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<td>Author(s)</td>
<td>Kamo, Shizuo</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1999), 1074: 1-11</td>
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<tr>
<td>Issue Date</td>
<td>1999-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62614">http://hdl.handle.net/2433/62614</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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Pseudo Dirichlet sets and a new cardinal invariant

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Abstract

Z. Bukovská [5] proved that \( p \leq \text{non}(\mathcal{P}D) \), where \( \mathcal{P}D \) denotes the set of all pseudo-Dirichlet sets. In this paper, we shall show that \( p \) can be replaced by \( h \) in this inequality. It is known that \( p < h \) is consistent (see [1]). So, the equality \( p = \text{non}(\mathcal{P}D) \) cannot be proved. This is a partial answer of problem 2 in [6].

Next, we shall introduce a certain cardinal invariant \( f \) and show that \( \text{add}(\mathcal{N}) \leq f \leq \text{non}(\mathcal{P}D) \). Also, we shall construct two generic models such that one satisfies the inequality \( b < f \) and another satisfies the inequality \( f < \text{non}(\mathcal{P}D) \).

1 Introduction

Throughout this paper, we shall use the standard terminologies for forcing of set theory and cardinal invariants on \( \omega \) (see [3]). For each \( a \in \mathbb{R} \), we denote by \( ||a|| \) the distance of \( a \) and the set of integers \( \mathbb{Z} \). Let \( A \) be a subset of the unit interval \([0,1] \).

\( A \) is called a pseudo Dirichlet set, if there exists an \( X \in [\omega]^{\omega} \) such that

\[
\forall a \in A \forall \infty n \in X \left( ||na|| < \frac{1}{|X \cap n| + 1} \right).
\]

We denote the set of all pseudo Dirichlet sets by \( \mathcal{P}D \). Z. Bukovská [5] showed that \( p \leq \text{non}(\mathcal{P}D) \). Let \( h \) be the least cardinal \( \kappa \) such that the boolean algebra \( \mathcal{P}(\omega)/\text{fin} \) does not satisfy the \( \kappa \)-distributive law.

Theorem 1.1 \( h \leq \text{non}(\mathcal{P}D) \).

Proof For each \( a \in [0,1) \), let \( ||a||^* \) denote the unique real number \( r \) such that \( 0 \leq r < 1 \) and \( a = r \) (mod \( \mathbb{Z} \)). To show this theorem, let \( A \subset [0,1] \) and \( |A| < h \).
For each $a \in A$, take a maximal almost disjoint set $W_a \subset [\omega]^{\omega}$ such that
\[ \forall n \in X \forall m \in X \setminus n \left( \| na \|^* - \| ma \|^* < \frac{1}{|X \cap n| + 1} \right), \text{ for all } X \in W_a. \]

Since $|A| < h$, there exists a maximal almost disjoint set $W$ such that $W$ is a refinement of all $W_a$'s. Take a $Y \in W$. Choose some $Y' = \{ y_i \mid i < \omega \} \subset [Y]^{\omega}$ such that
\[ |Y \cap [y_i, y_{i+1})| \geq i \text{ and } y_{i+1} - y_i < y_{i+2} - y_{i+1}, \text{ for all } i < \omega. \]

Let $Z = \{ y_{i+1} - y_i \mid i < \omega \}$. We complete the proof by showing that
\[ \forall \infty n \in Z \left( \| na \|^* < \frac{1}{|Z \cap n| + 1} \right), \text{ for all } a \in A. \]

Let $a \in A$. Since $W$ is a refinement of $W_a$, there exists an $X \in W_a$ such that $Y \subset X$. Take an $i < \omega$ such that $Y \setminus y_i \subset X$. Then, for any $j \in [i + 1, \omega)$, it holds that
\[ \|(y_{j+1} - y_j)a\|^* \leq \| y_{j+1}a \|^* - \| y_ja \|^* < \frac{1}{|X \cap y_j| + 1} < \frac{1}{j + 1}. \]

\[ \square \]

2 **Combinatorial principle w\text{In}_2**

T. Bartoszynski [2] introduced the notion of slalom and, using this, investigated systematically the relations between combinatorics and cardinal invariants which are associated by the null ideal $\mathcal{N}$ and the meager ideal $\mathcal{M}$. The following statement $\text{In}_2$ and the theorem are some of them.

**Definition 2.1** For $h \in \omega^\omega$ and $F \subset \omega^\omega$, define the statement $\text{In}_2(F, h)$ by
\[ \text{In}_2(F, h) \equiv \exists \varphi \in \prod_{n < \omega} [\omega]^{h(n)} \forall f \in F \forall n < \omega \left( f(n) \in \varphi(n) \right). \]

The statement $\text{In}_2(F, \text{id}_\omega)$ is denoted by $\text{In}_2(F)$, where $\text{id}_\omega$ is the identity function on $\omega$.

**Theorem 2.1** (Bartoszynski [2]) $\text{add}(\mathcal{N}) = \min\{ |F| \mid F \subset \omega^\omega \text{ and not } \text{In}_2(F) \}.$

In this section, we shall introduce the statement w$\text{In}_2$ which is some variant of $\text{In}_2$. And we shall study relations between w$\text{In}_2$ and non($\mathcal{P}\mathcal{D}$).
Definition 2.2  For \( H, h \in \omega\omega \) and \( F \subset \prod_{n<\omega}H(n) \), define the statement \( \text{wIn}_2(F, h, H) \) by

\[
\text{wIn}_2(F, h, H) \equiv \exists \varphi \in \prod_{n<\omega}H(n) \leq h(n) \forall f \in F \forall^\infty n<\omega (f(n) \in \varphi(n)).
\]

\( \text{wIn}_2(F, H) \) denotes the statement \( \text{wIn}_2(F, \text{id}_\omega, H) \). Let

\[
f = \min \{ |F| \mid \exists H \in \omega\omega (F \subset \prod_{n<\omega}H(n) \text{ and not } \text{wIn}_2(F, H)) \}.
\]

The following lemma can be easily proved by the result of Bartorszynski.

Lemma 2.2  \( \text{add}(\mathcal{N}) = \min \{ b, f \} \). \( \square \)

The main result of this section is the following theorem.

Theorem 2.3  \( f \leq \text{non}(\mathcal{P}\mathcal{D}) \).

To show this theorem, we need some notations and lemmas.

A sequence \( \langle I_n \mid n < \omega \rangle \) is called an interval partition of \( \omega \), if there exists an increasing function \( f \in \omega\omega \) such that \( f(0) = 0 \) and, for all \( n < \omega \), \( I_n = \{ k < \omega \mid f(n) \leq k < f(n+1) \} \).

The next lemma can be deduced from [6, Proposition 1]. But, for a convenience for the reader, we give a proof.

Lemma 2.4  Let \( n < \omega \) and \( 0 < m, k < \omega \). Then, there exists some \( p < \omega \) such that

\[
\forall a_0, \cdots, a_{m-1} \in [0,1] \exists \ s < \omega \ (n \leq s < p \text{ and } \forall i < m \ (\|sa_i\| < \frac{1}{k})).
\]

Proof  By induction on \( 1 \leq m < \omega \).

Case 1.  \( m = 1 \).

We claim that \( p = nk + 1 \) satisfies the condition. To show this, let \( a \in [0,1] \).

Define the mapping \( \sigma : X = \{ nj \mid j = 1, \cdots, k \} \rightarrow k \) by, for each \( j < k \),

\[
\sigma(nj) = \text{"the unique } i \text{ such that } \frac{i}{k} \leq \|ja\| < \frac{i+1}{k} \text{"}.
\]

If there exists some \( nj \) such that \( \sigma(nj) = 0 \) or \( k - 1 \), then \( s = nj \) is a required one.
Otherwise, there exist \( i < j \leq k \) such that \( \sigma(ni) = \sigma(nj) \) and \( s = n(j - i) \) is a required one.

Case 2. \( m = m' + 1. \)

By induction hypothesis, there exist \( 0 = p_0 < p_1 < \cdots < p_k \) such that

\[
\forall a_0, \ldots, a_{m'-1} \in [0, 1], \exists s < \omega (p_j + n \leq s < p_{j+1} \text{ and } \forall i < m' (\|s_{a_i}\| < \frac{1}{2k})),
\]

for \( j < k. \)

We show that \( p = p_k \) satisfies the condition. So, let \( a_0, \ldots, a_{m'} \in [0, 1]. \) By the choice of \( p_j \) (for \( j < k \)), there exist \( s_0, \ldots, s_{k-1} < \omega \) such that

\[
p_j + n \leq s_j < p_{j+1} \text{ and } \forall i < m' (\|s_{a_i}\| < \frac{1}{2k})), \text{ for } j < k.
\]

Then, it holds that

\[
\|s_{j}a_{m'}\| < \frac{1}{k}, \text{ for some } j < k
\]

or

\[
\|s_{j}a_{m'} - s_{j'}a_{m'}\| < \frac{1}{k}, \text{ for some } j < j' < k.
\]

In either cases, similar to case 1, we can take a required element \( s. \)

\[\square\]

**Corollary 2.5** There is an interval partition \( \langle I_n | n < \omega \rangle \) which satisfies

\[
(\ast) \left\{ \begin{array}{l}
\text{For any } n < \omega \text{ and } a_0, \ldots, a_{n-1} \in [0, 1], \text{ there exists some } k \in I_n \text{ such that } \\
\|ka_i\| < 2^{-n}, \text{ for all } i < n.
\end{array} \right.
\]

\[\square\]

**Proof of Theorem 2.3** Take an interval partition \( \langle I_n | n < \omega \rangle \) which satisfies \( (\ast) \) in the previous corollary. Define \( H \in \omega \omega \) by

\[
H(n) = 2^n \sum_{k \leq n} |I_k|, \text{ for all } n < \omega.
\]

To show the theorem, let \( A \subseteq [0, 1] \) and \( |A| < f. \) For each \( a \in A, \) define \( f_a \in \prod_{n<\omega} H(n) \)

by

\[
\frac{f_a(n)}{H(n)} \leq a < \frac{f_a(n) + 1}{H(n)}, \text{ for all } n < \omega.
\]

Since \( |A| < f, \) there exists a \( \varphi \in \prod_{n<\omega} [H(n)]^n \) such that
\[ \forall a \in A \forall n < \omega \ (f_a(n) \in \varphi(n)). \]

For each \( n < \omega \), take \( s_n \in I_n \) such that
\[ \| s_n \frac{j}{H(n)} \| < 2^{-n}, \text{ for all } j \in \varphi(n). \]

We complete the proof by showing that
\[ \forall a \in A \forall n < \omega \ (\| s_n a \| < 2^{-n+1}). \]

So, let \( a \in A \). Take an \( m < \omega \) such that
\[ \forall n \geq m \ (f_a(n) \in \varphi(n)). \]

Then, for any \( n \geq m \), since
\[ \frac{f_a(n)}{H(n)} \leq a < \frac{f_a(n) + 1}{H(n)}, \]

it holds that
\[ s_n \frac{f_a(n)}{H(n)} \leq s_n a < s_n \frac{f_a(n) + 1}{H(n)}. \]

So,
\[ \| s_n a \| \leq \| s_n \frac{f_a(n)}{H(n)} \| + \frac{s_n}{H(n)} < 2^{-n+1}. \]

Note that what we really proved is \( \min\{ |F| \ | \not\in \text{wIn}_2(F, H) \} \leq \text{non}(\mathcal{P}D) \), where \( H \) is a function defined in the proof of Theorem 2.3.

3 The cardinal invariant \( f \)

In the previous section, we introduced the cardinal invariant \( f \) and showed the equality \( \text{add}(\mathcal{N}) = \min\{ b, f \} \). Both of \( \text{add}(\mathcal{N}) \) and \( b \) appear in the Cichoń's diagram. It seems to be an interesting problem to check the relations between \( f \) and other cardinals in the diagram. Since it is known that \( \mathcal{P}D \subset \mathcal{N} \cap \mathcal{M} \), it holds that \( f \leq \min\{ \text{non}(\mathcal{N}), \text{non}(\mathcal{M}) \} \). So, \( f \) seems to be not so large. If the inequality \( f \leq b \) always holds, then \( f \) is equal to \( \text{add}(\mathcal{N}) \) and \( f \) does not become a new cardinal invariant. In this section, we shall show that there exists a generic model which satisfies the inequality \( b < f \).

**Definition 3.1** For each \( H \in \omega^{\omega} \), define the forcing notion \( Q(H) \) by
\[ Q(H) = \{ p \in \prod_{n < \omega} [H(n)]^{\mathbb{N}} \ | \exists k < \omega \ \forall n < \omega \ (|p(n)| \leq k) \}. \]
\[q \leq p \iff \forall n < \omega \left( p(n) \subset q(n) \right)\]

Define \(\tau_H : Q(H) \rightarrow \omega\) by
\[
\tau_H(p) = \min \{ k < \omega | \forall n < \omega (|p(n)| \leq k) \}.
\]

Using the density argument, the following lemma can be proved easily.

**Lemma 3.1** Let \(H \in \omega\omega\) and \(G\) be \(V\)-generic on \(Q(H)\). In \(V[G]\), define \(\varphi \in \prod_{n<\omega} \mathcal{P}(H(n))\) by
\[
\varphi(n) = \bigcup \{p(n) | p \in \mathcal{G}\}.
\]

Then, it holds that

1. \(|\varphi(n)| \leq n\), for all \(n < \omega\),
2. \(\forall g \in \left(\prod_{n<\omega} H(n)\right)^V \forall^\infty n < \omega (g(n) \in \varphi(n))\).

**Lemma 3.2** \(Q(H)\) satisfies the \(\omega_1\)-chain condition.

**Proof** Let \(W \subset Q(H)\) and \(|W| = \omega_1\). Replace \(W\) by a certain subset of \(W\), if necessary, we can assume that, for some \(k < \omega\),
\[
\tau_H(p) = k \text{ and } p|2k = p'|2k, \text{ for all } p, p' \in W.
\]

Then, every elements of \(W\) are mutually compatible.

**Lemma 3.3** Every unbounded family of functions in \(\omega\omega \cap V\) is still unbounded in \(V^Q(H)\).

Bartszinski and Judah [3, Theorem 6.4.13] proved that any finite support iteration by forcing notions which preserved the unboundedness in \(\omega\omega\) does not add a dominating function. So, starting a ground model which satisfies CH, by choosing appropriate \(H\)'s, we can construct an \(\omega_2\)-stage finite support iteration \(P\) such that \(V^P\) satisfies \(b = \omega_1\) and \(f = \omega_2\).

In order to prove Lemma 3.3, we need a result of Brendle and Judah [4]. Let \(P\) be a forcing notion which satisfies the \(\omega_1\)-chain condition and \(\tau : P \rightarrow \omega\) be a homomorphism. Following Brendle and Judah [4], we say that \((P, \tau)\) is nice, if it
satisfies
\[
\begin{cases}
\text{For any preence set } \{p_i \mid i < \omega\} \subset P, \text{ it holds that } \\
\forall m < \omega \exists n < \omega \forall q \in P \left( \text{if } \tau(q) \leq m, \text{ then } \exists i < n \left( q \upharpoonright p_i \right) \right).
\end{cases}
\]

**Theorem 3.4** (Brendle and Judah [4]) \( (P, \tau) \) be a nice forcing notion. Then, every unbounded family of functions in \( {}^\omega \omega \cap V \) is still unbounded in \( V^{Q(H)} \).

**Proof of Lemma 3.3** It suffices to show that \( (Q(H), \tau_H) \) is nice. So, let \( \{p_i \mid i < \omega\} \) be a preence subset of \( Q(H) \) and \( m < \omega \). To get a contradiction, assume that, for each \( n < \omega \), there exists a condition \( q_n \in Q(H) \) such that
\[
\tau_H(q_n) \leq m \quad \text{and} \quad \forall i < n \left( q_n \perp p_i \right).
\]
Since \( \{q_n \mid k \mid n < \omega\} \) is a finite set for every \( k < \omega \), we can choose \( X_k \in [\omega]^\omega \) by induction on \( k < \omega \) such that
\[
X_{k+1} \subseteq X_k \quad \text{and} \quad \forall n, n' \in X_k \left( q_n \upharpoonright (k+1) = q_{n'} \upharpoonright (k+1) \right).
\]
Define \( r \in Q(H) \) by
\[
r(k) = q_n(k), \ \text{for some/all } n \in X_k.
\]
Note that \( \tau_H(r) \leq m \). Since \( \{p_i \mid i < \omega\} \) is preence, there exists \( i < \omega \) such that \( r \) is compatible with \( p_i \). Let \( k = \tau_H(p_i) + m \). Take \( n \in X_k \) such that \( i < n \). Since \( i < n \), it holds that \( p_i \) and \( q_n \) are incompatible. Since \( \tau_H(p_i) + \tau_H(q_n) \leq k \), it holds that \( \exists j < k \left( |p_i(j) \cup q_n(j)| > j \right) \). By this and the fact that \( r \upharpoonright k = q_n \upharpoonright k \), \( r \) is incompatible with \( p_i \). This is a contradiction.

**4 Consistency of \( f < \text{non}(\mathcal{P}D) \)**

Concerning about the cardinal invariant associated by \( \text{In}_2 \), T. Bartoszynski [2] pointed out implicitly that, if two functions \( h_0, h_1 \in {}^\omega \omega \) satisfies that
\[
\lim_{n<\omega} h_i(n) = \infty, \ \text{for } i = 0, 1,
\]
then
\[
\min\{|F| \mid \text{not } \text{In}_2(F, h_0)\} = \min\{|F| \mid \text{not } \text{In}_2(F, h_1)\}.
\]
In this section, we shall show that, for any \( H \in {}^\omega \omega \), \( f \) may not be equal to
\[
\min\{|F| \mid \text{not } \text{wIn}_2(F, H)\}. \ \text{Using this, we shall prove the consistency of } f <
non(\mathcal{P}D).  Henceforce, \(H \in \omega\omega\) is an arbitrary, but fixed function on \(\omega\). For each \(k < \omega\), let
\[
T_k (= T_k^H) = \{ q \in Q(H) \mid \tau_H(q) \leq k \}.
\]
Define \(H_0, H_1 : \omega \times \omega \to \omega\) by
\[
H_0(k, m) = \min \left\{ l < \omega \left| \forall \delta : l \to [\omega_2]^{\leq k} \exists S \in [l]^m \exists v \in [\omega_2]^{\leq k} \forall i, j \in S (\text{if } i \neq j, \text{then } \delta(i) \cap \delta(j) = v) \right. \right\},
\]
\[
H_1(k, m) = \min \left\{ l < \omega \left| \forall \delta : l \to T_k \exists S \in [l]^m \exists q \in Q(H) \quad \forall i \in S (q \leq \delta(i)) \right. \right\}.
\]
Note that \(H_0\) is a recursive function. And, \(H_1\) is an \(H\)-recursive function, since it holds that
\[
\exists q' \in Q(H) \forall q \in S (q' \leq q) \iff \forall i < mk (|\bigcup_{q \in S} q(i)| \leq i), \text{ for any } S \in [T_k]^m.
\]

Define \(H_2, H^* : \omega \to \omega\) by
\[
H_2(k) = H_1(k, H_1(k, H_1(\ldots, H_1(k, k + 1) \ldots)))
\]
k times
\[
H^*(k) = H_0(k, H_2(k)).
\]

Define an \(\omega_2\)-stage finite support iteration \(P_\alpha\) (for \(\alpha \leq \omega_2\)) associated with \(Q_\alpha\) (for \(\alpha < \omega_2\)) by
\[
\models_\alpha Q_\alpha = Q(H), \text{ for all } \alpha < \omega_2.
\]
Let \(P(H) = P_{\omega_2}\). It holds that
\[
V^{P(H)} \models \forall F \subset \prod_{n<\omega} H(n) \quad (\text{if } |F| \leq \omega_1, \text{then } \text{wIn}_2(F, H)).
\]

The purpose of this section is to show

\textbf{Theorem 4.1} \quad V^{P(H)} \models \text{not wIn}_2((\prod_{n<\omega} H^*(n))^V, H^*).

\textbf{Corollary 4.2} \quad \text{Suppose that } V \models \text{CH}. \text{ Let } H \in \omega\omega \text{ be the function which is defined in the proof of Theorem 2.3. Then, it holds that}
\[
V^{P(H)} \models f = \omega_1 \text{ and non}(\mathcal{P}D) = \omega_2.
\]
To show Theorem 4.1, we need some definitions and lemmas. Let

\[ D = \{ p \in P(H) \mid \forall \alpha \in \text{supp}(p) \ ( p \models \alpha \text{ decides } \tau_H(p(\alpha))) \}. \]

The following lemma can be proved easily.

**Lemma 4.3**  \( D \) is dense in \( P(H) \). \( \square \)

Define \( \rho : D \to \omega \) by

\[
\rho(p) = \min \left\{ k < \omega \mid \begin{array}{|c|}
|\text{supp}(p)| \leq k \\
\forall \alpha \in \text{supp}(p) \ ( p \models \alpha \Rightarrow \tau_H(p(\alpha)) \leq k )
\end{array} \right\}.
\]

For each \( k < \omega \), let

\[ D_k = \{ p \in D \mid \rho(p) \leq k \} \]

**Lemma 4.4**  Let \( k < \omega \) and \( \delta : H^*(k) \to D_k \). Then, there exist \( p^+ \in P(H) \) and \( P(H) \)-name \( \dot{S} \) which satisfy (1), (2).

(1) \( \models \dot{S} \subseteq H^*(k) \) and \( |\dot{S}| \geq k + 1 \).

(2) \( \forall i < H^*(k) \forall p' \leq p^+ \ ( \text{if } p' \models i \in \dot{S}, \text{then } p' \leq \delta(i) ) \).

**Proof**  Let \( k < \omega \). Define \( l_m \) (for \( m \leq k \)) by

\[
\begin{align*}
l_0 &= k + 1 \\
l_{m+1} &= H_1(k, l_m)
\end{align*}
\]

Note that \( H^*(k) = H_0(k, l_k) \). Assume that \( \delta : H^*(k) \to D_k \). Since \( \langle \text{supp}(\delta(i)) \mid i < H^*(k) \rangle : H^*(k) \to [\omega_2]^k \), by the choice of \( H_0 \), there exist \( S_0 \in [H^*(k)]^k \) and \( v \in [\omega_2]^k \) such that

\[ \forall i, j \in S_0 \ ( \text{if } i \neq j, \text{then } \text{supp}(\delta(i)) \cap \text{supp}(\delta(j)) = v ) \].

Define \( p \in P(H) \) by

\[ \text{supp}(p) = \bigcup \{ \text{supp}(\delta(i)) \mid i \in S_0 \} \setminus v, \]

\[ p(\alpha) = \delta(i)(\alpha), \text{if } \alpha \in \text{supp}(\delta(i)) \text{ and } i \in S_0. \]

Let \( n = |v| \) and \( v = \{ \alpha_1, \cdots, \alpha_n \} \subseteq \). Note that \( n \leq k \). By induction on \( 1 \leq m \leq n \), choose \( P_{\alpha_m} \)-names \( \dot{S}_m, \dot{q}_m \) such that

(3) \( \models \alpha_m \dot{S}_m \in [\dot{S}_{m-1}]^{k-m} \) and \( \dot{q}_m \in \dot{Q}_{\alpha_m} \).

(4) \( p \models \alpha_m \cup ( \dot{q}_j \mid 1 \leq j < m ) \models \alpha_m \dot{q}_m \leq \delta(i)(\alpha_m), \text{for all } i \in \dot{S}_m. \)
We must show that these can be chosen. Assume that $m \leq n$ and $\dot{S}_j$, $\dot{q}_j$ were chosen, for $j < m$. Since $H_1$ is absolute and $H_1(k, l_{k-m}) = l_{k-m+1}$, it holds that

$$p \models \alpha_m \cup \{ \dot{q}_j \mid 1 \leq j < m \} \models_{\alpha_m} \exists q \in Q(H) \exists S \in [\dot{S}_{m-1}]^{l_{k-m}} \forall i \in S ( q \leq \delta(i)(\alpha_m))$$

Using this, it can be possible to choose $\dot{S}_m$ and $\dot{q}_m$.

Let $p^+ = p \cup \{ \dot{q}_m \mid 1 \leq m \leq n \}$, $\dot{S} = \dot{S}_n$. It is clear that this $p^+$ and $\dot{S}$ satisfy (1) in the lemma. In order to show that these satisfy (2), assume that

$$i < H^*(k) \text{ and } p' \leq p^+ \text{ and } p' \models i \in \dot{S}.$$ 

Since $\models_{P} \dot{S} = \dot{S}_n \subset \dot{S}_{n-1} \subset \cdots \subset S_0$, $i \in S_0$. For each $m = 1, \ldots, n$, since $\dot{S}_m$ is a $P_{\alpha_m}$-name, it holds that $p' \models_{\alpha_m} \dot{q}_m \leq \delta(i)(\alpha_m)$, for all $m = 1, \ldots, n$.

So, $p' \leq \delta(i)$.

\[ \square \]

**Lemma 4.5** Let $k < \omega$. Assume that a $P(H)$-name $\dot{a}$ satisfies

$$\models \dot{a} \in [H^*(k)]^{<k}.$$ 

Then, there exists some $j < H^*(k)$ such that

$$\forall p \in D_k ( \text{ not } p \models j \in \dot{a} ).$$

**Proof** Suppose not. Take $\delta : H^*(k) \rightarrow D_k$ such that

$$\delta(j) \models j \in \dot{a}, \text{ for all } j < H^*(k).$$

By the previous lemma, there exist $p^+ \in P(H)$ and $P(H)$-name $\dot{S}$ such that

$$\models \dot{S} \subset H^*(k) \text{ and } |\dot{S}| \geq k + 1,$$

$$\forall i < H^*(k) \forall p' \leq p^+ ( \text{ if } p' \models i \in \dot{S}, \text{ then } p' \leq \delta(i) ).$$

Then, it holds that $p^+ \models \dot{S} \subset \dot{a}$. This contradicts that $p^+ \models |\dot{S}| \geq k + 1$ and $|\dot{a}| \leq k$.

\[ \square \]

**Proof of Theorem 4.1** Assume that $\models_{P(H)} \dot{\varphi} \in \prod_{k<\omega} [H^*(k)]^k$. Using the previous lemma, for each $k < \omega$, take a $j_k < H^*(k)$ such that

$$\forall p \in D_k ( \text{ not } p \models j_k \in \dot{\varphi}(k) ).$$
We claim that $\models \exists^\infty k < \omega ( j_k \not\in \dot{\varphi}(k) )$. Suppose not. Then, there exist $p \in D$ and $n < \omega$ such that

$p \models \forall k > n ( j_k \not\in \dot{\varphi}(k) )$.

Take $k > n$ such that $p \in D_k$. Then, it holds that $p \models j_k \in \dot{\varphi}(k)$. But, this contradicts the choice of $j_k$.

\[ \square \]

**Added in proof:**

After the completion of this paper, Dr. Kada [7] have proved that $d < \text{non}(\mathcal{P}D)$ is consistent with ZFC.

**References**


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