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ASYMPTOTIC BEHAVIOR OF VARIATIONAL EIGENVALUES

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1. Introduction. We consider the nonlinear elliptic two-parameter problem

$$\begin{aligned} -\triangle u + \lambda g(u) = \mu f(u),\ \ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda, \mu > 0$ are parameters, and $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ is a bounded domain with an appropriately smooth boundary $\partial\Omega$. We assume

(A.1) $f, g \in C^1(\mathbf{R})$ are odd in u, and $f(u), g(u) > 0$ for $u > 0$. Furthermore, there exist constants $1 < q \leq p < (N+2)/(N-2)$ and $K_0, J_0, K_1, J_1 > 0$ such that

$$\frac{g(u)}{u} \to K_0, \quad \frac{f(u)}{u^p} \to K_1 \quad \text{as } u \to \infty, \tag{1.2}$$

$$\frac{g(u)}{u} \to J_0, \quad \frac{f(u)}{u^q} \to J_1 \quad \text{as } u \downarrow 0. \tag{1.3}$$

The typical example of f, g is

$$f(u) = |u|^{p-1}u + |u|^{q-1}u, \ \ g(u) = u, \quad (1 < q \leq p < (N+2)/(N-2)). \tag{1.4}$$

The purpose of this paper is to investigate and understand the structure of the set $\{(\lambda, \mu)\} \subset \mathbf{R}^2_+$ such that (1.1) has a solution $u \in W_0^{1,2}(\Omega)$ by variational methods, where $W_0^{1,2}(\Omega)$ is the usual real Sobolev space. To this end, viewing $\lambda > 0$ as a given parameter, we apply the following two variational problems subject to the constraints depending on positive parameters α, β and λ:

$$\text{Maximize } \int_\Omega \left(\int_0^{u(x)} f(s)ds \right) dx \text{ under the constraint} \tag{M.1}$$

$$u \in N_{\lambda,\alpha} := \left\{ u \in W_0^{1,2}(\Omega) : \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \lambda \int_\Omega \left(\int_0^{u(x)} g(s)ds \right) dx = \alpha \right\},$$

$$\text{Minimize } \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \lambda \int_\Omega \left(\int_0^{u(x)} g(s)ds \right) dx \text{ under the constraint} \tag{M.2}$$

$$u \in M_\beta := \left\{ u \in W_0^{1,2}(\Omega) : \int_\Omega \left(\int_0^{u(x)} f(s)ds \right) dx = \beta \right\}.$$

Then we obtain two solutions trio $(\lambda, \mu_1(\lambda,\alpha), u_{1,\lambda,\alpha})$, $(\lambda, \mu_2(\lambda,\beta), u_{2,\lambda,\beta}) \in \mathbf{R}_+^2 \times W_0^{1,2}(\Omega)$ corresponding to the problems **(M.1)** and **(M.2)**, respectively, by the Lagrange multiplier theorem. A natural problem in this context is to clarify the difference between $\mu_1(\lambda,\alpha)$ and $\mu_2(\lambda,\beta)$. To do this, we shall establish two asymptotic formulas for $\mu_1(\lambda,\alpha)$ and $\mu_2(\lambda,\beta)$ as $\lambda \to \infty$, respectively, which are explicitly represented by means of λ and α, β. Under the suitable conditions on (λ,α) (resp. (λ,β)), one of them for $\mu_1(\lambda,\alpha)$ (resp. $\mu_2(\lambda,\beta)$) depends only on the asymptotic behavior of f and g as $u \to \infty$, and another depends only on the behavior of f and g near 0. We emphasize that if $\alpha, \beta > 0$ are fixed, then $\mu_1(\lambda,\alpha) \to \infty$ faster than $\mu_2(\lambda,\beta)$ as $\lambda \to \infty$.

2. Main Results. We begin with notation. For $u, v \in W_0^{1,2}(\Omega)$ and $t \in \mathbf{R}$, let

$$\|u\|_d^d := \int_\Omega |u(x)|^d dx \ (d \geq 1), \ \ \|u\|_\infty := \sup_{x\in\Omega} |u(x)|, \ \ (u,v) := \int_\Omega u(x)v(x)dx,$$

$$F(t) := \int_0^t f(s)ds, \ \ G(t) := \int_0^t g(s)ds, \ \ \Phi(u) := \int_\Omega F(u(x))dx,$$

$$\Psi(u) := \int_\Omega G(u(x))dx, \ \ \Lambda_\lambda(u) := \frac{1}{2}\|\nabla u\|_2^2 + \lambda\Psi(u).$$

Furthermore, for any domain $D \subset \mathbf{R}^N$ the norm of $L^d(D)$ will be denoted by $\|\cdot\|_d$ for simplicity. For a given $\lambda, \alpha, \beta > 0$, $\mu = \mu_1(\lambda,\alpha)$ and $\mu = \mu_2(\lambda,\beta)$ are defined as the Lagrange multipliers associated with the problem **(M.1)** and **(M.2)**, respectively. Namely, $\mu_1(\lambda,\alpha)$ and $\mu_2(\lambda,\beta)$ are the Lagrange multipliers associated with the eigenfunctions $u_{1,\lambda,\alpha} \in N_{\lambda,\alpha}$ and $u_{2,\lambda,\beta} \in M_\beta$ which satisfy

$$\Phi(u_{1,\lambda,\alpha}) = \sup_{u\in N_{\lambda,\alpha}} \Phi(u). \tag{2.1}$$

$$\Lambda_\lambda(u_{2,\lambda,\beta}) = \inf_{u\in M_\beta} \Lambda_\lambda(u), \tag{2.2}$$

respectively. Then $(\lambda, \mu_1(\lambda,\alpha), u_{1,\lambda,\alpha})$ and $(\lambda, \mu_2(\lambda,\beta), u_{2,\lambda,\beta})$ satisfy (1.1) by the Lagrange multiplier theorem. Further, $\mu_1(\lambda,\alpha)$ and $\mu_2(\lambda,\beta)$ are represented as follows:

$$\mu_1(\lambda,\alpha) = \frac{2\alpha + \lambda\{(g(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha}) - 2\Psi(u_{1,\lambda,\alpha})\}}{(f(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha})}, \tag{2.3}$$

$$\mu_2(\lambda,\beta) = \frac{\|\nabla u_{2,\lambda,\beta}\|_2^2 + \lambda(g(u_{2,\lambda,\beta}), u_{2,\lambda,\beta})}{(f(u_{2,\lambda,\beta}), u_{2,\lambda,\beta})}. \tag{2.4}$$

Indeed, if $(\lambda,\mu,u) \in \mathbf{R}_+^2 \times W_0^{1,2}(\Omega)$ satisfies (1.1), then multiply (1.1) by u. Then integration by parts yields

$$\|\nabla u\|_2^2 + \lambda(g(u),u) = \mu(f(u),u). \tag{2.5}$$

(2.5) implies (2.4). Since $u_{1,\lambda,\alpha} \in N_{\lambda,\alpha}$, (2.5) also yields (2.3). Let $w \in H^1(\mathbf{R}^N)$ be the unique solution of the following nonlinear scalar field equation:

$$-\Delta w = w^p - w, \quad w > 0 \quad \text{in } \mathbf{R}^N, \quad w(0) = \max_{x\in\mathbf{R}^N} w(x). \tag{2.6}$$

Further, let W be the unique solution of (2.6), in which the exponent p is replaced by q.

In order to state our results, we define the several conditions for (un-indexed) sequences $\{(\lambda,\alpha)\} \subset \mathbf{R}_+^2$ and $\{(\lambda,\beta)\} \subset \mathbf{R}_+^2$:

$$\lambda \to \infty. \tag{B.1}$$

$$\alpha^2\lambda^{N-2} \to \infty. \tag{B.2}$$

$$\alpha^2\lambda^{N-2} \to 0. \tag{B.3}$$

$$\beta^2\lambda^{N} \to \infty. \tag{B.4}$$

$$\beta^2\lambda^{N} \to 0. \tag{B.5}$$

We explain the meaning of these conditions. In the problem **(M.1)**, $\|u_{1,\lambda,\alpha}\|_\infty$ behaves like $(\alpha^2\lambda^{N-2})^{1/4}$ for $\lambda \gg 1$. Therefore, if (B.2) (resp. (B.3)) is assumed, then $\|u_{1,\lambda,\alpha}\|_\infty \to \infty$ (resp. 0). Hence we see that the asymptotic behavior of $f(u), g(u)$ as $u \to \infty$ (resp. $u \to 0$) reflects mainly on the asymptotic formula for $\mu_1(\lambda,\alpha)$. Similarly, in the problem **(M.2)**, the growth order of $\|u_{2,\lambda,\beta}\|_\infty$ is $(\beta^2\lambda^N)^{1/(2(p+1))}$. Hence the condition (B.4) (resp. (B.5)) implies $\|u_{2,\lambda,\beta}\|_\infty \to \infty$ (resp. 0). Therefore, the asymptotic behavior of $f(u), g(u)$ at $u = \infty$ (resp. $u = 0$) gives effect mainly on the asymptotic behavior of $\mu_2(\lambda, \beta)$.

Now we state our main results.

Theorem 2.1. *Assume (A.1). If a sequence* $\{(\lambda, \alpha)\} \subset \mathbf{R}^2_+$ *satisfies (B.1) and (B.2), then the following asymptotic formula holds:*

$$\mu_1(\lambda, \alpha) = C_2 \alpha^{\frac{1-p}{2}} \lambda^{\frac{N+2-p(N-2)}{4}} + o(\alpha^{\frac{1-p}{2}} \lambda^{\frac{N+2-p(N-2)}{4}}), \tag{2.7}$$

where $C_2 = K_1^{-1} K_0^{\frac{N+2-p(N-2)}{4}} (\|w\|_{p+1}^{p+1}/2)^{\frac{p-1}{2}}$.

We note that $\alpha > 0$ *may not be fixed* in Theorem 2.1. If $\alpha > 0$ is fixed, then (B.1) implies (B.2) immediately. However, if $\alpha > 0$ is *not fixed*, then (B.1) does not imply (B.2) in general.

Theorem 2.2. *Assume (A.1). If a sequence* $\{(\lambda, \alpha)\} \subset \mathbf{R}^2_+$ *satisfies (B.1) and (B.3), then the following asymptotic formula holds:*

$$\mu_1(\lambda, \alpha) = C_3 \alpha^{\frac{1-q}{2}} \lambda^{\frac{N+2-q(N-2)}{4}} + o(\alpha^{\frac{1-q}{2}} \lambda^{\frac{N+2-q(N-2)}{4}}), \tag{2.8}$$

where $C_3 = J_1^{-1} J_0^{\frac{N+2-q(N-2)}{4}} (\|W\|_{q+1}^{q+1}/2)^{\frac{q-1}{2}}$.

We should notice that in the situation of Theorem 2.2, $\alpha > 0$ is *not fixed.* Clearly, if $\alpha > 0$ is fixed, then (B.1) contradicts (B.3). (B.1) and (B.3) are consistent, for example, if $\alpha = \lambda^{-m} (m > (N-2)/2)$.

Theorem 2.3. *Assume (A.1). If a sequence* $\{(\lambda, \beta)\} \subset \mathbf{R}^2_+$ *satisfies (B.1) and (B.4), then the following asymptotic formula holds:*

$$\mu_2(\lambda, \beta) = C_4 \beta^{-\frac{p-1}{p+1}} \lambda^{\frac{N+2-p(N-2)}{2(p+1)}} + o(\beta^{-\frac{p-1}{p+1}} \lambda^{\frac{N+2-p(N-2)}{2(p+1)}}), \tag{2.9}$$

where $C_4 = K_1^{-\frac{2}{p+1}} K_0^{\frac{N+2-p(N-2)}{2(p+1)}} (p+1)^{-\frac{p-1}{p+1}} \|w\|_{p+1}^{p-1}$.

Theorem 2.4. *Assume (A.1). If a sequence $\{(\lambda,\beta)\} \subset \mathbf{R}^2_+$ satisfies (B.1) and (B.5), then the following asymptotic formula holds:*

$$\mu_2(\lambda,\beta) = C_5\beta^{-\frac{q-1}{q+1}}\lambda^{\frac{N+2-q(N-2)}{2(q+1)}} + o(\beta^{-\frac{q-1}{q+1}}\lambda^{\frac{N+2-q(N-2)}{2(q+1)}}), \tag{2.10}$$

where $C_5 = J_1^{-\frac{2}{q+1}} J_0^{\frac{N+2-q(N-2)}{2(q+1)}} (q+1)^{-\frac{q-1}{q+1}}\|W\|_{q+1}^{q-1}$.

Remark 2.5. (1) Note that $\beta > 0$ *may not be fixed* in Theorem 2.3. If $\beta > 0$ is fixed, then (B.1) implies (B.4) immediately. However, if $\beta > 0$ is *not fixed*, then (B.1) does not imply (B.4) in general. Furthermore, in Theorem 2.4, $\beta > 0$ is *not fixed.* Clearly, if $\beta > 0$ is fixed, then (B.1) contradicts (B.5). (B.1) and (B.5) are consistent, for example, if $\beta = \lambda^{-m} (m > N/2)$.

(2) Theorem 2.1 and Theorem 2.3 imply that if $\alpha, \beta > 0$ *are fixed*, then

$$\frac{\mu_1(\lambda,\alpha)}{\mu_2(\lambda,\beta)} \to \infty \quad \text{as } \lambda \to \infty.$$

This phenomenon is explained as follows. We see that as $\lambda \to \infty$, $\|u_{1,\lambda,\alpha}\|_{p+1}^{p+1}$ behaves like $\alpha^{(p+1)/2}\lambda^{-(N+2-p(N-2))/4}$ (cf. (3.15) in Section 3). Therefore, if $\alpha, \beta > 0$ are fixed, then $\Phi(u_{1,\lambda,\alpha}) \to 0$ and consequently, $u_{1,\lambda,\alpha} \in M_\beta$ is impossible. Hence if $\beta > 0$ behaves like $\alpha^{(p+1)/2}\lambda^{-(N+2-p(N-2))/4}$ as $\lambda \to \infty$, then the growth order of $\mu_2(\lambda,\beta)$ as $\lambda \to \infty$ is the same as that of $\mu_1(\lambda,\alpha)$. More precisely (let $K_0 = K_1 = 1$ for simplicity), if the top term of $\mu_1(\lambda,\alpha)$ coincides with that of $\mu_2(\lambda,\beta)$, then by Theorem 2.1 and Theorem 2.3, $\beta = \beta_{\lambda,\alpha}$ must satisfy $\beta_{\lambda,\alpha} = C_2^{-\frac{p+1}{p-1}} C_4^{\frac{p+1}{p-1}} \alpha^{\frac{p+1}{2}} \lambda^{-(N+2-p(N-2))/4}$. This corresponds to the fact that

$$\begin{aligned}
\Phi(u_{1,\lambda,\alpha}) &= \frac{1}{p+1}(1+o(1))\|u_{1,\lambda,\alpha}\|_{p+1}^{p+1} \\
&= C_2^{-\frac{p+1}{p-1}} \frac{1}{p+1}(1+o(1))\|w\|_{p+1}^{p+1}\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}} \\
&= (1+o(1))\beta_{\lambda,\alpha},
\end{aligned}$$

which will be shown in Section 4.

Since the proof of Theorems 2.2–2.4 are similar to that of Theorem 2.1, we only prove Theorem 2.1 in the rest of this paper.

3. Lemmas. Since (1.1) is autonomous, by translation, we may assume without loss of generality that $0 \in \Omega$. In Section 3 and Section 4, we consider the problem **(M.1)**. For simplicity, C denotes various positive constants independent of (λ, α). In particular, the character C which may appear repeatedly in the same inequality sometimes denotes different constants independent of (λ, α). Further, a subsequence of a sequence will be denoted by the same notation as that of original sequence. Finally, for convenience, $K_0 = K_1 = J_0 = J_1 = 1$ in what follows. By (1.2) and (1.3), for $t \geq 0$ we have

$$C(t^p + t^q) \leq f(t) \leq C^{-1}(t^p + t^q), \tag{3.1}$$

$$Ct \leq g(t) \leq C^{-1}t, \tag{3.2}$$

$$C(\|u\|_{p+1}^{p+1} + \|u\|_{q+1}^{q+1}) \leq (f(u), u) \leq C^{-1}(\|u\|_{p+1}^{p+1} + \|u\|_{q+1}^{q+1}), \tag{3.3}$$

$$C(\|u\|_{p+1}^{p+1} + \|u\|_{q+1}^{q+1}) \leq \Phi(u) \leq C^{-1}(\|u\|_{p+1}^{p+1} + \|u\|_{q+1}^{q+1}), \tag{3.4}$$

$$C\|u\|_2^2 \leq (g(u), u) \leq C^{-1}\|u\|_2^2, \tag{3.5}$$

$$C\|u\|_2^2 \leq \Psi(u) \leq C^{-1}\|u\|_2^2. \tag{3.6}$$

We can prove the existence directly by choosing a maximizing sequence $\{u_n\} \subset N_{\lambda,\alpha}$ of (2.1), since $\sup_{u \in N_{\lambda,\alpha}} \Phi(u) < \infty$ for a fixed $(\lambda, \alpha) \in \mathbf{R}_+^2$. In fact, by (3.4) and the Gagliardo-Nirenberg inequality (cf. [7])

$$\|u\|_{\eta+1}^{\eta+1} \leq C\|u\|_2^{\frac{N+2-\eta(N-2)}{2}} \|u\|_X^{\frac{N(\eta-1)}{2}} \quad (1 < \eta < (N+2)/(N-2)) \tag{3.7}$$

for $u \in W_0^{1,2}(\Omega)$, we obtain that $\sup_{u \in N_{\lambda,\alpha}} \Phi(u) < \infty$.

The aim of this section is to estimate $\mu_1(\lambda, \alpha)$ from below and above by λ and α.

Lemma 3.1. *Assume that* $\{(\lambda, \alpha)\} \subset \mathbf{R}_+^2$ *satisfies (B.1) and (B.2). Then*

$$\mu_1(\lambda, \alpha) \leq C\alpha^{\frac{1-p}{2}} \lambda^{\frac{N+2-p(N-2)}{4}}. \tag{3.8}$$

To prove Lemma 3.1, we need some preparations.

Lemma 3.2. *For $\tau > 0$, let $w_\tau \in C^2(B_\tau)$ be the unique solution of the equation*

$$\begin{aligned} &\triangle w_\tau + w_\tau^p - w_\tau = 0 \quad \textit{in } B_\tau := \{x \in \mathbf{R}^N : |x| < \tau\}, \\ &w_\tau > 0 \quad \textit{in } B_\tau, \quad w_\tau = 0 \quad \textit{on } \partial B_\tau. \end{aligned} \tag{3.9}$$

Then $w_\tau \to w$ not only in $H^1(\mathbf{R}^N)$, but also uniformly on any compact subset in $\mathbf{R}^N$ as $\tau \to \infty$.

The unique existence of w_τ follows from Kwong [13], and the latter assertion can be proved by the similar arguments as those of Lemmas 4.5, 4.7 – 4.8 in Section 4. Hence we omit the proof. By [10], w_τ is radially symmetric, that is, $w_\tau(x) = w_\tau(r)$ $(r = |x|)$.

Lemma 3.3. *Assume that $\{(\lambda, \alpha)\} \subset \mathbf{R}_+^2$ satisfies (B.1) and (B.2). Let $w_{\sqrt{\lambda} r_0}$ be the solution of (3.9) for $\tau = \sqrt{\lambda} r_0$, where $0 < r_0 \ll 1$ is a constant. Put*

$$U_{\lambda,\alpha}(|x|) := \begin{cases} c_{\lambda,\alpha}\alpha^{1/2}\lambda^{(N-2)/4} w_{\sqrt{\lambda} r_0}(\sqrt{\lambda}|x|), & x \in B_{r_0} := \{x \in \mathbf{R}^N : |x| < r_0\} \subset \Omega, \\ 0, & x \in \Omega \setminus B_{r_0}, \end{cases}$$

where $c_{\lambda,\alpha} := \min\{c > 0 : c\alpha^{1/2}\lambda^{(N-2)/4} w_{\sqrt{\lambda} r_0}(\sqrt{\lambda}|x|) \in N_{\lambda,\alpha}\}$. Then $C \le c_{\lambda,\alpha} \le C^{-1}$.

Proof. For $t \ge 0$, let $m_{\lambda,\alpha}(t) := \Lambda_\lambda(tU_{\lambda,\alpha}) = \dfrac{1}{2}\|\nabla(tU_{\lambda,\alpha})\|_2^2 + \lambda\Psi(tU_{\lambda,\alpha})$. Then clearly $m_{\lambda,\alpha}(0) = 0$ and $m_{\lambda,\alpha}(t) \to \infty$ as $t \to \infty$ for a fixed (λ, α). Hence $c_{\lambda,\alpha} > 0$ exists. Since

$$\|\nabla U_{\lambda,\alpha}\|_2^2 = c_{\lambda,\alpha}^2 \alpha \|\nabla w_{\sqrt{\lambda} r_0}\|_2^2, \quad \lambda\|U_{\lambda,\alpha}\|_2^2 = c_{\lambda,\alpha}^2 \alpha \|w_{\sqrt{\lambda} r_0}\|_2^2,$$

by (3.6), we obtain

$$\alpha = \Lambda_\lambda(U_{\lambda,\alpha}) \sim c_{\lambda,\alpha}^2 \alpha \left(\frac{1}{2}\|\nabla w_{\sqrt{\lambda} r_0}\|_2^2 + C^{-1}\|w_{\sqrt{\lambda} r_0}\|_2^2 \right). \tag{3.10}$$

By Lemma 3.2 and (3.10) we obtain our conclusion. □

Proof of Lemma 3.1. By direct calculation we have

$$\|U_{\lambda,\alpha}\|_{p+1}^{p+1} = c_{\lambda,\alpha}^{p+1} \|w_{\sqrt{\lambda} r_0}\|_{p+1}^{p+1} \alpha^{\frac{p+1}{2}} \lambda^{-\frac{N+2-p(N-2)}{4}};$$

this along with (2.1), (3.3), (3.4) and Lemmas 3.2 – 3.3 implies

$$\begin{aligned} (f(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha}) &\ge C\Phi(u_{1,\lambda,\alpha}) \ge C\Phi(U_{\lambda,\alpha}) \ge C\|U_{\lambda,\alpha}\|_{p+1}^{p+1} \\ &\ge C\alpha^{\frac{p+1}{2}} \lambda^{-\frac{N+2-p(N-2)}{4}} \end{aligned} \tag{3.11}$$

Furthermore, since $u_{1,\lambda,\alpha} \in N_{\lambda,\alpha}$, we have

$$\|\nabla u_{1,\lambda,\alpha}\|_2^2,\ \ \lambda\|u_{1,\lambda,\alpha}\|_2^2 \le C\alpha. \tag{3.12}$$

Then, by (2.3), (3.6), (3.11) and (3.12)

$$\mu_1(\lambda,\alpha) \le \frac{2\alpha + C\lambda\|u_{1,\lambda,\alpha}\|_2^2}{(f(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha})} \le C\alpha^{\frac{(1-p)}{2}}\lambda^{\frac{N+2-p(N-2)}{4}}.$$

Thus the proof is complete. □

Lemma 3.4. *Assume that* $\{(\lambda,\alpha)\} \subset \mathbf{R}_+^2$ *satisfies (B.1) and (B.2). Then*

$$\mu_1(\lambda,\alpha) \ge C\alpha^{\frac{1-p}{2}}\lambda^{\frac{N+2-p(N-2)}{4}}. \tag{3.13}$$

Proof. Since $u_{1,\lambda,\alpha} \in N_{\lambda,\alpha}$, we obtain by (3.6) that there exists a constant $\delta > 0$ such that

$$\|\nabla u_{1,\lambda,\alpha}\|_2^2 + \lambda(g(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha}) \ge \delta\left\{\frac{1}{2}\|\nabla u_{1,\lambda,\alpha}\|_2^2 + \lambda\Psi(u_{1,\lambda,\alpha})\right\} = \delta\Lambda_\lambda(u_{1,\lambda,\alpha}) = \delta\alpha. \tag{3.14}$$

Then we obtain by (B.2), (3.7) and (3.12) that

$$\begin{aligned} \|u_{1,\lambda,\alpha}\|_{p+1}^{p+1} &\le C\|u_{1,\lambda,\alpha}\|_2^{\frac{N+2-p(N-2)}{2}}\|\nabla u_{1,\lambda,\alpha}\|_2^{\frac{N(p-1)}{2}} \le C\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}}, \\ \|u_{1,\lambda,\alpha}\|_{q+1}^{q+1} &\le C\|u_{1,\lambda,\alpha}\|_2^{\frac{N+2-q(N-2)}{2}}\|\nabla u_{1,\lambda,\alpha}\|_2^{\frac{N(q-1)}{2}} \le C\alpha^{\frac{q+1}{2}}\lambda^{-\frac{N+2-q(N-2)}{4}} \\ &\le C(\alpha^2\lambda^{N-2})^{\frac{q-p}{4}}\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}} \le C\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}}. \end{aligned} \tag{3.15}$$

Then by (3.3) and (3.15), we obtain

$$(f(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha}) \le C(\|u_{1,\lambda,\alpha}\|_{p+1}^{p+1} + \|u_{1,\lambda,\alpha}\|_{q+1}^{q+1}) \le C\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}}. \tag{3.16}$$

Then by (2.5), (3.14) and (3.16), we obtain

$$\begin{aligned} \mu_1(\lambda,\alpha) &= \frac{\|\nabla u_{1,\lambda,\alpha}\|_2^2 + \lambda(g(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha})}{(f(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha})} \\ &\ge \frac{\delta\alpha}{C\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}}} \ge C\alpha^{\frac{1-p}{2}}\lambda^{\frac{N+2-p(N-2)}{4}}. \end{aligned}$$

□

4. Proof of Theorem 2.1. We put

$$\xi_{1,\lambda,\alpha} := (\lambda/\mu_1(\lambda,\alpha))^{1/(p-1)}, \ v_{1,\lambda,\alpha}(x) := \xi_{1,\lambda,\alpha}^{-1} u_{1,\lambda,\alpha}(x),$$
$$\Omega_\lambda := \{y \in R^N : y = \sqrt{\lambda}x, x \in \Omega\}, \ w_{1,\lambda,\alpha}(y) := \xi_{1,\lambda,\alpha}^{-1} u_{1,\lambda,\alpha}(x) \ (y := \sqrt{\lambda}x),$$
$$h_0(t) := g(t) - t, \ H_0(t) := \int_0^t h_0(s)ds, \ h_1(t) := f(t) - |t|^{p-1}t, \ H_1(t) := \int_0^t h_1(s)ds.$$

Then by (1.1), we see that $v_{1,\lambda,\alpha}$ and $w_{1,\lambda,\alpha}$ satisfy the following equations, respectively:

$$-\frac{1}{\lambda}\Delta v_{1,\lambda,\alpha} = v_{1,\lambda,\alpha}^p + \xi_{1,\lambda,\alpha}^{-p} h_1(\xi_{1,\lambda,\alpha} v_{1,\lambda,\alpha}) - v_{1,\lambda,\alpha} - \xi_{1,\lambda,\alpha}^{-1} h_0(\xi_{1,\lambda,\alpha} v_{1,\lambda,\alpha}) \quad \text{in } \Omega,$$
$$v_{1,\lambda,\alpha} > 0 \quad \text{in } \Omega, \ v_{1,\lambda,\alpha} = 0 \quad \text{on } \partial\Omega, \tag{4.1}$$
$$-\Delta w_{1,\lambda,\alpha} = w_{1,\lambda,\alpha}^p - w_{1,\lambda,\alpha} + \xi_{1,\lambda,\alpha}^{-p} h_1(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}) - \xi_{1,\lambda,\alpha}^{-1} h_0(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}) \quad \text{in } \Omega_\lambda,$$
$$w_{1,\lambda,\alpha} > 0 \quad \text{in } \Omega_\lambda, \quad w_{1,\lambda,\alpha} = 0 \quad \text{on } \partial\Omega_\lambda. \tag{4.2}$$

If $\{\lambda, \alpha)\} \subset \mathbf{R}_+^2$ satisfies (B.1) and (B.2), then by Lemma 3.1, we obtain

$$\xi_{1,\lambda,\alpha}^{p-1} = \frac{\lambda}{\mu_1(\lambda,\alpha)} \geq C(\alpha^2\lambda^{N-2})^{\frac{p-1}{4}} \to \infty. \tag{4.3}$$

By Lemma 3.1, we easily obtain the following Lemma 4.1.

Lemma 4.1. *Assume that* $\{(\lambda,\alpha)\} \subset R_+^2$ *satisfies (B.1) and (B.2). Then*

$$\|\nabla w_{1,\lambda,\alpha}\|_2^2 \leq C, \tag{4.4}$$
$$\|w_{1,\lambda,\alpha}\|_2^2 \leq C, \tag{4.5}$$
$$\|w_{1,\lambda,\alpha}\|_{\eta+1}^{\eta+1} \leq C \quad (1 \leq \eta \leq (N+2)/(N-2)). \tag{4.6}$$

Lemma 4.2. *Assume that* $\{(\lambda,\alpha)\} \subset \mathbf{R}_+^2$ *satisfies (B.1) and (B.2). Then*

(i) $\sup_{x\in\Omega} v_{1,\lambda,\alpha}(x) \leq C$.

(ii) $c_\tau \lambda^{-N/2} \leq \int_\Omega v_{1,\lambda,\alpha}^\tau dx \leq C_\tau \lambda^{-N/2}$ *if* $1 \leq \tau < \infty$.

Proof. By (4.4) and (4.5), we obtain

$$\int_\Omega \left(\frac{1}{\lambda}|\nabla v_{1,\lambda,\alpha}|^2 + v_{1,\lambda,\alpha}^2\right) dx = \xi_{1,\lambda,\alpha}^{-2}\left(\frac{1}{\lambda}\|\nabla u_{1,\lambda,\alpha}\|_2^2 + \|u_{1,\lambda,\alpha}\|_2^2\right)$$
$$= (\|\nabla w_{1,\lambda,\alpha}\|^2 + \|w_{1,\lambda,\alpha}\|_2^2)\lambda^{-N/2} \le C\lambda^{-N/2}. \tag{4.7}$$

Furthermore, by (3.6) and Lemma 3.4, we obtain

$$\int_\Omega \left(\frac{1}{\lambda}|\nabla v_{1,\lambda,\alpha}|^2 + v_{1,\lambda,\alpha}^2\right) dx \ge C\xi_{1,\lambda,\alpha}^{-2}\lambda^{-1}\Lambda_\lambda(u_{1,\lambda,\alpha}) = C\xi_{1,\lambda,\alpha}^{-2}\lambda^{-1}\alpha$$
$$= C\{\mu_1(\lambda,\alpha)^{\frac{2}{p-1}}\alpha\lambda^{-\frac{N+2-p(N-2)}{2(p-1)}}\}\lambda^{-N/2} \ge C\lambda^{-N/2}. \tag{4.8}$$

Once (4.7) and (4.8) which correspond to Lin, Ni and Takagi [14, Corollary 2.1 (2.6), Proposition 2.2] are established, then (i) and (ii) follow from exactly the same arguments used in the proof of [14, Lemma 2.3 and Corollary 2.1 (2.7)] by using (4.7) and (4.8). Hence the proof is complete. □

Lemma 4.3. *Assume that* $\{(\lambda,\alpha)\} \subset \mathbf{R}_+^2$ *satisfies (B.1) and (B.2). Then* $\|v_{1,\lambda,\alpha}\|_\infty \ge C$.

Lemma 4.4. *Assume that* $\{(\lambda,\alpha)\} \subset \mathbf{R}_+^2$ *satisfies (B.1) and (B.2). Then*

$$\rho_{\lambda,\alpha} := \lambda^{1/2} dist(x_{1,\lambda,\alpha}, \partial\Omega) \to \infty.$$

Lemma 4.5. *Assume that* $\{(\lambda,\alpha)\} \subset \mathbf{R}_+^2$ *satisfies (B.1) and (B.2). Furthermore, let* $y_{1,\lambda,\alpha} := \sqrt{\lambda}x_{1,\lambda,\alpha} \in R^N$. *Then for any subsequence* $S \subset \{(\lambda,\alpha)\}$, *there exists a subsequence* $\{(\lambda_j,\alpha_j)_{j\in N}\}$ *of* S *such that* $z_j(y) := w_{1,\lambda_j,\alpha_j}(y+y_{1,\lambda_j,\alpha_j}) \to w(y)$ *on any compact subset in* $\mathbf{R}^N$ *as* $j \to \infty$.

Lemmas 4.3–4.5 follow from Lemma 4.1, Lemma 4.2 and exactly the same arguments used in the proof of Ni and Wei [16, Step 1 (proof of (3.2)), p. 737–738]. Furthermore, the following Lemma 4.6 is a direct consequence of (1.2), (4.3) and Lemma 4.2 (ii). Hence we omit the proofs.

Lemma 4.6. *Assume that* $\{(\lambda,\alpha)\} \subset \mathbf{R}^2_+$ *satisfies (B.1) and (B.2). Then*

$$\begin{aligned}
&\xi_{1,\lambda,\alpha}^{-p}\int_{\Omega_\lambda} h_1(\xi_{1,\lambda,\alpha}w_{1,\lambda,\alpha}(y))w_{1,\lambda,\alpha}(y)dy \to 0,\\
&\xi_{1,\lambda,\alpha}^{-(p+1)}\int_{\Omega_\lambda} H_1(\xi_{1,\lambda,\alpha,\alpha}w_{1,\lambda,\alpha}(y))dy \to 0,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\xi_{1,\lambda,\alpha}^{-1}\int_{\Omega_\lambda} h_0(\xi_{1,\lambda,\alpha}w_{1,\lambda,\alpha}(y))w_{1,\lambda,\alpha}(y)dy \to 0,\\
&\xi_{1,\lambda,\alpha}^{-2}\int_{\Omega_\lambda} H_0(\xi_{1,\lambda,\alpha,\alpha}w_{1,\lambda,\alpha}(y))dy \to 0.
\end{aligned} \tag{4.10}$$

Lemma 4.7. *Assume* $\{(\lambda,\alpha)\} \subset \mathbf{R}^2_+$ *satisfies (B.1) and (B.2). Then*

$$\|w\|_{p+1} \le \liminf \|w_{1,\lambda,\alpha}\|_{p+1} \le \limsup \|w_{1,\lambda,\alpha}\|_{p+1} \le \|w\|_{p+1}. \tag{4.11}$$

Proof. The first inequality in (4.11) follows from (4.6), Lemma 4.5 and Fatou's lemma. We show the last inequality. First, multiply (2.6) by w. Then integration by parts yields

$$\|\nabla w\|_2^2 + \|w\|_2^2 = \|w\|_{p+1}^{p+1}. \tag{4.12}$$

Let $B_{r_0} \subset \Omega$. Furthermore, let $\chi_\lambda \in C^2(\mathbf{R}^N)$ satisfy

$$\chi_\lambda(y) = \begin{cases} 1, & |y| \le \sqrt{\lambda}r_0 - 1,\\ 0, & |y| \ge \sqrt{\lambda}r_0, \end{cases}$$

and

$$0 \le \chi_\lambda(y) \le 1, \quad |\nabla\chi_\lambda(y)| \le C \quad \text{for } y \in \mathbf{R}^N, \ \lambda \gg 1.$$

Let $V_\lambda(y) = w(y)\chi_\lambda(y)$ for $y \in \mathbf{R}^N$. Then for $\lambda \gg 1$, clearly, we have

$$\|\nabla V_\lambda\|_2 = (1+o(1))\|\nabla w\|_2, \quad \|V_\lambda\|_2 = (1+o(1))\|w\|_2, \quad \|V_\lambda\|_{p+1} = (1+o(1))\|w\|_{p+1}. \tag{4.13}$$

Let $c_\lambda := \inf\{c > 0 : cV_\lambda(\sqrt{\lambda}x) \in N_{\lambda,\alpha}\}$ and $e_\lambda(x) := c_\lambda V_\lambda(\sqrt{\lambda}x)$. Then we can easily show that $c_\lambda \to \infty$ as $\lambda \to \infty$. By using this and (1.2), we obtain

$$\left|\int_\Omega H_0(e_\lambda(x))dx\right| = o(1)\|e_\lambda\|_2^2.$$

By this and (4.13), we obtain

$$\begin{aligned}\alpha = \Lambda_\lambda(e_\lambda) &= \frac{1}{2}\|\nabla e_\lambda\|_2^2 + \frac{1}{2}\lambda\left(\|e_\lambda\|_2^2 + \int_\Omega H_0(e_\lambda(x))dx\right)\\ &= \frac{1}{2}c_\lambda^2\lambda^{\frac{2-N}{2}}(\|\nabla V_\lambda\|_2^2 + (1+o(1))\|V_\lambda\|_2^2) = \frac{1}{2}c_\lambda^2\lambda^{\frac{2-N}{2}}(\|\nabla w\|_2^2 + (1+o(1))\|w\|_2^2)\\ &= \frac{1}{2}c_\lambda^2\lambda^{\frac{2-N}{2}}(1+o(1))\|w\|_{p+1}^{p+1}.\end{aligned} \tag{4.14}$$

Similarly, we also obtain

$$\int_\Omega H_1(e_\lambda(x))dx = o(1)\|e_\lambda\|_{p+1}^{p+1} = o(1)c_\lambda^{p+1}\lambda^{-N/2}\|V_\lambda\|_{p+1}^{p+1}. \tag{4.15}$$

By (2.1) we have $\Phi(u_{1,\lambda,\alpha}) \geq \Phi(e_\lambda)$, namely,

$$\frac{1}{p+1}\|u_{1,\lambda,\alpha}\|_{p+1}^{p+1} + \int_\Omega H_1(u_{1,\lambda,\alpha}(x))dx \geq \frac{1}{p+1}\|e_\lambda\|_{p+1}^{p+1} + \int_\Omega H_1(e_\lambda(x))dx.$$

This along with (4.10), (4.13) and (4.15) yields

$$\begin{aligned}&(1+o(1))\xi_{\lambda,\alpha}^{p+1}\lambda^{-N/2}\|w_{1,\lambda,\alpha}\|_{p+1}^{p+1} = (1+o(1))\|u_{1,\lambda,\alpha}\|_{p+1}^{p+1} \geq (1+o(1))\|e_\lambda\|_{p+1}^{p+1}\\ &= (1+o(1))c_\lambda^{p+1}\lambda^{-N/2}\|V_\lambda\|_{p+1}^{p+1} = (1+o(1))c_\lambda^{p+1}\lambda^{-N/2}\|w\|_{p+1}^{p+1}.\end{aligned}$$

This along with (4.14) implies that

$$\|w_{1,\lambda,\alpha}\|_{p+1}^{p+1} \geq (1+o(1))(2\alpha)^{\frac{p+1}{2}}\lambda^{-\frac{(p+1)(N+2-p(N-2))}{4(p-1)}}\mu_1(\lambda,\alpha)^{\frac{p+1}{p-1}}\|w\|_{p+1}^{-\frac{(p+1)(p-1)}{2}}. \tag{4.16}$$

Finally, by Lemma 3.4, (4.9) and (4.10), we obtain

$$\begin{aligned}&\lambda\{(g(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha}) - 2\Psi(u_{1,\lambda,\alpha})\}\\ &= \lambda\left\{\int_\Omega h_0(u_{1,\lambda,\alpha}(x))u_{1,\lambda,\alpha}(x)dx - 2\int_\Omega H_0(u_{1,\lambda,\alpha}(x))dx\right\}\\ &= \xi_{1,\lambda,\alpha}\lambda^{\frac{2-N}{2}}\int_{\Omega_\lambda} h_0(\xi_{1,\lambda,\alpha}w_{1,\lambda,\alpha}(y))w_{1,\lambda,\alpha}(y)dy - 2\lambda^{\frac{2-N}{2}}\int_{\Omega_\lambda} H_0(\xi_{1,\lambda,\alpha}w_{1,\lambda,\alpha}(y))dy\\ &= o(1)\xi_{1,\lambda,\alpha}^2\lambda^{\frac{2-N}{2}} = o(1)\mu_1(\lambda,\alpha)^{-\frac{2}{p-1}}\lambda^{\frac{N+2-p(N-2)}{2(p-1)}} = o(1)\alpha.\end{aligned}$$

This along with (2.3) and (4.9) yields

$$\mu_1(\lambda,\alpha) = \frac{2(1+o(1))\alpha}{(1+o(1))\|u_{1,\lambda,\alpha}\|_{p+1}^{p+1}} = \frac{2(1+o(1))\alpha}{(1+o(1))\xi_{\lambda,\alpha}^{p+1}\lambda^{-N/2}\|w_{1,\lambda,\alpha}\|_{p+1}^{p+1}}.$$

This implies

$$\mu_1(\lambda,\alpha)^{\frac{2}{p-1}} = \frac{(1+o(1))\lambda^{\frac{N+2-p(N-2)}{2(p-1)}}\|w_{1,\lambda,\alpha}\|_{p+1}^{p+1}}{2(1+o(1))\alpha}. \tag{4.17}$$

By substituting (4.17) into (4.16), we obtain

$$\|w\|_{p+1}^{\frac{(p+1)(p-1)}{2}} \geq (1-o(1))\|w_{1,\lambda,\alpha}\|_{p+1}^{\frac{(p+1)(p-1)}{2}}.$$

Thus we obtain the last inequality in (4.11). □

By Lemma 4.7, we easily obtain:

Lemma 4.8. *Assume that* $\{(\lambda,\alpha)\} \subset \mathbf{R}_+^2$ *satisfies (B.1) and (B.2). Then*

$$\|w_{1,\lambda,\alpha}\|_2 \to \|w\|_2, \quad \|\nabla w_{1,\lambda,\alpha}\|_2 \to \|\nabla w\|_2. \tag{4.18}$$

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1: By Lemma 4.6 and Lemma 4.8, we obtain

$$\begin{aligned}\Psi(u_{1,\lambda,\alpha}) &= \frac{1}{2}\|u_{1,\lambda,\alpha}\|_2^2 + \int_\Omega H_0(u_{1,\lambda,\alpha}(x))dx \\ &= \frac{1}{2}\lambda^{-N/2}\xi_{1,\lambda,\alpha}^2\|w_{1,\lambda,\alpha}\|_2^2 + \lambda^{-N/2}\int_{\Omega_\lambda} H_0(\xi_{1,\lambda,\alpha}w_{1,\lambda,\alpha}(y))dy \\ &= \frac{1}{2}\lambda^{-N/2}\xi_{1,\lambda,\alpha}^2(\|w\|_2^2 + o(1)).\end{aligned} \tag{4.19}$$

Then by (4.12) and (4.19)

$$\begin{aligned}\alpha = \Lambda_\lambda(u_{1,\lambda,\alpha}) &= \frac{1}{2}\|\nabla u_{1,\lambda,\alpha}\|_2^2 + \lambda\Psi(u_{1,\lambda,\alpha}) \\ &= \frac{1}{2}\xi_{1,\lambda,\alpha}^2\lambda^{(2-N)/2}\left((1+o(1))\|\nabla w\|_2^2 + (1+o(1))\|w\|_2^2\right) \\ &= \frac{1}{2}(1+o(1))\xi_{1,\lambda,\alpha}^2\lambda^{(2-N)/2}\|w\|_{p+1}^{p+1};\end{aligned}$$

this implies

$$\mu_1(\lambda,\alpha)^{-\frac{2}{p-1}}\lambda^{\frac{N+2-p(N-2)}{2(p-1)}} = \frac{2\alpha}{(1+o(1))\|w\|_{p+1}^{p+1}}. \tag{4.20}$$

Now, Theorem 2.1 is a direct consequence of (4.20). For the case where $K_0 \neq 1$, $K_1 \neq 1$, we have only to replace $\lambda, \mu_1(\lambda,\alpha)$ by $K_0\lambda, K_1\mu_1(\lambda,\alpha)$, respectively. □

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