Title: Stability of solitary waves for coupled Klein-Gordon-Schrödinger equations in one space dimension (Variational Problems and Related Topics)

Author(s): Ohta, Masahito

Citation: 数理解析研究所講究録 (1999), 1076: 83-92

Issue Date: 1999-02

URL: http://hdl.handle.net/2433/62635

Type: Departmental Bulletin Paper

Textversion: publisher

Kyoto University
Stability of solitary waves for coupled Klein-Gordon-Schrödinger equations in one space dimension

Masahito Ohta (太田 雅人)
Faculty of Engineering, Shizuoka University, Hamamatsu 432-8561, Japan
Email:tsmoota@eng.shizuoka.ac.jp

1 Introduction and Main Result

In this note we consider the stability of solitary wave solutions for the Yukawa coupled Klein-Gordon-Schrödinger equations in one space dimension:

\[
\begin{align}
  i\partial_t u + \partial_x^2 u &= uv, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \\
  \partial_t^2 v - \partial_x^2 v + v &= -|u|^2, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.
\end{align}
\] (1.1) (1.2)

Here, \( u = u(t, x) \) and \( v = v(t, x) \) describe a complex scalar nucleon field and a real scalar meson field, respectively (see Fukuda and M. Tsutsumi [3] and Yukawa [11]). In Section 5 of [3], Fukuda and M. Tsutsumi showed that (1.1)--(1.2) admits the following two types of exact solitary wave solutions (1.3)--(1.4) and (1.5)--(1.6):

(I) when \( \lambda^2 < 1 \) and \( \mu = \lambda^2/4 + 1/(1 - \lambda^2) \),

\[
\begin{align}
  u(t,x) &= \frac{3}{2\sqrt{1-\lambda^2}} \text{sech}^2 \left( \frac{x-\lambda t}{2\sqrt{1-\lambda^2}} \right) \exp[i\mu t + i(\lambda/2)(x-\lambda t)], \\
  v(t,x) &= -\frac{3}{2(1-\lambda^2)} \text{sech}^2 \left( \frac{x-\lambda t}{2\sqrt{1-\lambda^2}} \right),
\end{align}
\] (1.3) (1.4)
(II) when $\lambda^2 = 1$ and $\mu > 1/4$, 

$$u(t, x) = \sqrt{2(\mu - 1/4)} \operatorname{sech} \left( \sqrt{\mu - 1/4} (x - \lambda t) \right) \times \exp[i \mu t + i(\lambda/2)(x - \lambda t)],$$

$$v(t, x) = -2(\mu - 1/4) \operatorname{sech}^2 \left( \sqrt{\mu - 1/4} (x - \lambda t) \right),$$

and they proposed a problem of whether the solitary wave solutions are stable or not. The purpose of this note is to give a partial answer to the problem.

To explain our results precisely, we prepare some function spaces and functionals. Let $X = H^1(\mathbb{R}; \mathbb{C}) \times H^1(\mathbb{R}; \mathbb{R}) \times L^2(\mathbb{R}; \mathbb{R})$ be a real Hilbert space with the inner product

$$(u, v, w), (\psi, \phi, \varphi) \in X.$$

Then, (1.1)–(1.2) is written as abstract Hamiltonian system in the form

$$\frac{d}{dt} \vec{u}(t) = JE'(\vec{u}(t)),$$

where

$$\vec{u}(t) = \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} \in X, \quad J = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

and $E$ is the energy functional on $X$ defined by

$$E(u, v, w) = \int_{\mathbb{R}} \left\{ |\partial_x u|^2 + |u|^2 v + \frac{1}{2} \left( w^2 + (\partial_x v)^2 + v^2 \right) \right\} dx.$$  

The following global existence of solutions of the Cauchy problem for (1.7) is known.
Proposition 1.1 For any $\vec{u}_0 \in X$ there exists a unique solution $\vec{u} \in C(\mathbb{R}; X)$ of (1.7) with $\vec{u}(0) = \vec{u}_0$ satisfying
\[ E(\vec{u}(t)) = E(\vec{u}_0), \quad P(\vec{u}(t)) = P(\vec{u}_0), \quad Q(\vec{u}(t)) = Q(\vec{u}_0), \quad t \in \mathbb{R}. \] (1.11)

For $\lambda, \mu \in \mathbb{R}$, we put
\[ S_{\lambda, \mu}(\vec{u}) = E(\vec{u}) + \lambda P(\vec{u}) + \mu Q(\vec{u}), \quad \vec{u} \in X. \]

Then, we have
\[ S_{\lambda, \mu}(u, v, w) = \int_{\mathbb{R}} \left\{ \left| \partial_x (e^{-i\lambda x/2} u) \right|^2 + (\mu - \lambda^2/4) |u|^2 + |u|^2 v 
+ \frac{1}{2} \left( (w + \lambda \partial_x v)^2 + (1 - \lambda^2)(\partial_x v)^2 + v^2 \right) \right\} dx, \] (1.12)

and $T(\lambda t, \mu t) \vec{\psi}_{\lambda, \mu}$ is a solution of (1.7) if $\vec{\psi}_{\lambda, \mu}$ is a solution of $S'_{\lambda, \mu}(\vec{\psi}) = 0$. Note that $S'_{\lambda, \mu}(\psi, \phi, \varphi) = E'(\psi, \phi, \varphi) + \lambda P'(\psi, \phi, \varphi) + \mu Q'(\psi, \phi, \varphi) = 0$ is equivalent to
\[ -\partial_x^2 \psi + i\lambda \partial_x \psi + \mu \psi + \psi \phi = 0, \] (1.13)
\[ -(1 - \lambda^2) \partial_x^2 \phi + \phi + |\psi|^2 = 0, \] (1.14)
\[ \varphi + \lambda \partial_x \phi = 0, \] (1.15)

and also that by $\psi(x) = e^{i(\lambda/2)x} \tilde{\psi}(x)$, (1.13) is transformed into
\[ -\partial_x^2 \tilde{\psi} + (\mu - \lambda^2/4) \tilde{\psi} + \tilde{\psi} \phi = 0. \] (1.16)

Thus, when $\lambda^2 < 1$ and $\mu = \lambda^2/4 + 1/(1 - \lambda^2)$, if we put
\[ \psi_{\lambda, \mu}(x) = \frac{3}{2\sqrt{1 - \lambda^2}} \text{sech} \left( \frac{x}{2\sqrt{1 - \lambda^2}} \right) \exp[i(\lambda/2)x], \] (1.17)
\[ \phi_{\lambda, \mu}(x) = -\frac{3}{2(1 - \lambda^2)} \text{sech}^2 \left( \frac{x}{2\sqrt{1 - \lambda^2}} \right), \] (1.18)
\[ \varphi_{\lambda, \mu}(x) = -\lambda \partial_x \phi_{\lambda, \mu}(x), \] (1.19)

then $\vec{\psi}_{\lambda, \mu} = (\psi_{\lambda, \mu}, \phi_{\lambda, \mu}, \varphi_{\lambda, \mu})$ is a solution of $S'_{\lambda, \mu}(\vec{\psi}) = 0$, and when $\lambda^2 = 1$ and $\mu > 1/4$, if we put
\[ \psi_{\lambda, \mu}(x) = \sqrt{2(\mu - 1/4)} \text{sech} \left( \sqrt{\mu - 1/4} x \right) \exp[i(\lambda/2)x], \] (1.20)
\[ \phi_{\lambda, \mu}(x) = -2(\mu - 1/4) \text{sech}^2 \left( \sqrt{\mu - 1/4} x \right), \] (1.21)
\[ \varphi_{\lambda, \mu}(x) = -\lambda \partial_x \phi_{\lambda, \mu}(x), \] (1.22)
then $\vec{\psi}_{\lambda,\mu} = (\psi_{\lambda,\mu}, \phi_{\lambda,\mu}, \varphi_{\lambda,\mu})$ is a solution of $S'_{\lambda,\mu}(\vec{\psi}) = 0$.

**Definition.** We say that a subset $\Sigma$ of $X$ is **stable** if for any $\varepsilon > 0$ there exists a $\delta > 0$ with the following property. If $\vec{u}_0 \in X$ satisfies $\inf\{\|\vec{u}_0 - \vec{\psi}\|_X : \vec{\psi} \in \Sigma\} < \delta$, then the solution $\vec{u}(t)$ of (1.7) with $\vec{u}(0) = \vec{u}_0$ exists for all $t \in \mathbb{R}$ and satisfies

$$\sup_{t \in \mathbb{R}} \inf_{\vec{\psi} \in \Sigma}\{\|\vec{u}(t) - \vec{\psi}\|_X : \vec{\psi} \in \Sigma\} < \varepsilon.$$ 

Moreover, let $\vec{\psi}_{\lambda,\mu}$ be a solution of $S'_{\lambda,\mu}(\vec{\psi}) = 0$. We say that the solitary wave solution $T(\lambda t, \mu t)\vec{\psi}_{\lambda,\mu}$ is **stable** if $\{T(\alpha, \beta)\vec{\psi}_{\lambda,\mu} : \alpha, \beta \in \mathbb{R}\}$ is stable.

We are now in a position to state our main result in this note.

**Theorem II.** Let $\lambda^2 = 1$ and $\mu > 1/4$. Then, the solitary wave solution $T(\lambda t, \mu t)\vec{\psi}_{\lambda,\mu}$ given by (1.20)–(1.22) is stable for any $\mu > 1/4$.

**Remark.** In my lecture at the conference, I announced that when $\lambda^2 < 1$ and $\mu = \lambda^2/4 + 1/(1 - \lambda^2)$, the solitary wave solution $T(\lambda t, \mu t)\vec{\psi}_{\lambda,\mu}$ given by (1.17)–(1.19) is stable if $\lambda^2$ is sufficiently close to 1. However, after the conference, I found a mistake in the proof. So, the stability of $T(\lambda t, \mu t)\vec{\psi}_{\lambda,\mu}$ given by (1.17)–(1.19) seems to be still an open problem.

## 2 Proof of Theorem II

In this section, we give the proof of Theorem II, basically along the argument in [7].

When $\lambda^2 = 1$ and $\mu > 1/4$, we obtain the following basic identity from (1.12).

$$S_{\lambda,\mu}(u, v, w) = \int_{\mathbb{R}} \left\{ \left| \partial_x (e^{-i\lambda x/2}u) \right|^2 + (\mu - 1/4)|u|^2 - \frac{1}{2}|u|^4 \right\} dx + \frac{1}{2} \int_{\mathbb{R}} \left\{ (w + \lambda \partial_x v)^2 + (|u|^2 + v)^2 \right\} dx. \quad (2.1)$$

Associated with the identity (2.1), we define for $\rho > 0$

$$S_{\lambda,\mu}^0(u) = \int_{\mathbb{R}} \left\{ \left| \partial_x (e^{-i\lambda x/2}u) \right|^2 + (\mu - 1/4)|u|^2 - \frac{1}{2}|u|^4 \right\} dx, \quad (2.2)$$

$$Q^0(u) = \int_{\mathbb{R}} |u|^2 dx, \quad (2.3)$$

$$I^0(\rho) = \inf\{S_{\lambda,\mu}^0(u) : u \in H^1(\mathbb{R}), Q^0(u) = \rho\}, \quad (2.4)$$

$$\Sigma^0(\rho) = \{u \in H^1(\mathbb{R}) : S_{\lambda,\mu}^0(u) = I^0(\rho), Q^0(u) = \rho\}. \quad (2.5)$$
For $\alpha, \beta \in \mathbb{R}$ and $u \in L^2(\mathbb{R})$, we define

$$T_1(\alpha)u(x) = u(x + \alpha), \quad T_2(\beta)u(x) = e^{i\beta}u(x).$$

**Lemma 2.1** Assume that $\lambda^2 = 1$ and $\mu > 1/4$. Let $\psi_{\lambda, \mu}$ be the function defined by (1.20). Then, we have

$$\Sigma^0(\rho(\mu)) = \{T_1(\alpha)T_2(\beta)\psi_{\lambda, \mu} : \alpha, \beta \in \mathbb{R}\},$$

where $\rho(\mu) = Q^0(\psi_{\lambda, \mu}) = 4\sqrt{\mu - 1/4}$.

**Lemma 2.2** Let $\rho > 0$. If $\{u_j\} \subset H^1(\mathbb{R})$ satisfies $S^0_{\lambda, \mu}(u_j) \to I^0(\rho)$ and $Q(\rho) \to \rho$, then there exist $\{\alpha_j\} \subset \mathbb{R}$, a subsequence of $\{T_1(\alpha_j)u_j\}$ (we still denote it by the same letter) and $\psi \in \Sigma^0(\rho)$ such that

$$T_1(\alpha_j)u_j \to \psi \text{ strongly in } H^1(\mathbb{R}).$$

Lemma 2.2 is proved by using the concentration compactness method introduced by Lions [6]. For the proofs of Lemmas 2.1 and 2.2, see Cazenave and Lions [1]. From Lemmas 2.1 and 2.2 and the conservation laws, one can show the stability of solitary wave solutions for the single nonlinear Schrödinger equation (for details, see [1]).

Following the idea by Cazenave and Lions [1], we consider the following minimization problem:

$$I(\rho) = \inf\{S_{\lambda, \mu}(\vec{u}) : \vec{u} \in X, Q(\vec{u}) = \rho\}, \quad (2.6)$$

$$\Sigma(\rho) = \{\vec{u} \in X : S_{\lambda, \mu}(\vec{u}) = I(\rho), Q(\vec{u}) = \rho\}. \quad (2.7)$$

From Lemma 2.1 and (2.1), we have

**Lemma 2.3** Assume that $\lambda^2 = 1$ and $\mu > 1/4$. For any $\rho > 0$, we have $I(\rho) = I^0(\rho)$ and

$$\Sigma(\rho) = \{\vec{\psi} = (\psi, \phi, \varphi) : \psi \in \Sigma^0(\rho), \phi = -|\psi|^2, \varphi = -\lambda \partial_x \phi\}. \quad (2.8)$$

Moreover, let $\vec{\psi}_{\lambda, \mu} = (\psi_{\lambda, \mu}, \phi_{\lambda, \mu}, \varphi_{\lambda, \mu})$ be the vector in $X$ given by (1.20)–(1.22). Then, we have

$$\Sigma(\rho(\mu)) = \{T_{1}(\alpha, \beta)\vec{\psi}_{\lambda, \mu} : \alpha, \beta \in \mathbb{R}\}, \quad (2.9)$$

where $\rho(\mu) = Q(\vec{\psi}_{\lambda, \mu}) = 4\sqrt{\mu - 1/4}$. 

87
Proof. First, we note that $S_{\lambda,\mu}^0(u) \leq S_{\lambda,\mu}(\vec{u})$ holds for all $\vec{u} = (u, v, w) \in X$, so that we have $I^0(\rho) \leq I(\rho)$. We put

$$\Sigma_1(\rho) = \{\vec{\psi} = (\psi, \phi, \varphi) : \psi \in \Sigma^0(\rho), \phi = -|\psi|^2, \varphi = -\lambda \partial_x \phi\}.$$ 

If $\vec{\psi} = (\psi, \phi, \varphi) \in \Sigma_1(\rho)$, then we have $Q(\vec{\psi}) = Q^0(\psi) = \rho$ and

$$I(\rho) \leq S_{\lambda,\mu}(\vec{\psi}) = S_{\lambda,\mu}^0(\psi) = I^0(\rho) \leq I(\rho).$$ 

Thus, we have $I(\rho) = I^0(\rho)$ and $\vec{\psi} \in \Sigma(\rho)$. Conversely, if $\vec{\psi} = (\psi, \phi, \varphi) \in \Sigma(\rho)$, then we have $Q(\vec{\psi}) = Q^0(\psi) = \rho$ and

$$I^0(\rho) \leq S_{\lambda,\mu}^0(\psi) \leq S_{\lambda,\mu}(\vec{\psi}) = I(\rho) = I^0(\rho).$$ 

Thus, we have $\vec{\psi} \in \Sigma_1(\rho)$. Hence, we obtain (2.8). (2.9) follows from Lemma 2.1 and (2.8). This completes the proof. \(\square\)

Lemma 2.4 Let $\rho > 0$ and $\vec{\psi}_0 \in \Sigma(\rho)$. If $\{\vec{u}_j\} = \{(u_j, v_j, w_j)\} \subset X$ satisfies

$$E(\vec{u}_j) \to E(\vec{\psi}_0), \quad P(\vec{u}_j) \to P(\vec{\psi}_0), \quad Q(\vec{u}_j) \to Q(\vec{\psi}_0). \tag{2.10}$$

then there exist $\{\alpha_j\} \subset \mathbb{R}$, a subsequence of $\{T(\alpha_j, 0)\vec{u}_j\}$ (we still denote it by the same letter) and $\vec{\psi}_1 = (\psi_1, \phi_1, \varphi_1) \in \Sigma(\rho)$ such that

$$T(\alpha_j, 0)\vec{u}_j \to \vec{\psi}_1 \quad \text{strongly in } X.$$ 

Proof. First, we note that by the Gagliardo-Nirenberg-Sobolev inequality and (2.10), we see that $\{\vec{u}_j\}$ is a bounded sequence in $X$, and

$$S_{\lambda,\mu}(\vec{u}_j) = E(\vec{u}_j) + \lambda P(\vec{u}_j) + \mu Q(\vec{u}_j) \to S_{\lambda,\mu}(\vec{\psi}_0) = I(\rho). \tag{2.11}$$ 

Since we have

$$I(\rho) = I^0(\rho) \leq S_{\lambda,\mu}^0(u_j) \leq S_{\lambda,\mu}(\vec{u}_j),$$

it follows from (2.11) and (2.10) that

$$S_{\lambda,\mu}^0(u_j) \to I^0(\rho), \quad Q^0(u_j) = Q(\vec{u}_j) \to \rho.$$
Thus, by Lemma 2.2, there exist $\{\alpha_j\} \subset \mathbb{R}$ and a subsequence of $\{T_1(\alpha_j)u_j\}$ (we still denote it by the same letter) and $\tilde{\psi} \in \Sigma^0(\rho)$ such that
\[
T_1(\alpha_j)u_j \rightarrow \tilde{\psi} \quad \text{strongly in } H^1(\mathbb{R})).
\] (2.12)

Since $\{\vec{u}_j\}$ is bounded in $X$, so is $\{T(\alpha_j, 0)\vec{u}_j\}$. Thus, there exists a subsequence $\{\vec{u}_j^1\} = \{(u_j^1, v_j^1, w_j^1)\}$ of $\{T(\alpha_j, 0)\vec{u}_j\}$ and $\vec{\psi}_1 = (\psi_1, \phi_1, \varphi_1) \in X$ such that
\[
\vec{u}_j^1 \rightharpoonup \vec{\psi}_1 \quad \text{weakly in } X.
\] (2.13)

By (2.12) and (2.13), we have $\psi_1 = \tilde{\psi} \in \Sigma^0(\rho)$ and
\[
u_j^1 \rightarrow \psi_1 \quad \text{strongly in } H^1(\mathbb{R}).
\] (2.14)

Moreover, from (2.1) and (2.14), we have
\[
|u_j^1|^2 + v_j^1 \rightarrow 0 \quad \text{strongly in } L^2(\mathbb{R}),
\] (2.15)
\[
w_j^1 + \lambda \partial_x v_j^1 \rightarrow 0 \quad \text{strongly in } L^2(\mathbb{R}).
\] (2.16)

From (2.13)–(2.16), we have
\[
v_j^1 \rightarrow \phi_1 = -|\psi_1|^2 \quad \text{strongly in } L^2(\mathbb{R}),
\] (2.17)
\[
\partial_x v_j^1 \rightharpoonup \partial_x \phi_1 \quad \text{weakly in } L^2(\mathbb{R}),
\] (2.18)
\[
w_j^1 \rightarrow \varphi_1 = -\lambda \partial_x \phi_1 \quad \text{weakly in } L^2(\mathbb{R}).
\] (2.19)

Since $\psi_1 \in \Sigma^0(\rho)$, $\phi_1 = -|\psi_1|^2$ and $\varphi_1 = -\lambda \partial_x \phi_1$, it follows from Lemma 2.3 that $\vec{\psi}_1 = (\psi_1, \phi_1, \varphi_1) \in \Sigma(\rho)$. Finally, we have to show the strong convergence of $\{\partial_x v_j^1\}$ and $\{w_j^1\}$ in $L^2(\mathbb{R})$. By the definition (1.9) and the convergences in (2.10) and (2.14), we have
\[
\int_{\mathbb{R}} w_j^1 \partial_x v_j^1 dx = P(\vec{u}_j^1) - \int_{\mathbb{R}} i\bar{u}_j^1 \partial_x u_j^1 dx = P(\vec{u}_j) - \int_{\mathbb{R}} i\bar{u}_j \partial_x u_j dx \rightarrow P(\vec{\psi}_0) - \int_{\mathbb{R}} i\bar{\psi}_1 \partial_x \psi_1 dx.
\]

Since $\vec{\psi}_0, \vec{\psi}_1 \in \Sigma(\rho)$, from Lemma 2.3, we have $P(\vec{\psi}_0) = P(\vec{\psi}_1)$. Thus, we have
\[
\int_{\mathbb{R}} w_j^1 \partial_x v_j^1 dx \rightarrow \int_{\mathbb{R}} \varphi_1 \partial_x \phi_1 dx.
\] (2.20)
Therefore, by (2.16) and (2.18)-(2.20), we have
\[
\lambda^2 \|\partial_x \phi_1\|_{L^2}^2 + \|\varphi_1\|_{L^2}^2 \leq \liminf_{j \to \infty} \left( \lambda^2 \|\partial_x v_j^1\|_{L^2}^2 + \|w_j^1\|_{L^2}^2 \right)
\]
\[
= \liminf_{j \to \infty} \left( \|w_j^1 + \lambda \partial_x v_j^1\|_{L^2}^2 - 2\lambda \int_{\mathbb{R}} w_j^1 \partial_x v_j^1 \, dx \right)
\]
\[
= -2\lambda \int_{\mathbb{R}} \varphi_1 \partial_x \phi_1 \, dx = \lambda^2 \|\partial_x \phi_1\|_{L^2}^2 + \|\varphi_1\|_{L^2}^2.
\]
Here we have used the fact that \( \varphi_1 + \lambda \partial_x \phi_1 = 0 \) in the last identity. Hence, we obtain
\[
\partial_x v_j^1 \to \partial_x \phi_1 \quad \text{strongly in} \quad L^2(\mathbb{R}),
\]
\[
w_j^1 \to \varphi_1 \quad \text{strongly in} \quad L^2(\mathbb{R}).
\]
This completes the proof. \( \square \)

**Proof of Theorem II.** By Lemma 2.3, it is enough to show that \( \Sigma(\rho) \) is stable for any \( \rho > 0 \). We prove it by contradiction. Suppose that \( \Sigma(\rho) \) is not stable. Then, by the definition, there exist a constant \( \varepsilon_0 > 0 \) and sequences \( \{\vec{u}_{0j}\} \subset X \) and \( \{t_j\} \subset \mathbb{R} \) such that
\[
\liminf_{j \to \infty} \{\|\vec{u}_{0j} - \vec{\psi}\|_X : \vec{\psi} \in \Sigma(\rho)\} = 0
\]
and
\[
\inf\{\|\vec{u}_{j(t_j)} - \vec{\psi}\|_X : \vec{\psi} \in \Sigma(\rho)\} \geq \varepsilon_0,
\]
where \( \vec{u}_{j}(t) \) is the solution of (1.7) with \( \vec{u}_{j}(0) = \vec{u}_{0j} \). By (2.23) and the conservation laws (1.11), we have
\[
E(\vec{u}_{j(t_j)}) = E(\vec{u}_{0j}) \to E(\vec{\psi}_0),
\]
\[
P(\vec{u}_{j(t_j)}) = P(\vec{u}_{0j}) \to P(\vec{\psi}_0),
\]
\[
Q(\vec{u}_{j(t_j)}) = Q(\vec{u}_{0j}) \to Q(\vec{\psi}_0)
\]
for some \( \vec{\psi}_0 \in \Sigma(\rho) \). Thus, by Lemma 2.4, there exist \( \{\alpha_j\} \subset \mathbb{R} \), a subsequence of \( \{T(\alpha_j, 0)\vec{u}_{j(t_j)}\} \) (we still denote it by the same letter) and \( \vec{\psi}_1 \in \Sigma(\rho) \) such that
\[
T(\alpha_j, 0)\vec{u}_{j(t_j)} \to \vec{\psi}_1 \quad \text{strongly in} \quad X.
\]
However, this contradicts (2.24). Hence, \( \Sigma(\rho) \) is stable. \( \square \)
3 Concluding Remarks

When $\lambda^2 > 1$ and $\mu = \lambda^2/4 + 1/(2(\lambda^2 - 1))$, (1.1)-(1.2) also admit the following exact solitary wave solutions (see [10]):

\[
\begin{align*}
    u(t, x) &= \frac{3}{\sqrt{\lambda^2 - 1}} \operatorname{sech} \left( \frac{x - \lambda t}{\sqrt{2(\lambda^2 - 1)}} \right) \tanh \left( \frac{x - \lambda t}{\sqrt{2(\lambda^2 - 1)}} \right) \\
    &\quad \times \exp[\mu t + i(\lambda/2)(x - \lambda t)], \\
    v(t, x) &= -\frac{3}{\lambda^2 - 1} \operatorname{sech}^2 \left( \frac{x - \lambda t}{\sqrt{2(\lambda^2 - 1)}} \right). 
\end{align*}
\]

(3.1) (3.2)

The variational characterizations and the stability problem for (3.1)-(3.2) seem to be open problems.

References


