcal{G}^{(\lambda)},\mathbb{G}_{m,A});\ \boldsymbol{a}\mapsto E_p(\boldsymbol{a},\lambda;X).$$

Moreover, by (12), we can define homomorphisms

$$\xi_1^1 : \mathrm{Coker}(W(A)\xrightarrow{F^{(\lambda)}}W(A))\longrightarrow \mathrm{H}^2_0(\widehat{\mathcal{G}}^{(\lambda)},\widehat{\mathbb{G}}_{m,A});\ \boldsymbol{a}\mapsto F_p(\boldsymbol{a},\lambda;X,Y)$$

and, when $\lambda$ is nilpotent,

$$\xi_1^1 : \mathrm{Coker}(\widehat{W}(A)\xrightarrow{F^{(\lambda)}}\widehat{W}(A))\longrightarrow \mathrm{H}^2_0(\mathcal{G}^{(\lambda)},\mathbb{G}_{m,A});\ \boldsymbol{a}\mapsto F_p(\boldsymbol{a},\lambda;X,Y).$$

Under these notations, we gave the result in the one-dimensional case in the previous paper [16] as in the following style.

**Theorem 4.1.** *Let $A$ be a $\mathbb{Z}_{(p)}$-algebra and $\lambda\in A$. Then the homomorphisms*

$$\xi_0^1 : \mathrm{Ker}(W(A)\xrightarrow{F^{(\lambda)}}W(A))\longrightarrow \mathrm{Hom}_{A\text{-}gr}(\widehat{\mathcal{G}}^{(\lambda)},\widehat{\mathbb{G}}_{m,A}),$$

$$\xi_1^1 : \mathrm{Coker}(W(A)\xrightarrow{F^{(\lambda)}}W(A))\longrightarrow \mathrm{H}^2_0(\widehat{\mathcal{G}}^{(\lambda)},\widehat{\mathbb{G}}_{m,A})$$

*are bijective. Moreover, if* $\lambda$ *is nilpotent, the homomorphisms*

$$\xi_0^1 : \mathrm{Ker}(\widehat{W}(A) \xrightarrow{F^{(\lambda)}} \widehat{W}(A)) \to \mathrm{Hom}_{A-gr}(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A}),$$
$$\xi_1^1 : \mathrm{Coker}(\widehat{W}(A) \xrightarrow{F^{(\lambda)}} \widehat{W}(A)) \to \mathrm{H}_0^2(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A})$$

*are bijective.*

For general $n$, we can cosider the both of $\mathrm{Hom}(\mathcal{W}_n, \mathbb{G}_{m,A})$, $\mathrm{H}_0^2(\mathcal{W}_n, \mathbb{G}_{m,A})$ and $\mathrm{Hom}(\widehat{\mathcal{W}}_n, \widehat{\mathbb{G}}_{m,A})$, $\mathrm{H}_0^2(\widehat{\mathcal{W}}_n, \widehat{\mathbb{G}}_{m,A})$, but for simplicity hereafter we treat the first them only.

Next we look at $n = 2$ case. Let $\lambda_1, \lambda_2 \in A$, and assume that $\lambda_1$ is nilpotent in $A/\lambda_2$. By (9) and Theorem 4.1, an extension

$$0 \to \mathcal{G}^{(\lambda_2)} \to \mathcal{W}_2 \to \mathcal{G}^{(\lambda_1)} \to 0 \quad \in \mathrm{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)}) \tag{13}$$

is given by

$$\mathcal{W}_2 = \mathrm{Spec}A[X, Y, \frac{1}{\lambda_1 X + 1}, \frac{1}{\lambda_2 Y + D(X)}] \tag{14}$$

where $D(X) = E_p(\boldsymbol{a}, \lambda_1; X)$ and $\boldsymbol{a} \in \widehat{W}(A/\lambda_2)$ with $F^{(\lambda_1)}\boldsymbol{a} = 0 \in \widehat{W}(A/\lambda_2)$. Now we put $F^{(\lambda_1)}\boldsymbol{a} = (\lambda_2 a_0', \lambda_2 a_1', \dots)$ and $\boldsymbol{a}' = (a_0', a_1', \dots)$. We define an endomorphism $U_2 = U_2(\lambda_1, \lambda_2; \boldsymbol{a}') : \widehat{W}(A)^2 \to \widehat{W}(A)^2$ by

$$U_2 \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = \begin{pmatrix} F^{(\lambda_1)} & -T_{\boldsymbol{a}'} \\ 0 & F^{(\lambda_2)} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = \begin{pmatrix} F^{(\lambda_1)}\boldsymbol{\alpha} - T_{\boldsymbol{a}'}\boldsymbol{\beta} \\ F^{(\lambda_2)}\boldsymbol{\beta} \end{pmatrix}.$$

For $\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} \in \widehat{W}(A)^2$, we define a formal power series $E_p(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X, Y)$ by

$$E_p(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X, Y) := E_p(\boldsymbol{\alpha}, \lambda_1; X) E_p(\boldsymbol{\beta}, \lambda_2; \frac{Y}{D(X)}).$$

When we put $(s, t) := (x_1, y_1) + (x_2, y_2) \in \mathcal{W}_2$, we can easily see that

$$s = x_1 + x_2 + \lambda_1 x_1 x_2,$$
$$\frac{t}{D(s)} = \frac{y_1}{D(x_1)} + \frac{y_2}{D(x_2)} + H_1(x_1, x_2) \in \mathcal{G}^{(\lambda_2)},$$

where

$$H_1(x_1, x_2) = \frac{1}{\lambda_2}\left\{\frac{D(x_1)D(x_2)}{D(x_1 + x_2 + \lambda_1 x_1 x_2)} - 1\right\}$$
$$= \frac{1}{\lambda_2}\left\{F_p(F^{(\lambda_1)}\boldsymbol{a}, \lambda_1; (x_1, x_2)) - 1\right\}.$$

Moreover, for $F = F_p(\boldsymbol{b}, \lambda_1; X_1, X_2)$, we define

$$[p]F := F_p([p]\boldsymbol{b}, \lambda_1; X_1, X_2),$$

and

$$G_p(\boldsymbol{\delta}, \lambda_2; F) := \prod_{k=1}^{\infty} \left( \frac{1+(F-1)^{p^k}}{[p]^k F} \right)^{\frac{1}{p^k \lambda_2^{p^k}} \Phi_{k-1}\boldsymbol{\delta}}.$$

Then using again the integrality lemma, we can see that

$$G_p(\boldsymbol{\delta}, \lambda_2; F) \in \mathbb{Z}_{(p)}[\boldsymbol{a}', \lambda_1, \lambda_2, \boldsymbol{\delta}][[X_1, X_2]].$$

Under these notations, we can show that the boundary of $E_p(\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}; X, Y)$ is given by

(15)

$$\begin{aligned}
&(\partial E_p(\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}; \cdot, \cdot))(X_1, Y_1, X_2, Y_2)\\
&= \frac{E_p(\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}; X_1, Y_1) E_p(\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}; X_2, Y_2)}{E_p(\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}; (X_1, Y_1) + (X_2, Y_2))}\\
&= F_p(F^{(\lambda_1)}\boldsymbol{\alpha} - T_{\boldsymbol{a}'}\boldsymbol{\beta}, \lambda_1; (X_1, X_2)) F_p(F^{(\lambda_2)}\boldsymbol{\beta}, \lambda_2; (\frac{Y_1}{D(X_1)}, \frac{Y_2}{D(X_2)})) \times\\
&\qquad F_p(F^{(\lambda_2)}\boldsymbol{\beta}, \lambda_2; (H_1, \frac{Y_1}{D(X_1)} + \frac{Y_2}{D(X_2)})) G_p(-F^{(\lambda_2)}\boldsymbol{\beta}, \lambda_2; F).
\end{aligned}$$

Now arranging the equality (15), we define a cocycle by

(16)

$$\begin{aligned}
&F_p(\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{\delta}\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}; (X_1, Y_1), (X_2, Y_2))\\
&= F_p(\boldsymbol{\gamma}, \lambda_1; (X_1, X_2)) F_p(\boldsymbol{\delta}, \lambda_2; (\frac{Y_1}{D(X_1)}, \frac{Y_2}{D(X_2)})) \times\\
&\qquad F_p(\boldsymbol{\delta}, \lambda_2; (H_1, \frac{Y_1}{D(X_1)} + \frac{Y_2}{D(X_2)})) G_p(-\boldsymbol{\delta}, \lambda_2; F).
\end{aligned}$$

By the equality (15), we can define the homomorphisms

$$\begin{aligned}
\xi_0^2 : &\mathrm{Ker}\left(\widehat{W}(A)^2 \xrightarrow{U_2} \widehat{W}(A)^2\right) \to \mathrm{Hom}(\mathcal{W}_2, \mathbb{G}_{m,A});\\
&\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix} \mapsto E_p(\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}; X, Y)
\end{aligned}$$

and

$$\xi_1^2 : \mathrm{Coker}\left(\widehat{W}(A)^2 \xrightarrow{U_2} \widehat{W}(A)^2\right) \longrightarrow \mathrm{H}_0^2(\mathcal{W}_2, \mathbb{G}_{m,A});$$

$$\begin{pmatrix}\gamma\\ \delta\end{pmatrix} \mapsto F_p(\begin{pmatrix}\gamma\\ \delta\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}; (X_1, Y_1), (X_2, Y_2)).$$

Then we can obtain a commutative diagram:

(17)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}F^{(\lambda_1)} & \longrightarrow & \mathrm{Ker}U_2 & \longrightarrow & \mathrm{Ker}F^{(\lambda_2)} & \longrightarrow & \\
 & & \Big\downarrow{\scriptstyle \xi_0^1} & & \Big\downarrow{\scriptstyle \xi_0^2} & & \Big\downarrow{\scriptstyle \xi_0^1} & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathcal{G}^{(\lambda_1)}, \mathbb{G}_{m,A}) & \longrightarrow & \mathrm{Hom}(\mathcal{W}_2, \mathbb{G}_{m,A}) & \longrightarrow & \mathrm{Hom}(\mathcal{G}^{(\lambda_2)}, \mathbb{G}_{m,A}) & \xrightarrow{\partial} & \\
 & \longrightarrow & \mathrm{Coker}F^{(\lambda_2)} & \longrightarrow & \mathrm{Coker}U_2 & \longrightarrow & \mathrm{Coker}F^{(\lambda_2)} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \xi_1^1} & & \Big\downarrow{\scriptstyle \xi_1^2} & & \Big\downarrow{\scriptstyle \xi_1^1} & & \\
 & \xrightarrow{\partial} & \mathrm{H}_0^2(\mathcal{G}^{(\lambda_1)}, \mathbb{G}_{m,A}) & \longrightarrow & \mathrm{H}_0^2(\mathcal{W}_2, \mathbb{G}_{m,A}) & \longrightarrow & \mathrm{H}_0^2(\mathcal{G}^{(\lambda_2)}, \mathbb{G}_{m,A}), & &
\end{array}$$

where the second horizontal line is the exact sequence deduced from (13), and the first horizontal line is the exact sequence defined by the following maps in order:

$$\alpha \mapsto \begin{pmatrix}\alpha\\ 0\end{pmatrix}, \quad \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \mapsto \beta, \quad \beta \mapsto T_{a'}\beta,$$

$$\gamma \mapsto \begin{pmatrix}\gamma\\ 0\end{pmatrix}, \quad \begin{pmatrix}\gamma\\ \delta\end{pmatrix} \mapsto \delta.$$

By the commutative diagram (17) and Theorem 4.1, we can show the following.

**Theorem 4.2.** *The homonmrphisms*

$$\xi_0^2 : \mathrm{Ker}\left(\widehat{W}(A)^2 \xrightarrow{U_2} \widehat{W}(A)^2\right) \longrightarrow \mathrm{Hom}(\mathcal{W}_2, \mathbb{G}_{m,A})$$

*and*

$$\xi_1^2 : \mathrm{Coker}\left(\widehat{W}(A)^2 \xrightarrow{U_2} \widehat{W}(A)^2\right) \longrightarrow \mathrm{H}_0^2(\mathcal{W}_2, \mathbb{G}_{m,A})$$

*are bijective.*

Befor we give the final form of the theorem, we will explain the situation by looking at the $n = 3$ case.

By (9), an extension

(18) $$0 \longrightarrow \mathcal{G}^{(\lambda_3)} \longrightarrow \mathcal{W}_3 \longrightarrow \mathcal{W}_2 \longrightarrow 0$$

is given by

$$\mathcal{W}_2 = \mathrm{Spec}A[X, Y, \frac{1}{1+\lambda_1 X}, \frac{1}{D_1(X)+\lambda_2 Y}]$$

and

$$\mathcal{W}_3 = \mathrm{Spec}A[X, Y, Z, \frac{1}{1+\lambda_1 X}, \frac{1}{D_1(X)+\lambda_2 Y}, \frac{1}{D_2(X,Y)+\lambda_3 Z}].$$

Moreover, by Theorems 4.1, 4.2, $D_1$ and $D_2$ are given by

$$D_1(X) = E_p(\boldsymbol{a}, \lambda_1; X),$$
$$D_2(X,Y) = E_p(\begin{pmatrix}\boldsymbol{b}\\ \boldsymbol{c}\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}; X, Y),$$

with

$$F^{(\lambda_1)}\boldsymbol{a} = 0 \in \widehat{W}(A/\lambda_2),$$
$$U_2(\begin{pmatrix}\boldsymbol{b}\\ \boldsymbol{c}\end{pmatrix}) = \begin{pmatrix}F^{(\lambda_1)} & -T_{\boldsymbol{a}'}\\ 0 & F^{(\lambda_2)}\end{pmatrix}\begin{pmatrix}\boldsymbol{b}\\ \boldsymbol{c}\end{pmatrix} = 0 \in \widehat{W}(A/\lambda_3),$$

where $\boldsymbol{a}'$ is defined by $(\lambda_2 a_0', \lambda_2 a_2', \dots) = F^{(\lambda_1)}\boldsymbol{a}$. Now we define $\boldsymbol{b}'$ and $\boldsymbol{c}'$ by $(\lambda_3 b_0', \lambda_3 b_2', \dots) = F^{(\lambda_2)}\boldsymbol{b} - T_{\boldsymbol{a}'}\boldsymbol{c}$ and $(\lambda_3 c_0', \lambda_3 c_1', \dots) = F^{(\lambda_3)}\boldsymbol{c}$.

Again we define a new power series by

$$E_p(\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}; X, Y, Z) = E_p(\boldsymbol{\alpha}, \lambda_1; X)E_p(\boldsymbol{\beta}, \lambda_2; \frac{Y}{D_1(X)})E_p(\boldsymbol{\gamma}, \lambda_3; \frac{Z}{D_2(X,Y)}).$$

Then the boundary of this series can be caluculated as follows.

$$\left(\partial E_p(\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}; \cdot, \cdot, \cdot)\right)(X_1, Y_1, Z_1, X_2, Y_2, Z_2) \tag{19}$$
$$= \frac{E_p(\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}; X_1, Y_1, Z_1)E_p(\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}; X_2, Y_2, Z_2)}{E_p(\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}; (X_1, Y_1, Z_1) + (X_2, Y_2, Z_2))}$$

$$= F_p(F^{(\lambda_1)}\boldsymbol{\alpha}, \lambda_1; X_1, X_2) F_p(F^{(\lambda_2)}\boldsymbol{\beta}, \lambda_2; \frac{Y_1}{D_1(X_1)}, \frac{Y_2}{D_1(X_2)}) \times$$
$$F_p(F^{(\lambda_2)}\boldsymbol{\beta}, \lambda_2; (H_1, \frac{Y_1}{D_1(X_1)} + \frac{Y_2}{D_1(X_2)}) E_p(\boldsymbol{\beta}, \lambda_2; H_1)^{-1} \times$$
$$F_p(F^{(\lambda_3)}\boldsymbol{\gamma}, \lambda_3; \frac{Z_1}{D_2(X_1, Y_1)}, \frac{Z_2}{D_2(X_2, Y_2)}) \times$$
$$F_p(F^{(\lambda_3)}\boldsymbol{\gamma}, \lambda_3; (H_2, \frac{Z_1}{D_2(X_1, Y_1)} + \frac{Z_2}{D_2(X_2, Y_2)})) E_p(\boldsymbol{\gamma}, \lambda_3; H_2)^{-1}$$

$$= F_p(F^{(\lambda_1)}\boldsymbol{\alpha} - T_{\boldsymbol{a}'}\boldsymbol{\beta} - T_{\boldsymbol{b}'}\boldsymbol{\gamma}, \lambda_1; X_1, X_2) \times$$
$$F_p(F^{(\lambda_2)}\boldsymbol{\beta} - T_{\boldsymbol{b}'}\boldsymbol{\beta}, \lambda_2; \frac{Y_1}{D_1(X_1)}, \frac{Y_2}{D_1(X_2)}) \times$$
$$F_p(F^{(\lambda_2)}\boldsymbol{\beta} - T_{\boldsymbol{b}'}\boldsymbol{\gamma}, \lambda_2; (H_1, \frac{Y_1}{D_1(X_1)} + \frac{Y_2}{D_1(X_2)})) \times$$
$$F_p(F^{(\lambda_3)}\boldsymbol{\gamma}, \lambda_3, \frac{Z_1}{D_2(X_1, Y_1)}, \frac{Z_2}{D_2(X_2, Y_2)}) \times$$
$$F_p(F^{(\lambda_3)}\boldsymbol{\gamma}, \lambda_3, (H_2, \frac{Z_1}{D_2(X_1, Y_1)} + \frac{Z_2}{D_2(X_2, Y_2)})) \times$$
$$G_p(F^{(\lambda_2)}\boldsymbol{\beta} - T_{\boldsymbol{b}'}\boldsymbol{\gamma}, \lambda_2; F_p(F^{(\lambda_1)}\boldsymbol{a}, \lambda_1; X_1, X_2))^{-1} \times$$
$$G_p(F^{(\lambda_3)}\boldsymbol{\gamma}, \lambda_3; \lambda_3 H_2 + 1)^{-1}.$$

Again arranging the equation (19), we define a cocycle by

$$F_p(\begin{pmatrix} \boldsymbol{\delta} \\ \boldsymbol{\epsilon} \\ \boldsymbol{\zeta} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; X_1, Y_1, Z_1, X_2, Y_2, Z_2)$$
$$= F_p(\boldsymbol{\delta}, \lambda_1; X_1, X_2) F_p(\boldsymbol{\epsilon}, \lambda_2; \frac{Y_1}{D_1(X_1)}, \frac{Y_2}{D_1(X_2)}) \times$$
$$F_p(\boldsymbol{\epsilon}, \lambda_2; (H_1, \frac{Y_1}{D_1(X_1)} + \frac{Y_2}{D_1(X_2)})) F_p(\boldsymbol{\zeta}, \lambda_3, \frac{Z_1}{D_2(X_1, Y_1)}, \frac{Z_2}{D_2(X_2, Y_2)}) \times$$
$$F_p(\boldsymbol{\zeta}, \lambda_3, (H_2, \frac{Z_1}{D_2(X_1, Y_1)} + \frac{Z_2}{D_2(X_2, Y_2)})) \times$$
$$G_p(\boldsymbol{\epsilon}, \lambda_2; F_p(F^{(\lambda_1)}\boldsymbol{a}, \lambda_1; X_1, X_2))^{-1} G_p(\boldsymbol{\zeta}, \lambda_3; \lambda_3 H_2 + 1)^{-1}.$$

Now we define an endomorphism $U_3 = U(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}') : \widehat{W}(A)^3 \longrightarrow \widehat{W}(A)^3$

by

$$U_3\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = (\begin{pmatrix}F^{(\lambda_1)} & -T_{\boldsymbol{a}'} & -T_{\boldsymbol{b}'}\\ 0 & F^{(\lambda_1)} & -T_{\boldsymbol{c}'}\\ 0 & 0 & F^{(\lambda_3)}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}F^{(\lambda_1)}\alpha - T_{\boldsymbol{a}'}\beta - T_{\boldsymbol{b}'}\gamma\\ F^{(\lambda_1)}\beta - T_{\boldsymbol{c}'}\gamma\\ F^{(\lambda_3)}\gamma\end{pmatrix}.$$

Then by (19), we can define homomorphisms

$$\xi_0^3 : \mathrm{Ker}(\widehat{W}(A)^3 \xrightarrow{U_3} \widehat{W}(A)^3) \longrightarrow \mathrm{Hom}(\mathcal{W}_3, \mathbb{G}_{m,A});$$

$$\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \mapsto E_p(\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}; X, Y, Z),$$

$$\xi_1^3 : \mathrm{Coker}(\widehat{W}(A)^3 \xrightarrow{U_3} \widehat{W}(A)^3) \longrightarrow \mathrm{H}_0^2(\mathcal{W}_3, \mathbb{G}_{m,A});$$

$$\begin{pmatrix}\delta\\ \epsilon\\ \zeta\end{pmatrix} \mapsto F_p(\begin{pmatrix}\delta\\ \epsilon\\ \zeta\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}; X_1, Y_1, Z_1, X_2, Y_2, Z_2).$$

Then similarly as in the case of $n = 2$, we have a commutative diagram:

(20)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}U_2 & \longrightarrow & \mathrm{Ker}U_3 & \longrightarrow & \mathrm{Ker}F^{(\lambda_3)} & \longrightarrow & \\
 & & \downarrow{\scriptstyle \xi_0^2} & & \downarrow{\scriptstyle \xi_0^3} & & \downarrow{\scriptstyle \xi_0^1} & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathcal{W}_2, \mathbb{G}_{m,A}) & \longrightarrow & \mathrm{Hom}(\mathcal{W}_3, \mathbb{G}_{m,A}) & \longrightarrow & \mathrm{Hom}(\mathcal{G}^{(\lambda_3)}, \mathbb{G}_{m,A}) & \xrightarrow{\partial} & \\
 & \longrightarrow & \mathrm{Coker}U_2 & \longrightarrow & \mathrm{Coker}U_3 & \longrightarrow & \mathrm{Coker}F^{(\lambda_3)} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \xi_1^2} & & \downarrow{\scriptstyle \xi_1^3} & & \downarrow{\scriptstyle \xi_1^1} & & \\
 & \xrightarrow{\partial} & \mathrm{H}_0^2(\mathcal{W}_2, \mathbb{G}_{m,A}) & \longrightarrow & \mathrm{H}_0^2(\mathcal{W}_3, \mathbb{G}_{m,A}) & \longrightarrow & \mathrm{H}_0^2(\mathcal{G}^{(\lambda_3)}, \mathbb{G}_{m,A}), & &
\end{array}$$

with the second exact horizontal line obtained by the exact sequence (18) and the first horizontai line defined in order as follows:

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \mapsto \begin{pmatrix}\alpha\\ \beta\\ 0\end{pmatrix}, \quad \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \mapsto \gamma, \quad \gamma \mapsto \begin{pmatrix}T_{\boldsymbol{b}'}\gamma\\ T_{\boldsymbol{c}'}\gamma\end{pmatrix},$$

$$\begin{pmatrix}\delta\\ \epsilon\end{pmatrix} \mapsto \begin{pmatrix}\delta\\ \epsilon\\ 0\end{pmatrix}, \quad \begin{pmatrix}\delta\\ \epsilon\\ \zeta\end{pmatrix} \mapsto \zeta.$$

By the commutative diagram (20) and Theorem 4.2, we have the result in the three-dimensional case.

**Theorem 4.3.** *The homonmrphisms*

$$\xi_0^3 : \mathrm{Ker}\left(\widehat{W}(A)^3 \xrightarrow{U_3} \widehat{W}(A)^3\right) \longrightarrow \mathrm{Hom}(\mathcal{W}_3, \mathbb{G}_{m,A})$$

*and*

$$\xi_1^3 : \mathrm{Coker}\left(\widehat{W}(A)^3 \xrightarrow{U_3} \widehat{W}(A)^3\right) \longrightarrow \mathrm{H}_0^2(\mathcal{W}_3, \mathbb{G}_{m,A})$$

*are bijective.*

Then as one can guess the general result, we have the final form as follows.

Let $\mathcal{W}_n$ is a group scheme over $A$ obtained succsessively by

$$0 \longrightarrow \mathcal{G}^{(\lambda_{i+1})} \longrightarrow \mathcal{W}_{i+1} \longrightarrow \mathcal{W}_i \longrightarrow 0, \tag{21}$$

for $i = 1, 2, \ldots, n-1$, where $\mathcal{W}_1 = \mathcal{G}^{(\lambda_1)}$. Then by (9), each $\mathcal{W}_i$ $(i = 1, 2, \ldots, n)$ is given by

$$\mathcal{W}_i = \mathrm{Spec} A[X_0, X_1, \ldots, X_{i-1}, \frac{1}{1+\lambda_1 X_0}, \frac{1}{D_1(X_0) + \lambda_2 X_1}, \ldots, \frac{1}{D_{i-1}(X_0, \ldots, X_{i-2}) + \lambda_i X_{i-1}}],$$

where $D_k(X_0, \ldots, X_{k-1}) : \mathcal{W}_{k, A/\lambda_{k+1}} \longrightarrow \mathbb{G}_{m, A/\lambda_{k+1}}$ is a homomorphism for $k = 1, \ldots, i-1$. Here we understand that $D_0 = 1$. Now we assume that for $1 \le i \le n-1$, each $D_i(X_0, X_1, \ldots, X_{i-1})$ is given by

$$D_i(X_0, X_1, \ldots, X_{i-1}) = E_p(\begin{pmatrix} \boldsymbol{a}_1^i \\ \boldsymbol{a}_2^i \\ \vdots \\ \boldsymbol{a}_i^i \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{pmatrix}; X_0, X_1, \ldots, X_{i-1})$$

$$:= \prod_{\ell=1}^{i} E_p(\boldsymbol{a}_i^\ell, \lambda_\ell; \frac{X_{\ell-1}}{D_{\ell-1}(X_0, \ldots, X_{\ell-2})}),$$

and

$$\begin{pmatrix} \boldsymbol{a}_1^i \\ \boldsymbol{a}_2^i \\ \vdots \\ \boldsymbol{a}_i^i \end{pmatrix} \in \mathrm{Ker}\left(U_i := \begin{pmatrix} F^{(\lambda_1)} & -T_{\boldsymbol{a}'^1_1} & -T_{\boldsymbol{a}'^2_1} & \cdots & -T_{\boldsymbol{a}'^{i-1}_1} \\ 0 & F^{(\lambda_2)} & -T_{\boldsymbol{a}'^2_2} & \cdots & -T_{\boldsymbol{a}'^{i-1}_2} \\ 0 & 0 & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & -T_{\boldsymbol{a}'^{i-1}_{i-1}} \\ bm0 & 0 & 0 & \ddots & F^{(\lambda_{i-1})} \end{pmatrix} : \widehat{W}(A/\lambda_i)^i \longrightarrow \widehat{W}(A/\lambda_i)^i\right).$$

Here the $\ell$-th components of $\boldsymbol{a}'^{i}_{j}$'s are defined inductively by

$$\lambda_i(\boldsymbol{a}'^{i}_{j})_\ell = \left(F^{(\lambda_j)}\boldsymbol{a}^i_j - \sum_{\ell=j}^{i-1} T_{\boldsymbol{a}'^{\ell}_{j}}\boldsymbol{a}^i_\ell\right)_\ell .$$

We put

$$H_i := \frac{1}{\lambda_{i+1}}\left(\frac{D_i(X_0,\dots,X_{i-1})D_i(Y_0,\dots,Y_{i-1})}{D_i((X_0,\dots,X_{i-1})+(Y_0,\dots,Y_{i-1}))} - 1\right).$$

Furthermore we define a formal power series by

$$F_p(\begin{pmatrix} \boldsymbol{b}^i_1 \\ \boldsymbol{b}^i_2 \\ \vdots \\ \boldsymbol{b}^i_i \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{pmatrix}; X_0, X_1, \dots, X_{i-1}, Y_0, Y_1, \dots, Y_{i-1})$$

$$:= \prod_{j=1}^{i} F_p(\boldsymbol{b}^i_j, \lambda_j : \frac{X_{j-1}}{D_{j-1}(X_0,\dots,X_{j-2})}, \frac{Y_{j-1}}{D_{j-}(Y_0,\dots,Y_{j-2})}) \times$$

$$\prod_{k=2}^{i} F_p(\boldsymbol{b}^i_k, \lambda_k; H_{k-1}, \frac{X_{k-1}}{D_{k-1}(X_0,\dots,X_{k-2})} + \frac{Y_{k-1}}{D_{k-1}(Y_0,\dots,Y_{k-2})}) \times$$

$$G_p(-\boldsymbol{b}^i_k, \lambda_k; \lambda_k H_{k-1} + 1).$$

Then we can show the following theorem inductively.

**Theorem 4.4.** *The homomorphisms*

$$\xi^i_0 : \mathrm{Ker}(\widehat{W}(A)^i \xrightarrow{U_i} \widehat{W}(A)^i) \longrightarrow \mathrm{Hom}(\mathcal{W}_i, \mathbb{G}_{m,A});$$

$$\begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \alpha^i \end{pmatrix} \mapsto E_p(\begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \beta^i \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{pmatrix}; X_0, X_1, \dots, X_{i-1})$$

*and*

$$\xi^i_1 : \mathrm{Coker}(\widehat{W}(A)^i \xrightarrow{U_i} \widehat{W}(A)^i) \longrightarrow \mathrm{H}^2_0(\mathcal{W}_i, \mathbb{G}_{m,A});$$

$$\begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^i \end{pmatrix} \mapsto F_p(\begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^i \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{pmatrix}; X_0, X_1, \dots, X_{i-1}, Y_0, Y_1, \dots, Y_{i-1})$$

*are bijective, for each i.*

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