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GEOMETRIC TOPOLOGY OF BANACH-MAZUR COMPACTA

by

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ABSTRACT. This is a survey on geometric topological properties of Banach-Mazur compacta Q(n). We begin by an introduction of this interesting class of spaces which has recently witnessed an intensive new development. Next, we list the main new results in this area, concerning local homotopical and general position properties of Q(n). In the last part we present the key ideas of the proofs. Also included are some unsolved problems and related conjectures.

1. Introduction

Banach-Mazur compacta lie in the intersection of two mathematical disciplines, namely *geometry of Banach spaces* [21] [25] [26] [31] and *infinite-dimensional topology* [16] [19] [24] [34]. Historically, first studies of these spaces concentrated on their *metric* properties, e.g. their *diameters*, *radii* at various centers, *distances* between particular points, etc. [31]. On the other hand, their *topological* structure was not well understood, except for the fact that they are contractible spaces. Notably the Polish school set forth some of the most challenging questions, e.g. are Q(n) absolute retracts (Q-manifolds) [19][34]? Recently, we have seen an upsurge of interest in this area and as a result some of these problems have been successfully solved (and as usually, several new appeared). This presented an opportunity for this survey.

Identify the set of all n-dimensional Banach spaces $BAN(n)$ with the set of all norms in $\mathbb{R}^n$. Define the *Banach-Mazur distance* $\rho : BAN(n) \times BAN(n) \longrightarrow \mathbb{R}_+$ for arbitrary pairs of Banach spaces $X = (\mathbb{R}^n, \|\ \|_X)$, $Y = (\mathbb{R}^n, \|\ \|_Y) \in BAN(n)$ as follows:

$$\rho(X,Y) = \inf\{\|T\| \cdot \|T^{-1}\| \mid T : X \to Y \text{ is an isomorphism}\}.$$

Then for every triple $X, Y, Z \in BAN(n)$, the following properties hold:

(1) $\rho(X,Z) \leq \rho(X,Y) \cdot \rho(Y,Z)$;

(2) $\rho(X,Y) = \rho(Y,X)$;

(3) $\rho(X,Y) \geq 1$; and

(4) $\rho(X,Y) = 1$ if and only if X and Y are *isometric*, $X \cong Y$, i.e. there is an isomorphism $T : X \to Y$ which preserves the norm: $\|x\|_X = \|T(x)\|_Y$, for every $x \in X$.

Clearly $d = \ln \rho$ is a *pseudo-metric* (cf.[17]) on $BAN(n)$, hence the equivalence $d(x,y) > 0 \Longleftrightarrow x \neq y$ need not be true. Let us verify the properties (1)-(4):

Ad(1) Let $X = (\mathbb{R}^n, \|\ \|_X)$, $Y = (\mathbb{R}^n, \|\ \|_Y)$ and $Z = (\mathbb{R}^n, \|\ \|_Z) \in BAN(n)$. Then for any pair of linear operators $X \xrightarrow{T} Y \xrightarrow{S} Z$ one has the inequality $\|S \circ T\| \leq \|S\| \cdot \|T\|$, since, by definition,

$$\|S\| = \sup_{\|y\|_Y \leq 1} \|S(y)\|_Z \quad \text{and} \quad \|T\| = \sup_{\|x\|_X \leq 1} \|T(x)\|_Y.$$

Hence, for isomorphisms S and T, we clearly get

$$\rho(X,Z) = \inf\{\|S \circ T\| \cdot \|(S\circ T)^{-1}\|\} \leq \inf\{\|S\| \cdot \|S^{-1}\| \cdot \|T\| \cdot \|T^{-1}\|\} =$$

$$= \inf\{\|S\| \cdot \|S^{-1}\|\} \cdot \inf\{\|T\| \cdot \|T^{-1}\|\} = \rho(X,Y) \cdot \rho(Y,Z).$$

Ad(2) This is obvious - replace T by T^{-1}.

Ad(3) For any isomorphism $T : X \to Y$ we have the inequality

$$1 = \|\mathrm{Id}_X\| = \|T \circ T^{-1}\| \leq \|T\| \cdot \|T^{-1}\|,$$

thus we get the inequality

$$\rho(X,Y) = \inf\{\|T\| \cdot \|T^{-1}\|\} \geq 1.$$

Ad(4) This is also obvious - recall that $\|T\| \cdot \|T^{-1}\| = 1$ if and only if $\|T(x)\|_Y = \|T\| \, \|x\|_X$, for every $x \in X$.

Define now an equivalence relation on $BAN(n)$ as follows: $X \sim Y$ if and only if $\rho(X,Y) = 1$ (equivalently, $\ln \rho(X,Y) = 0$) and introduce a metric into the quotient space

$$Q(n) = BAN(n) \,/_{\sim} = \{\text{all isometry classes of } n\text{−dim Banach spaces}\}$$

by $d([X],[Y]) = \ln \rho(X,Y)$.

It is easy to check that the function $d : Q(n) \times Q(n) \longrightarrow \mathbb{R}_+$ is *well-defined*, i.e. independent of the choice of representatives X and Y. Function d is indeed a *metric*. Let us check only the *Triangle inequality*: Given any $[X],[Y],[Z] \in Q(n)$, one calculates

$$d([X],[Z]) = \ln \rho(X,Z) \leq \ln(\rho(X,Y) \cdot \rho(Y,Z)) =$$

$$= \ln \rho(X,Y) + \ln \rho(Y,Z) = d([X],[Y]) + d([Y],[Z]).$$

The resulting metric space $(Q(n), d)$ turns out to be compact [22]. It is called the *Banach-Mazur compactum* and is usually written simply as $Q(n)$. ■

2. Representing $Q(n)$ as the orbit space

We shall present a different way of introducing $Q(n)$, namely as a decomposition (orbit) space of $C(n)$, where $C(n)$ is the space of all compact convex bodies V in $\mathbb{R}^n$, symmetric with respect to the origin 0 (see Figure 1).

We shall measure the distance between arbitrary subsets $A, B \subset \mathbb{R}^n$ by the *Hausdorff metric* $\rho_H(A,B) = \max_{a\in A, b\in B}\{\sup d(a,B), \sup d(A,b)\}$, where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is a fixed Euclidean metric [17] and we shall define linear combinations $\sum_{i=0}^n \lambda_i A_i$, for any $A_1, A_2, ..., A_n \in C(n)$, using the *Minkowski operation* [33], as follows: $\Sigma\lambda_i A_i = \{\Sigma\lambda_i a_i | a_i \in A_i\}$.

Then $(C(n), \rho_H)$ is a locally compact, convex infinite-dimensional space. Moreover, there exists an action $GL(n) \times C(n) \to C(n)$, of the general linear group, defined by $(T,V) \mapsto T(V)$, for any $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n \in GL(n)$ and $V \in C(n)$, which agrees with the

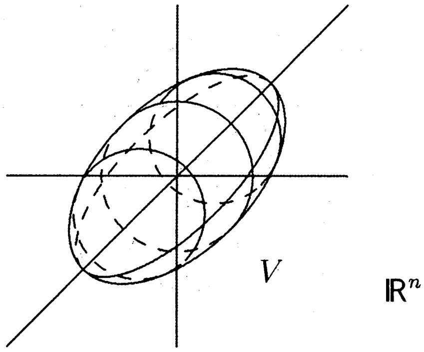

Figure 1

convex structure on $C(n)$. Hence $C(n)$ can be viewed as a disjoint union of the orbits $G(x) = \{g \cdot x \,|\, g \in GL(n)\}$.

We shall establish the existence of a homeomorphism $C(n)/GL(n) \cong Q(n)$. Given an arbitrary body $V \in C(n)$, define the *Minkowski functional* $p_V : \mathbb{R}^n \to \mathbb{R}_+$ by $p_V(x) = \inf\{\frac{1}{t} \mid t \cdot x \in V\}$ (see Figure 2) [30].

This yields a norm on $\mathbb{R}^n$, $p_V : \mathbb{R}^n \to \mathbb{R}$, given by $\|x\| = p_V(x)$, for every $x \in \mathbb{R}^n$. Define $M : C(n) \to BAN(n)$ by $M(V) = (\mathbb{R}^n, p_V)$. Notice that the inverse map is defined by sending $(\mathbb{R}^n, \|\ \ \|)$ to the unit ball B^n with respect to $\|\ \ \|$. Then M is a continuous surjective map, in fact a bijection.

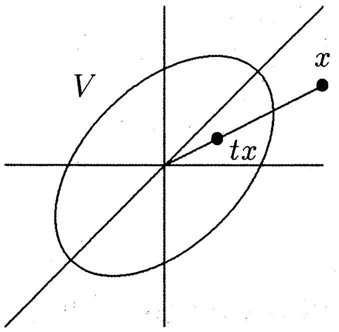

Figure 2

Clearly (see (4) above)), for any two n-dimensional Banach spaces X and Y, X and Y are isometric, $X \cong Y$ if and only if there exists $T \in GL(n)$ such that $V = T(W)$, where $X = (\mathbb{R}^n, p_V)$ and $Y = (\mathbb{R}^n, p_W)$ (see Figure 3).

Observe that $M(V) \sim M(W) \iff V = T(W)$. Therefore M induces a continuous bijection, hence a homeomorphism

$$\tilde{M} : C(n)/GL(n) \longrightarrow BAN(n)/_{\sim} \equiv Q(n).$$

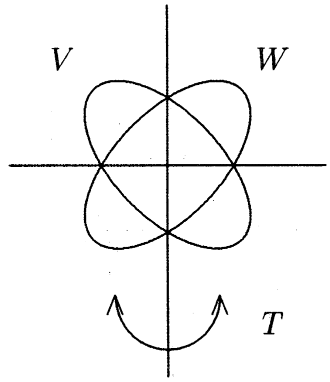

Figure 3

For an illustration, think of V and W as points on the same $GL(n)$-orbit. Then along this orbit, containing V and W, one can move from V to W via an appropriate linear operator $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $T(V) = W$, so we clearly have an isometry $T : (\mathbb{R}^n, \| \ \|_V) \to (\mathbb{R}^n, \| \ \|_W)$ (see Figure 4).

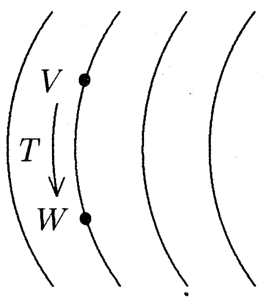

Figure 4

3. The Löwner ellipsoid

The benefit of the alternative presentation of $Q(n)$ is that it becomes possible to study Banach-Mazur compacta via *convex bodies* [15][21][23], i.e. instead of Banach spaces we study spaces of convex bodies, where a significant tool has existed since 1930's – the *Löwner ellipsoid* [22].

For any $V \in C(n)$ there exists (a unique) ellipsoid $E_V \subset \mathbb{R}^n$ such that

(1) $V \subset L_V$ (there is also a version where $J_V \subset V$);

(2) E_V has the minimal (resp. maximal) volume; and

(3) E_V is centrally symmetric.

Therefore we have a correspondence $\mathcal{L} : C(n) \to \mathcal{E} = \{\text{ellipsoids}\}$, given by $V \mapsto E_V$ such that:

(4) $\mathcal{L}$ is continuous in the Hausdorff metric ρ_H on $C(n)$.
(5) $\mathcal{L}$ is $GL(n)$-invariant, i.e. if $T : V \to W$ then $T(E_V) = E_{T(V)}$.

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, "T"'] \arrow[dr, phantom, "\circlearrowright"] & E_V \arrow[d] \\ W \arrow[r, hook] & E_W \end{tikzcd}$$

So $\mathcal{L}$ preserves the action of $GL(n)$. Let $\mathcal{E}$ be the orbit of a special convex body - the unit ball B^n. Hence, $\mathcal{L} : C(n) \to GL(n) \cdot B^n$ is a retraction onto the elliptic orbit. Let $E(n) = \mathcal{L}^{-1}(B^n)$. Then every $V \in E(n)$ embeds in $E_V = B^n$. Thus

$$L(n) = \{\text{all convex bodies } V \text{ whose Löwner ellipsoids coincide with } B^n\}$$

and hence $E(n)$ preserves the action of the subgroup $O(n) \subset GL(n)$ and $(GL(n)-\text{orbits}) \cap (E(n) = O(n) - \text{orbits})$. Therefore $C(n)/GL(n) = E(n)/O(n) = Q(n)$ (see Figure 5).

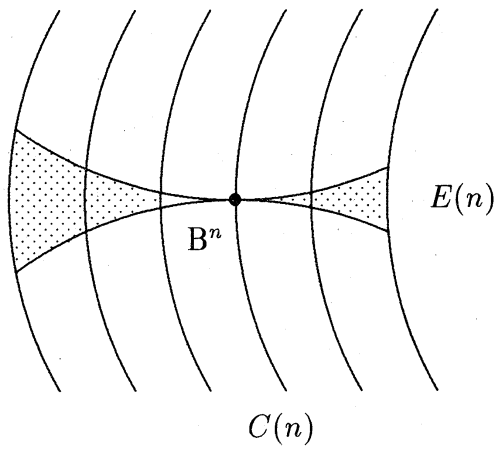

Figure 5

Question (3.1) *Is $\mathcal{L} : C(n) \to \mathcal{E}$ a Lipschitz map?*

4. Main questions concerning $Q(n)$

Question (4.1) *Evaluation of the diameter of $Q(n)$:* A classical result [22] asserts that diam $Q(n) \leq \ln n$, for every n. An asymptotic estimate due to Gluškin [20] is that for some constant $c > 0$, $c \ln n \leq$ diam $Q(n) \leq \ln n$. For more on this and related problems see [31].

Question (4.2) *Contractibility of $Q(n)$.* Solved by Milman in the 1960's - he proved that $Q(n) \simeq *$.

Question (4.3) *Is $Q(n)$ a retract of the Hilbert cube?* The answer is affirmative, since $Q(n)$ is AE: for $n = 2$ this is due to Fabel [18], for any $n \geq 3$ due independently, to Antonyan [11] and Ageev-Bogatyi-Fabel [6] (for an alternative proof see [7]).

Question (4.4) *Is $Q(n)$ homeomorphic to the Hilbert cube?* The answer is negative for (at least) $n = 2$, since $Q(2) \not\cong I^\infty$, as shown by Ageev-Bogatyi [4][5].

Question (4.5) *Is $Q(n) \setminus \{\mathcal{E}\}$, where $\mathcal{E}$ is the Euclidean point, a Hilbert cube manifold?* The answer is affirmative for (at least) $n = 2$ as shown by Ageev-Repovš [9].

Question (4.6) *Is $Q(n)$ a topologically homogeneous space?* The answer is negative for (at least) $n = 2$, as shown by Ageev-Repovš [9].

5. Outlines of the proofs

Theorem (5.1) $Q(n) \simeq *$.

Proof. Recall that $\mathcal{L} : C(n) \longrightarrow C(n)$ is a continuous map, it preserves the $GL(n)$ - action and is a retraction onto the set of all ellipsoids. We shall invoke now the following:

Millman trick (5.2) *For any convex body $V \in C(n)$ and any $t \in [0,1]$ define $H(V,t) = t \cdot V + (1-t) \cdot E_V$ (i.e. Minkowski linear combination): Then the map $H : C(n) \times [0,1] \longrightarrow C(n)$ has the following properties: (1) H is continuous; (2) $H_0 = \mathcal{L}$; (3) $H_1 = Id$; (4) H preserves the $GL(n)$ action; and (5) $H_t|_{\mathcal{E}} = \mathrm{Id}$, for every $t \in [0,1]$.* ∎

Then H induces a map on the orbit space

$$\tilde{H} : C(n)/GL(n) \times [0,1] \longrightarrow C(n)/GL(n) = Q(n)$$

such that

$$\tilde{H}([V],t) = [H(V,t)]\ , \text{ for every } V \in C(n) \text{ and } t \in [0,1]$$

Clearly, $\tilde{H}$ is continuous and has the following properties: (1) $\tilde{H}_0$ is constant; (2) $\tilde{H}_1$ is identity; and (3) $\tilde{H}_t|_{[\mathcal{E}]}$ is identity, for every t. Hence $\tilde{H}$ is a contraction of $Q(n)$ to a point. ∎

Theorem (5.3) *$Q(n)$ is an AR.*

Proof. Consider the following commutative diagram:

$$Q(n) = C(n)/GL(n) = E(n)/O(n) \hookrightarrow C(n)/O(n)$$

where

$$r(O(n) \cdot V) = GL(n) \cdot V$$

Therefore $Q(n)$ is a retract of $C(n)/O(n)$. So in order to prove that $Q(n)$ is indeed an AR it suffices to verify the following:

Assertion (5.4) *$C(n)/O(n)$ is an AR.*

Proof. Recall the following facts:
(1) $O(n)$ is a compact Lie group; and
(2) $C(n)$ is a space with a convex structure (defined via the Minkowski operation) and this convex structure preserves the action of the group $GL(n)$.

Murayama [27] proved that $C(n)$ is an $O(n)$-AR and Antonyan [10] proved that $X \in G$-AR, for any compact Lie group G implies $X/G \in$ AR. These two results together yield that $C(n)/O(n) \in$ AR, as asserted. ∎

The key here is that the group $O(n)$ is compact, because [10] and [27] treated only the compact case. Ageev-Repovš [8] (see also [7]) proved a more general fact, namely that
(1) $C(n)$ is $GL(n)$-AR; and
(2) $C(n)/GL(n) \in$ AR
and they also gave an alternative proof of Theorem (5.3).

Theorem (5.5) $Q(2) \not\cong \mathrm{I}^\infty$.

Proof. The argument consists of seven steps (every assertion is reduced to the next one). Let $Q'(2) = Q(2) \setminus \{\mathcal{E}\}$ and $C'(2) = C(2) \setminus \mathcal{E}$.

Assertion (5.6) $Q'(2) \not\simeq *$.

Assertion (5.7) $H^4(Q'(2); \mathbb{Q}) \neq 0$.

Assertion (5.8) $Q'(2) = C'(2)/GL(2)$ *is the orbit space of* $C'(2)/GL^+(2)$*, which is the Eilenberg-MacLane complex* $K(\mathbb{Q}, 2)$*, with respect to the action of* $\mathbb{Z}_2 = GL(2)/GL^+(2)$.

Assertion (5.9) *The orbit space of arbitrary involution on the Eilenberg-MacLane complex* $K(\mathbb{Q}, 2)$ *has nontrivial cohomology,* $H^4(K(\mathbb{Q}, 2)/\mathbb{Z}_2; \mathbb{Q}) \neq 0$.

Assertion (5.10) $C'(2)/GL^+(2) = K(\mathbb{Q}, 2)$.

Assertion (5.11) $C'(2)/SO(2) = K(\mathbb{Q}, 2)$.

Assertion (5.12) $C'(2)/SO(2) = \bigcup_{k=1}^{\infty} F_k$, $F_k^{\text{open}} \subset C'(2)$, $F_k = K(\mathbb{Z}, 2)$, $F_k \subset F_l \iff l|k$, *and the homomorphism* $\Pi_2(F_k) \to \Pi_2(F_l)$ *of the homotopy groups coincides with multiplication on* $\mathbb{Z}$ *by* $l|k$.

Suppose now that to the contrary, $Q(2)$ were homeomorphic to I^∞. Then one would have (invoking Assertion (5.6)) the following contradiction

$$* \not\simeq Q(2) \setminus \{[\mathcal{E}]\} \cong \mathrm{I}^\infty \setminus \{\text{point}\} \cong \mathrm{I}^\infty \times [0,1) \simeq *. \ \blacksquare$$

Remark (5.13) This result has also been announced by Antonyan [12].

Recall that the Hilbert cube $\mathrm{I}^\infty = \prod_{i=1}^\infty [0,1]$ (originally defined as $\{(x_i) | \sum_{i=1}^\infty x_i^2 < \infty$ and $|x_i| \le \frac{1}{2^i}$, for every $i\}$) has the following two properties: (1) $\mathrm{I}^\infty \in \mathrm{AR}$; and (2) I^∞ possesses the Disjoint m-disks property, for every m, i.e. for every $\varepsilon > 0$ and $f_i : D^m \to \mathrm{I}^\infty$, $i \in 1,2$, there exist $f_i' : D^m \to \mathrm{I}^\infty$ such that $d(f_i, f_i') < \varepsilon$ and $\mathrm{Im} f_1' \cap \mathrm{Im} f_2' = \emptyset$.

Indeed, since obviously for every $\varepsilon > 0$ there exist $f_i : \mathrm{I}^\infty \to \mathrm{I}^\infty$, $i \in \{1,2\}$, such that $d(f_i, id) < \varepsilon$ and $\mathrm{Im} f_1 \cap \mathrm{Im} f_2 = \emptyset$: just map once into

$$(\prod_1^N [0,1]) \times \{0\} \times \{0\} \times \{0\} \times \ldots$$

and the second time to

$$(\prod_1^N [0,1]) \times \{1\} \times \{1\} \times \{1\} \times \ldots$$

where N is chosen big enough, $N = N(\varepsilon)$.

Toruńczyk [32] proved that the properties (1) and (2) actually detect I^∞ among all compacta.

Remark (5.14) Note that $C(n)$ has both properties locally, hence $C(n)$ is an I^∞-manifold. That $C(n)$ is AR follows by the Dugundji theorem [16], whereas $\mathrm{DD}^m\mathrm{P}$ is checked in a straightforward fashion.

X is called an I^∞-*manifold* if for every $x \in X$ there exists a closed neighborhood $F(x) \subset X$ such that $F(x) \cong \mathrm{I}^\infty$. Clearly, every I^∞-manifold possesses the following properties: (i) $X \in \mathrm{ANR}$; (ii) X is locally compact; and (iii) $X \in \mathrm{DD}^m\mathrm{P}$, for every m.

Toruńczyk [32] proved that properties (i) - (iii) are in fact characteristic for I^∞-manifolds. Now, it follows from Theorem (5.3) that $Q(n) \in \mathrm{AR}$, hence $Q(n) \setminus \{\mathcal{E}\} \in \mathrm{ANR}$. So, in order to prove that $Q(n) \setminus \{\mathcal{E}\}$ is an I^∞-manifold it suffices to verify that it has $\mathrm{DD}^m\mathrm{P}$, for every m. We are now ready to prove:

Theorem (5.15) *$Q(2) \setminus \{\mathcal{E}\}$ is a Hilbert cube manifold.*

Proof. Let $Q'(2) = Q(2) \setminus \{\mathcal{E}\}$. Recall the map $\mathcal{L} : C(2) \to \mathcal{E} = GL(2) \cdot \mathrm{B}^2$, given by $\mathcal{L}(V) = E_V$ (Löwner ellipsoid). Define $L(2) = \mathcal{L}^{-1}(\mathrm{B}^2) \subset C(2)$, that is $L(2) = \{V \in C(2) | E_V = \mathrm{B}^2\}$. Then the following properties hold:

(a) $L(2)$ is compact and preserves the $O(2)$-action: for every $A \in O(2)$ and every $V \in L(2)$, $A(V) \in L(2)$; and

(b) $L(2)/O(2) = C(2)/GL(2) = Q(2)$, hence $Q'(2) = (L(2)/O(2)) \setminus \{\mathrm{B}^2\}$.

So it suffices to show that this is an I^∞-manifold.

(c) Given $V \subset W \subset \mathrm{B}^2$, where $V \in L(2)$ (i.e. $E_V = \mathrm{B}^2$) it follows that also $W \in L(2)$, i.e. $E_W = \mathrm{B}^2$ (see Figure 6).

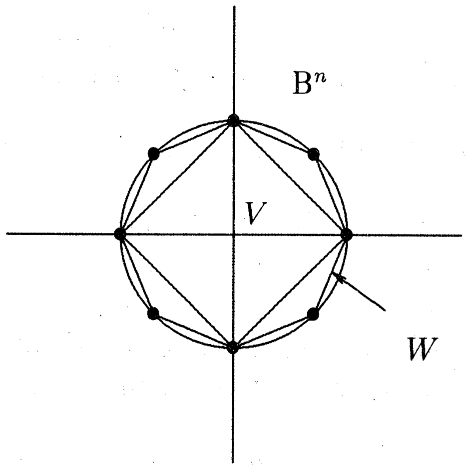

Figure 6

Assertion (5.16) *For every $\delta > 0$, there exist $O(2)$-equivariant maps $f_1, f_2 : L(2) \to L(2)$ such that $d(f_i, \mathrm{Id}) < \delta$, $i \in \{1,2\}$, $f_i(L(2) \setminus \{\mathrm{B}^2\}) \subset (L(2) \setminus \{\mathrm{B}^2\}$ and* $\mathrm{Im} f_1 \cap \mathrm{Im} f_2 = \mathrm{B}^2$.

Let us show that this assertion implies that $Q'(2) \in \mathrm{DD}^m\mathrm{P}$ (and so by Toruńczyk Characterization theorem we will prove Theorem (5.15)).

The maps f_i induce maps $\tilde{f}_i : L(2)/O(2) \longrightarrow L(2)/O(2)$ such that for every i:

(1) $d(\tilde{f}_i, \mathrm{Id}_{Q(2)}) < \delta$;

(2) $\tilde{f}_i((L(2) \setminus \{\mathrm{B}^2\})/O(2)) \subset Q'(2)$, i.e. $\tilde{f}_i(Q'(2)) \subset Q'(2)$; and

(3) $\mathrm{Im}\tilde{f}_1 \cap \mathrm{Im}\tilde{f}_2 = \mathcal{E}$.

So define $\hat{f}_i = \tilde{f}_i | Q'(2) : Q'(2) \to Q'(2)$ and conclude that $\mathrm{Im}\hat{f}_1 \cap \mathrm{Im}\hat{f}_2 = \emptyset$. ∎

To construct f_1, let us consider for every $\varepsilon > 0$, the following map $T_\varepsilon : L(2) \to L(2)$, given by $T(V) = \mathrm{Conv}(V_\varepsilon)$, where $V_\varepsilon = V \cup \{x \in B^2 \setminus \{0\} \mid$ there exists $y \in V$ with $\|x\| = \|y\|$ and the nonoriented angle $\widehat{x0y}$ between the rays $[0x)$ and $[0y)$ is less than or equal to $\varepsilon\}$.

It is clear that V_ε preserves the action of $O(2) : (g \cdot V)_\varepsilon = g \cdot V_\varepsilon$, for every $g \in O(2)$, $V \in L_\varepsilon(2)$. The compactness of V implies that V_ε is compact; the inequality $\|x - y\| < \widehat{x0y}$, for every $\|x\| = \|y\|$, implies that

(4) $V \subseteq V_\varepsilon \subseteq \overline{N}(V; \varepsilon)$, where $\overline{N}(V; \varepsilon)$ is a closed ε-neighborhood of V in B^2.

Besides,

(5) V_ε is continuously dependent on V and ε: if $\varepsilon_k \to \varepsilon > 0$ and $V_k \in L(2) \to V$, then $(V_k)_{\varepsilon_k} \to V_\varepsilon$.

Applying the Dowker theorem [29] for the lower semicontinuous function $g : L_\varepsilon(2) \to \mathbb{R}^+$, $g(V) = \sup\{t > 0 \mid B^2 \setminus N(V; t) \neq \emptyset\}$, we get a continuous function $\gamma : L(2) \to \mathbb{R}^+$

with $0 < \gamma(V) < \delta \cdot g(V)$, $V \in L_\varepsilon(2)$ and $\gamma(B^2) = 0$. The desired continuous $O(2)$-map $f_1 : L_\varepsilon(2) \to L_\varepsilon(2)$ is defined by setting $f_1(V) = \mathrm{Conv}(V_{\gamma(V)})$. By (4), f_1 and $\mathrm{Id}_{L_G(2)}$ are δ-closed.

A so-called contact map $\alpha : L(2) \to \exp(S^1)$ is defined by $\alpha(V) = V \cap S^1$. The discontinuouty properties of α is discussed in [3]. The most significant property of α is that

(6) $\alpha(\mathrm{Conv}(A)) = \mathrm{Conv}(A) \cap S^1 = A \cap S^1$, for every subset $A \subseteq B^2$.

Therefore:

(7) $f_1(V) \cap S^1 = \alpha(f_1(V)) = V_{\gamma(V)} \cap S^1$ contains an nonempty open subset of S^1, for every $V \in L_\varepsilon(2)$.

A mapping f_2 will be constructed in such manner that property (7) does not satisfy: $f_2(V) \cap S^1$ does not contain an open subset of S^1 for every $V \in L_\varepsilon(2)$. Therefore $\mathrm{Im} f_1 \cap \mathrm{Im} f_2 = 0$. To construct f_2, we first need a special mapping F.

Assertion (5.17) *For every $\varepsilon > 0$, there exists an $O(2)$-mapping $F : L(2) \to C(2)$ such that:*

(1) $d(F, \mathrm{Id}_{L(2)}) < \varepsilon$; and

(2) If $V \neq B^2$ then $F(V) = \mathrm{Conv}(\sum_{i=1}^m \lambda_i D_i)$, where D_i is a d_i-dimensional disk, $d_i < 2$, with the center at the origin ($F(B^2)$ in fact coincides with B^2) and $\sum_{i=1}^m \lambda_i = 1$, $\lambda_i \geq 0$.

In connection with this theorem we formulate a geometric conjecture, which is trivially true in dimension 2. Once this conjecture is verified, our Theorem (5.15) will immediately generalize to all $n \geq 2$, and the proof will be essentially the same as above, modulo the replacement everywhere of $n = 2$ by $n \geq 2$.

Conjecture (5.18) *The body $\sum_{i=1}^m \lambda_i D_i$ in Assertion(5.17)(2) differs* essentially *from the ball, i.e. its boundary does not contain any open subset of the sphere.*

It is well-known (cf. [1][2]) that there exists an $O(2)$-retraction $R : C(2) \to L(2)$, which takes $C_\mathcal{E}(2)$ exactly into $L_\mathcal{E}(2)$. But we need the following more precise result which follows by geometric considerations:

Assertion (5.19) *There exists a $O(2)$-retraction $R : C(2) \to L(2)$, such that V and $R(V)$ are affinely equivalent, for every $V \in C(2)$.*

Since $L_G(2)$ is compact, $R|_{L(2)}$ is uniformly continuous for every V. By Assertion (5.16) there is a function $F : L(2) \to C(2)$, sufficiently close to $\mathrm{Id}_{L(2)}$, such that $\mathrm{dist}(\mathrm{Id}, R \circ F) < \delta$.

Since the boundary $F(V), V \neq B^2$, does not contain an open subset of the sphere, $R \circ F(V)$, which is affine by equivalent $F(V)$, does not also contain an open subset of the sphere. The map $f_2 = R \circ F$ is thus as desired. ∎

Corollary (5.20) *$Q(2)$ is nonhomogeneous.*

Proof. It follows from the proof of Theorem (5.5) that $Q(2) \setminus \{\mathcal{E}\}$ is noncontractible. On the other hand, for every $x \in Q(2) \setminus \{\mathcal{E}\}$, $Q(2) \setminus \{x\}$ is contractible. Therefore there is no homeomorphism $h : (Q(2), \mathcal{E}) \longrightarrow (Q(2), x)$, for any $x \neq \mathcal{E}$. ■

Conjecture (5.21) $Q(n \geq 3) \not\cong I^\infty$.

Conjecture (5.22) $Q'(n \geq 3) \not\simeq *$.

Conjecture (5.23) $Q'(2) = K(\mathbb{Q}, 2)$.

6. Direct limits of $Q(n)$

We conclude by stating a recent interesting related result of Banakh, Kawamura and Sakai [14], concerning the topology of the direct limit of $Q(n)$'s (as $n \to \infty$) defined below. Let $1 \leq p \leq \infty$. For each n-dimensional Banach space $E = (E, \|\,.\,\|)$, we define a norm $\|\,.\,\|_p$ on $E \times \mathbb{R}$ as follows:

$$\|(x,t)\|_p = \begin{cases} (\|x\|^p + |t|^p)^{1/p} & \text{if} \quad p < \infty \\ \max\{\|x\|, |t|\} & \text{if} \quad p = \infty \end{cases}$$

Theorem (6.1) *(1) The correspondence $(E, \|\,.\,\|) \to (E \times \mathbb{R}, \|\,.\,\|_p)$ defines a topological embedding of $Q(n)$ into $Q(n+1)$, and hence we obtain a tower of the Banach-Mazur compacta: $Q(1) \subset Q(2) \subset Q(3) \subset \cdots$.*
(2) Let Q_p be the direct limit of this tower. Then Q_p is homeomorphic to $Q^\infty = \lim_\to Q^n$, where Q^n denotes the n-fold product of I^∞ so that Q^n is identified with the subspace $Q^n \times 0 \subset Q^{n+1}$.

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