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1 Introduction and Main Result

In this note we consider the stability of solitary wave solutions for the Yukawa coupled Klein-Gordon-Schrödinger equations in one space dimension:

\[
\begin{align*}
    i\partial_t u + \partial_x^2 u &= uv, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \\
    \partial_t^2 v - \partial_x^2 v + v &= -|u|^2, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.
\end{align*}
\]

Here, \( u = u(t, x) \) and \( v = v(t, x) \) describe a complex scalar nucleon field and a real scalar meson field, respectively (see Fukuda and M. Tsutsumi [3] and Yukawa [11]). In Section 5 of [3], Fukuda and M. Tsutsumi showed that (1.1)–(1.2) admits the following two types of exact solitary wave solutions (1.3)–(1.4) and (1.5)–(1.6):

(I) when \( \lambda^2 < 1 \) and \( \mu = \lambda^2/4 + 1/(1 - \lambda^2) \),

\[
\begin{align*}
    u(t, x) &= \frac{3}{2\sqrt{1-\lambda^2}} \text{sech}^2 \left( \frac{x - \lambda t}{2\sqrt{1-\lambda^2}} \right) \exp[i\mu t + i(\lambda/2)(x - \lambda t)], \\
    v(t, x) &= -\frac{3}{2(1-\lambda^2)} \text{sech}^2 \left( \frac{x - \lambda t}{2\sqrt{1-\lambda^2}} \right),
\end{align*}
\]
(II) when $\lambda^2 = 1$ and $\mu > 1/4$,

$$
\begin{align*}
    u(t, x) &= \sqrt{2(\mu - 1/4)} \ \text{sech} \left( \sqrt{\mu - 1/4} \ (x - \lambda t) \right) \\
    &\quad \times \exp\left[i\mu t + i(\lambda/2)(x - \lambda t)\right], \\
    v(t, x) &= -2(\mu - 1/4) \ \text{sech}^2 \left( \sqrt{\mu - 1/4} \ (x - \lambda t) \right),
\end{align*}
$$

and they proposed a problem of whether the solitary wave solutions are stable or not. The purpose of this note is to give a partial answer to the problem.

To explain our results precisely, we prepare some function spaces and functionals. Let $X = H^1(\mathbb{R}; \mathbb{C}) \times H^1(\mathbb{R}; \mathbb{R}) \times L^2(\mathbb{R}; \mathbb{R})$ be a real Hilbert space with the inner product

$$
((u, v, w), (\psi, \phi, \varphi))_X = 2 \Re \int_{\mathbb{R}} \left( \partial_x u(x) \overline{\partial_x \psi(x)} + u(x) \overline{\psi(x)} \right) dx
$$

for $(u, v, w), (\psi, \phi, \varphi) \in X$. Then, (1.1)-(1.2) is written as abstract Hamiltonian system in the form

$$
\frac{d}{dt} \vec{u}(t) = JE'(\vec{u}(t)),
$$

where

$$
\vec{u}(t) = \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} \in X,
$$

$$
J = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},
$$

and $E$ is the energy functional on $X$ defined by

$$
E(u, v, w) = \int_{\mathbb{R}} \left\{ |\partial_x u|^2 + |u|^2 v + \frac{1}{2} \left( w^2 + (\partial_x v)^2 + v^2 \right) \right\} dx.
$$

The energy functional $E(\vec{u})$ is invariant under the action $T(\alpha, \beta)$ of the group defined by

$$
T(\alpha, \beta)(u, v, w)(x) = (e^{i\beta}u(x + \alpha), v(x + \alpha), w(x + \alpha))
$$

for $\alpha, \beta \in \mathbb{R}$ and $(u, v, w) \in X$. Associated with the group action $T(\alpha, \beta)$, we define two conserved functionals on $X$, the momentum $P$ and the charge $Q$, by

$$
P(u, v, w) = \int_{\mathbb{R}} (i\bar{u}\partial_x u + w\partial_x v) \ dx,
$$

$$
Q(u, v, w) = \int_{\mathbb{R}} |u|^2 dx.
$$

The following global existence of solutions of the Cauchy problem for (1.7) is known.
Proposition 1.1 For any $\vec{u}_0 \in X$ there exists a unique solution $\vec{u} \in C(\mathbb{R}; X)$ of (1.7) with $\vec{u}(0) = \vec{u}_0$ satisfying

\[ E(\vec{u}(t)) = E(\vec{u}_0), \quad P(\vec{u}(t)) = P(\vec{u}_0), \quad Q(\vec{u}(t)) = Q(\vec{u}_0), \quad t \in \mathbb{R}. \tag{1.11} \]

For $\lambda, \mu \in \mathbb{R}$, we put

\[ S_{\lambda, \mu}(\vec{u}) = E(\vec{u}) + \lambda P(\vec{u}) + \mu Q(\vec{u}), \quad \vec{u} \in X. \]

Then, we have

\[ S_{\lambda, \mu}(u, v, w) = \int_{\mathbb{R}} \left\{ \left| \partial_x (e^{-i\lambda x/2} u) \right|^2 + (\mu - \lambda^2/4)|u|^2 + |u|^2 v \right. \]
\[ + \frac{1}{2} \left( (w + \lambda \partial_x v)^2 + (1 - \lambda^2)(\partial_x v)^2 + v^2 \right) \} \, dx, \tag{1.12} \]

and $T(\lambda t, \mu t)\vec{\psi}_{\lambda, \mu}$ is a solution of (1.7) if $\vec{\psi}_{\lambda, \mu}$ is a solution of $S'_{\lambda, \mu}(\vec{\psi}) = 0$. Note that $S'_{\lambda, \mu}(\psi, \phi, \varphi) = E'(\psi, \phi, \varphi) + \lambda P'(\psi, \phi, \varphi) + \mu Q'(\psi, \phi, \varphi) = 0$ is equivalent to

\[ -\partial_x^2 \psi + i\lambda \partial_x \psi + \mu \psi + \psi \phi = 0, \tag{1.13} \]
\[ -(1 - \lambda^2) \partial_x^2 \phi + \phi + |\psi|^2 = 0, \tag{1.14} \]
\[ \varphi + \lambda \partial_x \phi = 0, \tag{1.15} \]

and also that by $\psi(x) = e^{i(\lambda/2)x}\tilde{\psi}(x)$, (1.13) is transformed into

\[ -\partial_x^2 \tilde{\psi} + (\mu - \lambda^2/4)\tilde{\psi} + \tilde{\psi} \phi = 0. \tag{1.16} \]

Thus, when $\lambda^2 < 1$ and $\mu = \lambda^2/2 + 1/(1 - \lambda^2)$, if we put

\[ \psi_{\lambda, \mu}(x) = \frac{3}{2\sqrt{1 - \lambda^2}} \text{sech}^2 \left( \frac{x}{2\sqrt{1 - \lambda^2}} \right) \exp[i(\lambda/2)x], \tag{1.17} \]
\[ \phi_{\lambda, \mu}(x) = -\frac{3}{2(1 - \lambda^2)} \text{sech}^2 \left( \frac{x}{2\sqrt{1 - \lambda^2}} \right), \tag{1.18} \]
\[ \varphi_{\lambda, \mu}(x) = -\lambda \partial_x \phi_{\lambda, \mu}(x), \tag{1.19} \]

then $\vec{\psi}_{\lambda, \mu} = (\psi_{\lambda, \mu}, \phi_{\lambda, \mu}, \varphi_{\lambda, \mu})$ is a solution of $S'_{\lambda, \mu}(\vec{\psi}) = 0$, and when $\lambda^2 = 1$ and $\mu > 1/4$, if we put

\[ \psi_{\lambda, \mu}(x) = \sqrt{2(\mu - 1/4)} \text{sech} \left( \sqrt{\mu - 1/4} \, x \right) \exp[i(\lambda/2)x], \tag{1.20} \]
\[ \phi_{\lambda, \mu}(x) = -2(\mu - 1/4) \text{sech}^2 \left( \sqrt{\mu - 1/4} \, x \right), \tag{1.21} \]
\[ \varphi_{\lambda, \mu}(x) = -\lambda \partial_x \phi_{\lambda, \mu}(x), \tag{1.22} \]
then \( \vec{\psi}_{\lambda,\mu} = (\psi_{\lambda,\mu}, \phi_{\lambda,\mu}, \varphi_{\lambda,\mu}) \) is a solution of \( S'_{\lambda,\mu}(\vec{\psi}) = 0 \).

**Definition.** We say that a subset \( \Sigma \) of \( X \) is **stable** if for any \( \epsilon > 0 \) there exists a \( \delta > 0 \) with the following property. If \( \vec{u}_0 \in X \) satisfies \( \inf \{ \| \vec{u}_0 - \vec{\psi} \|_X : \vec{\psi} \in \Sigma \} < \delta \), then the solution \( \vec{u}(t) \) of (1.7) with \( \vec{u}(0) = \vec{u}_0 \) exists for all \( t \in \mathbb{R} \) and satisfies

\[
\sup_{t \in \mathbb{R}} \inf_{\vec{\psi} \in \Sigma} \{ \| \vec{u}(t) - \vec{\psi} \|_X \} < \epsilon.
\]

Moreover, let \( \vec{\psi}_{\lambda,\mu} \) be a solution of \( S'_{\lambda,\mu}(\vec{\psi}) = 0 \). We say that the solitary wave solution \( T(\lambda t, \mu t) \vec{\psi}_{\lambda,\mu} \) is **stable** if \( \{ T(\alpha, \beta) \vec{\psi}_{\lambda,\mu} : \alpha, \beta \in \mathbb{R} \} \) is stable.

We are now in a position to state our main result in this note.

**Theorem II.** Let \( \lambda^2 = 1 \) and \( \mu > 1/4 \). Then, the solitary wave solution \( T(\lambda t, \mu t) \vec{\psi}_{\lambda,\mu} \) given by (1.20)–(1.22) is stable for any \( \mu > 1/4 \).

**Remark.** In my lecture at the conference, I announced that when \( \lambda^2 < 1 \) and \( \mu = \lambda^2/4 + 1/(1 - \lambda^2) \), the solitary wave solution \( T(\lambda t, \mu t) \vec{\psi}_{\lambda,\mu} \) given by (1.17)–(1.19) is stable if \( \lambda^2 \) is sufficiently close to 1. However, after the conference, I found a mistake in the proof. So, the stability of \( T(\lambda t, \mu t) \vec{\psi}_{\lambda,\mu} \) given by (1.17)–(1.19) seems to be still an open problem.

### 2 Proof of Theorem II

In this section, we give the proof of Theorem II, basically along the argument in [7].

When \( \lambda^2 = 1 \) and \( \mu > 1/4 \), we obtain the following basic identity from (1.12).

\[
S_{\lambda,\mu}(u, v, w) = \int_{\mathbb{R}} \left\{ \left| \partial_x (e^{-i\lambda x/2}u) \right|^2 + \left( \mu - 1/4 \right)|u|^2 - \frac{1}{2}|u|^4 \right\} dx + \frac{1}{2} \int_{\mathbb{R}} \left\{ (w + \lambda \partial_x v)^2 + \left( |u|^2 + v \right)^2 \right\} dx. \tag{2.1}
\]

Associated with the identity (2.1), we define for \( \rho > 0 \)

\[
S^0_{\lambda,\mu}(u) = \int_{\mathbb{R}} \left\{ \left| \partial_x (e^{-i\lambda x/2}u) \right|^2 + \left( \mu - 1/4 \right)|u|^2 - \frac{1}{2}|u|^4 \right\} dx, \tag{2.2}
\]

\[
Q^0(u) = \int_{\mathbb{R}} |u|^2 dx, \tag{2.3}
\]

\[
I^0(\rho) = \inf \{ S^0_{\lambda,\mu}(u) : u \in H^1(\mathbb{R}), \ Q^0(u) = \rho \}, \tag{2.4}
\]

\[
\Sigma^0(\rho) = \{ u \in H^1(\mathbb{R}) : S_{\lambda,\mu}(u) = I^0(\rho), \ Q^0(u) = \rho \}. \tag{2.5}
\]
For $\alpha, \beta \in \mathbb{R}$ and $u \in L^2(\mathbb{R})$, we define

$$T_1(\alpha)u(x) = u(x + \alpha), \quad T_2(\beta)u(x) = e^{i\beta}u(x).$$

**Lemma 2.1** Assume that $\lambda^2 = 1$ and $\mu > 1/4$. Let $\psi_{\lambda,\mu}$ be the function defined by (1.20). Then, we have

$$\Sigma^0(\rho(\mu)) = \{T_1(\alpha)T_2(\beta)\psi_{\lambda,\mu} : \alpha, \beta \in \mathbb{R}\},$$

where $\rho(\mu) = Q^0(\psi_{\lambda,\mu}) = 4\sqrt{\mu - 1/4}$.

**Lemma 2.2** Let $\rho > 0$. If $\{u_j\} \subset H^1(\mathbb{R})$ satisfies $S_{\lambda,\mu}^0(u_j) \rightharpoonup I^0(\rho)$ and $Q^0(u_j) \rightharpoonup \rho$, then there exist $\{\alpha_j\} \subset \mathbb{R}$, a subsequence of $\{T_1(\alpha_j)u_j\}$ (we still denote it by the same letter) and $\psi \in \Sigma^0(\rho)$ such that

$$T_1(\alpha_j)u_j \rightharpoonup \psi \text{ strongly in } H^1(\mathbb{R}).$$

Lemma 2.2 is proved by using the concentration compactness method introduced by Lions [6]. For the proofs of Lemmas 2.1 and 2.2, see Cazenave and Lions [1]. From Lemmas 2.1 and 2.2 and the conservation laws, one can show the stability of solitary wave solutions for the single nonlinear Schrödinger equation (for details, see [1]).

Following the idea by Cazenave and Lions [1], we consider the following minimization problem:

$$I(\rho) = \inf\{S_{\lambda,\mu}(\vec{u}) : \vec{u} \in X, \ Q(\vec{u}) = \rho\},$$

$$\Sigma(\rho) = \{\vec{u} \in X : S_{\lambda,\mu}(\vec{u}) = I(\rho), \ Q(\vec{u}) = \rho\}.$$  \hspace{1cm} (2.6) \hspace{1cm} (2.7)

From Lemma 2.1 and (2.1), we have

**Lemma 2.3** Assume that $\lambda^2 = 1$ and $\mu > 1/4$. For any $\rho > 0$, we have $I(\rho) = I^0(\rho)$ and

$$\Sigma(\rho) = \{\vec{\psi} = (\psi, \phi, \varphi) : \psi \in \Sigma^0(\rho), \ \phi = -|\psi|^2, \ \varphi = -\lambda \partial_x \phi\}.$$  \hspace{1cm} (2.8)

Moreover, let $\vec{\psi}_{\lambda,\mu} = (\psi_{\lambda,\mu}, \phi_{\lambda,\mu}, \varphi_{\lambda,\mu})$ be the vector in $X$ given by (1.20)–(1.22). Then, we have

$$\Sigma(\rho(\mu)) = \{T(\alpha, \beta)\vec{\psi}_{\lambda,\mu} : \alpha, \beta \in \mathbb{R}\},$$  \hspace{1cm} (2.9)

where $\rho(\mu) = Q(\vec{\psi}_{\lambda,\mu}) = 4\sqrt{\mu - 1/4}$. 

Proof. First, we note that $S_{\lambda,\mu}^{0}(u) \leq S_{\lambda,\mu}(\vec{u})$ holds for all $\vec{u} = (u, v, w) \in X$, so that we have $I^{0}(\rho) \leq I(\rho)$. We put

$$
\Sigma_{1}(\rho) = \{ \vec{\psi} = (\psi, \phi, \varphi) : \psi \in \Sigma^{0}(\rho), \phi = -|\psi|^{2}, \varphi = -\lambda \partial_{x}\phi \}. 
$$

If $\vec{\psi} = (\psi, \phi, \varphi) \in \Sigma_{1}(\rho)$, then we have $Q(\vec{\psi}) = Q^{0}(\psi) = \rho$ and

$$
I(\rho) \leq S_{\lambda,\mu}(\vec{\psi}) = S_{\lambda,\mu}^{0}(\psi) = I^{0}(\rho) \leq I(\rho).
$$

Thus, we have $I(\rho) = I^{0}(\rho)$ and $\vec{\psi} \in \Sigma(\rho)$. Conversely, if $\vec{\psi} = (\psi, \phi, \varphi) \in \Sigma(\rho)$, then we have $Q(\vec{\psi}) = Q^{0}(\psi) = \rho$ and

$$
I^{0}(\rho) \leq S_{\lambda,\mu}^{0}(\psi) \leq S_{\lambda,\mu}(\vec{\psi}) = I(\rho) = I^{0}(\rho).
$$

Thus, we have $\vec{\psi} \in \Sigma_{1}(\rho)$. Hence, we obtain (2.8). (2.9) follows from Lemma 2.1 and (2.8). This completes the proof. $\square$

Lemma 2.4 Let $\rho > 0$ and $\vec{\psi}_{0} \in \Sigma(\rho)$. If $\{\vec{u}_{j}\} = \{(u_{j}, v_{j}, w_{j})\} \subset X$ satisfies

$$
E(\vec{u}_{j}) \to E(\vec{\psi}_{0}), \quad P(\vec{u}_{j}) \to P(\vec{\psi}_{0}), \quad Q(\vec{u}_{j}) \to Q(\vec{\psi}_{0}). \tag{2.10}
$$

then there exist $\{\alpha_{j}\} \subset \mathbb{R}$, a subsequence of $\{T(\alpha_{j}, 0)\vec{u}_{j}\}$ (we still denote it by the same letter) and $\vec{\psi}_{1} = (\psi_{1}, \phi_{1}, \varphi_{1}) \in \Sigma(\rho)$ such that

$$
T(\alpha_{j}, 0)\vec{u}_{j} \to \vec{\psi}_{1} \quad \text{strongly in} \quad X.
$$

Proof. First, we note that by the Gagliardo-Nirenberg-Sobolev inequality and (2.10), we see that $\{\vec{u}_{j}\}$ is a bounded sequence in $X$, and

$$
S_{\lambda,\mu}(\vec{u}_{j}) = E(\vec{u}_{j}) + \lambda P(\vec{u}_{j}) + \mu Q(\vec{u}_{j}) \to S_{\lambda,\mu}(\vec{\psi}_{0}) = I(\rho). \tag{2.11}
$$

Since we have

$$
I(\rho) = I^{0}(\rho) \leq S_{\lambda,\mu}^{0}(u_{j}) \leq S_{\lambda,\mu}(\vec{u}_{j}),
$$

it follows from (2.11) and (2.10) that

$$
S_{\lambda,\mu}^{0}(u_{j}) \to I^{0}(\rho), \quad Q^{0}(u_{j}) = Q(\vec{u}_{j}) \to \rho.
$$
Thus, by Lemma 2.2, there exist \( \{ \alpha_j \} \subset \mathbb{R} \) and a subsequence of \( \{ T_1(\alpha_j)u_j \} \) (we still denote it by the same letter) and \( \tilde{\psi} \in \Sigma^0(\rho) \) such that

\[
T_1(\alpha_j)u_j \to \tilde{\psi} \quad \text{strongly in } \ H^1(\mathbb{R}).
\]

(2.12)

Since \( \{ \tilde{u}_j \} \) is bounded in \( X \), so is \( \{ T(\alpha_j, 0)\tilde{u}_j \} \). Thus, there exists a subsequence \( \{ \tilde{u}_j^1 \} = \{ (u_j^1, v_j^1, w_j^1) \} \) of \( \{ T(\alpha_j, 0)\tilde{u}_j \} \) and \( \tilde{\psi}_1 = (\psi_1, \phi_1, \varphi_1) \in X \) such that

\[
\tilde{u}_j^1 \to \tilde{\psi}_1 \quad \text{weakly in } \ X.
\]

(2.13)

By (2.12) and (2.13), we have \( \psi_1 = \tilde{\psi} \in \Sigma^0(\rho) \) and

\[
u_j^1 \to \psi_1 \quad \text{strongly in } \ H^1(\mathbb{R}).
\]

(2.14)

Moreover, from (2.1) and (2.14), we have

\[
|u_j^1|^2 + v_j^1 \to 0 \quad \text{strongly in } \ L^2(\mathbb{R}),
\]

(2.15)

\[
w_j^1 + \lambda \partial_x v_j^1 \to 0 \quad \text{strongly in } \ L^2(\mathbb{R}).
\]

(2.16)

From (2.13)-(2.16), we have

\[
v_j^1 \to \phi_1 = -|\psi_1|^2 \quad \text{strongly in } \ L^2(\mathbb{R}),
\]

(2.17)

\[
\partial_x v_j^1 \to \partial_x \phi_1 \quad \text{weakly in } \ L^2(\mathbb{R}),
\]

(2.18)

\[
w_j^1 \to \varphi_1 = -\lambda \partial_x \phi_1 \quad \text{weakly in } \ L^2(\mathbb{R}).
\]

(2.19)

Since \( \psi_1 \in \Sigma^0(\rho) \), \( \phi_1 = -|\psi_1|^2 \) and \( \varphi_1 = -\lambda \partial_x \phi_1 \), it follows from Lemma 2.3 that \( \tilde{\psi}_1 = (\psi_1, \phi_1, \varphi_1) \in \Sigma(\rho) \). Finally, we have to show the strong convergence of \( \{ \partial_x v_j^1 \} \) and \( \{ w_j^1 \} \) in \( L^2(\mathbb{R}) \). By the definition (1.9) and the convergences in (2.10) and (2.14), we have

\[
\int_{\mathbb{R}} w_j^1 \partial_x v_j^1 dx = P(\tilde{u}_j^1) - \int_{\mathbb{R}} iu_j^1 \partial_x u_j^1 dx
\]

\[
= P(\tilde{u}_j) - \int_{\mathbb{R}} iu_j^1 \partial_x u_j^1 dx \to P(\tilde{\psi}_0) - \int_{\mathbb{R}} i\tilde{\psi}_1 \partial_x \psi_1 dx.
\]

Since \( \tilde{\psi}_0, \tilde{\psi}_1 \in \Sigma(\rho) \), from Lemma 2.3, we have \( P(\tilde{\psi}_0) = P(\tilde{\psi}_1) \). Thus, we have

\[
\int_{\mathbb{R}} w_j^1 \partial_x v_j^1 dx \to \int_{\mathbb{R}} \varphi_1 \partial_x \phi_1 dx.
\]

(2.20)
Therefore, by (2.16) and (2.18)–(2.20), we have
\[
\lambda^2 \|\partial_x \phi_1\|_{L^2}^2 + \|\varphi_1\|_{L^2}^2 \leq \liminf_{j \to \infty} \left( \lambda^2 \|\partial_x v_j^1\|_{L^2}^2 + \|w_j^1\|_{L^2}^2 \right)
\]
\[
= \liminf_{j \to \infty} \left( \|w_j^1 + \lambda \partial_x v_j^1\|_{L^2}^2 - 2\lambda \int_{\mathbb{R}} v_j^1 \partial_x v_j^1 \, dx \right)
\]
\[
= -2\lambda \int_{\mathbb{R}} \varphi_1 \partial_x \phi_1 \, dx = \lambda^2 \|\partial_x \phi_1\|_{L^2}^2 + \|\varphi_1\|_{L^2}^2.
\]
Here we have used the fact that \( \varphi_1 + \lambda \partial_x \phi_1 = 0 \) in the last identity. Hence, we obtain
\[
\partial_x v_j^1 \to \partial_x \phi_1 \quad \text{strongly in} \quad L^2(\mathbb{R}),
\]
\[
\tag{2.21}
\]
\[
 w_j^1 \to \varphi_1 \quad \text{strongly in} \quad L^2(\mathbb{R}).
\]
\[
\tag{2.22}
\]
This completes the proof. \(\square\)

**Proof of Theorem II.** By Lemma 2.3, it is enough to show that \( \Sigma(\rho) \) is stable for any \( \rho > 0 \). We prove it by contradiction. Suppose that \( \Sigma(\rho) \) is not stable. Then, by the definition, there exist a constant \( \varepsilon_0 > 0 \) and sequences \( \{\vec{u}_{0j}\} \subset X \) and \( \{t_j\} \subset \mathbb{R} \) such that
\[
\lim_{j \to \infty} \inf\{\|\vec{u}_{0j} - \vec{\psi}\|_X : \vec{\psi} \in \Sigma(\rho)\} = 0 \quad \tag{2.23}
\]
and
\[
\inf\{\|\vec{u}_{j}(t_j) - \vec{\psi}\|_X : \vec{\psi} \in \Sigma(\rho)\} \geq \varepsilon_0, \quad \tag{2.24}
\]
where \( \vec{u}_{j}(t) \) is the solution of (1.7) with \( \vec{u}_{j}(0) = \vec{u}_{0j} \). By (2.23) and the conservation laws (1.11), we have
\[
E(\vec{u}_{j}(t_j)) = E(\vec{u}_{0j}) \to E(\vec{\psi}_0),
\]
\[
P(\vec{u}_{j}(t_j)) = P(\vec{u}_{0j}) \to P(\vec{\psi}_0),
\]
\[
Q(\vec{u}_{j}(t_j)) = Q(\vec{u}_{0j}) \to Q(\vec{\psi}_0)
\]
for some \( \vec{\psi}_0 \in \Sigma(\rho) \). Thus, by Lemma 2.4, there exist \( \{\alpha_j\} \subset \mathbb{R} \), a subsequence of \( \{T(\alpha_j, 0)\vec{u}_{j}(t_j)\} \) (we still denote it by the same letter) and \( \vec{\psi}_1 \in \Sigma(\rho) \) such that
\[
T(\alpha_j, 0)\vec{u}_{j}(t_j) \to \vec{\psi}_1 \quad \text{strongly in} \quad X.
\]
However, this contradicts (2.24). Hence, \( \Sigma(\rho) \) is stable. \(\square\)
3 Concluding Remarks

When $\lambda^2 > 1$ and $\mu = \lambda^2/4 + 1/(2(\lambda^2 - 1))$, (1.1)–(1.2) also admit the following exact solitary wave solutions (see [10]):

\[
\begin{align*}
    u(t, x) &= \frac{3}{\sqrt{\lambda^2 - 1}} \operatorname{sech}\left( \frac{x - \lambda t}{\sqrt{2(\lambda^2 - 1)}} \right) \tanh\left( \frac{x - \lambda t}{\sqrt{2(\lambda^2 - 1)}} \right) \\
    &\quad \times \exp[i\mu t + i(\lambda/2)(x - \lambda t)], \\
    v(t, x) &= -\frac{3}{\lambda^2 - 1} \operatorname{sech}^2\left( \frac{x - \lambda t}{\sqrt{2(\lambda^2 - 1)}} \right).
\end{align*}
\tag{3.1}
\]

(3.2)

The variational characterizations and the stability problem for (3.1)–(3.2) seem to be open problems.

References


