

# Noncommutative Topological Entropy

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## 1 Introduction

Recall that a pair  $(A, \alpha)$ , where  $A$  is a  $C^*$ -algebra and  $\alpha \in \text{Aut}(A)$  is an automorphism, is called a  $C^*$ -dynamical system. In the classical (i.e. abelian) setting we would have  $A = C(X)$  for a compact Hausdorff space  $X$  and  $\alpha$  would be induced by a homeomorphism  $\varphi : X \rightarrow X$  via the formula  $\alpha(f) = f \circ \varphi^{-1}$ . An important invariant in the study of classical dynamical systems is *entropy*. Roughly speaking, entropy is a measure of how much a homeomorphism  $\varphi : X \rightarrow X$  “mixes up” the space  $X$  (i.e. a measure of ergodicity).

In the classical setting, there are two notions of entropy. With *classical topological entropy* one attempts to count the minimal number of open sets required to cover the (compact) space  $X$  (see [Wal]). With *classical measurable entropy* one is given a  $\varphi$ -invariant probability measure  $\mu$  on  $X$  and one takes a certain weighted measure of a partition (see [Wal]). At first glance, it is not apparent that these notions are particularly well related to each other. But indeed they are and the following *variational principle* provides the bridge: If  $\varphi : X \rightarrow X$  is a homeomorphism of a compact metric space  $X$  and  $h_{\text{Top}}(\varphi)$  denotes the topological entropy of  $\varphi$  then

$$h_{\text{Top}}(\varphi) = \sup_{\mu} h_{\mu}(\varphi),$$

where  $h_{\mu}(\varphi)$  denotes the measurable entropy of  $\varphi$  with respect to  $\mu$  and the supremum is taken over all  $\varphi$ -invariant probability measures  $\mu$ .

There are dozens of papers dealing with various notions of *noncommutative measurable entropy* (i.e. notions of entropy of automorphisms (or endomorphisms) of noncommutative  $C^*$  and  $W^*$ -algebras taken with respect to an

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invariant state or trace). However, there has been relatively little work done in noncommutative generalizations of topological entropy. Unfortunately, noncommutative entropy is a highly technical subject and it requires a fair amount of diligence just to understand some of the definitions. Another difficult aspect is the large number of competing definitions (which probably best illustrates the mathematical infancy of the subject). Indeed, an incomplete list of notions of noncommutative entropy is [CS], [CNT], [ST], [Ch2], [Th], [Hu] and [Vo]. (It should be remarked that some of these various definitions are known to agree when restricted to suitably nice classes of operator algebras.) However, of these papers, only [Th], [Hu] and section 4 of [Vo] really deal with noncommutative *topological* entropy.

In this note we will briefly survey Voiculescu's *approximation approach* to topological entropy (cf. [Vo], [Br1]) and try to point out some of the open questions which we believe to be of particular interest. This approach also yields notions of measurable entropy (see [Vo]) and has the added feature of being conceptually simpler than some of the other approaches listed above.

## 2 The Approximation Approach

If  $A$  is a unital nuclear  $C^*$ -algebra and  $\alpha \in \text{Aut}(A)$  then the topological entropy of  $\alpha$  (in the sense of [Vo]) is denoted by  $ht(\alpha)$ . This definition has recently been modified so as to apply to automorphisms of arbitrary exact  $C^*$ -algebras as well. As previously mentioned, this approach is relatively simple (compared to [CNT], for example) but is still too technical to recall the precise definition (please see [Br1]).

The class of exact  $C^*$ -algebras is the largest class for which the approximation approach yields a reasonable definition. It is also not too hard to show that all of the properties which were known in the nuclear case extend to the exact setting as well. For example, we have the following basic properties.

**Proposition 2.1** (cf. [Vo, Prop. 4.2], [Br1, Prop. 2.5]) *If  $A$  is exact and  $\alpha \in \text{Aut}(A)$  then  $ht(\alpha^k) = |k|ht(\alpha)$  for all  $k \in \mathbb{Z}$ .*

**Proposition 2.2** (cf. [Vo, Prop. 4.3], [Br1, Prop. 2.6]) *If  $A$  is exact and*

$\{\omega_\lambda\}_{\lambda \in \Lambda}$  is a net of finite sets (partially ordered by inclusion) such that  $\text{span} \bigcup_{\lambda \in \Lambda, n \in \mathbb{Z}} \alpha^n(\omega_\lambda)$  is dense in  $A$  then

$$ht(\alpha) = \sup_{\lambda} ht(\alpha, \omega_\lambda)$$

**Proposition 2.3** (cf. [Vo, Prop. 4.9], [Br1, Prop. 2.7]) *If  $A_i$  are exact,  $\alpha_i \in \text{Aut}(A_i)$  for  $i = 1, 2$  then  $ht(\alpha_1 \otimes \alpha_2) \leq ht(\alpha_1) + ht(\alpha_2)$ . If  $A_1, A_2$  are unital then we also have  $ht(\alpha_1 \otimes \alpha_2) \geq \max\{ht(\alpha_1), ht(\alpha_2)\}$ .*

It is still an open problem whether or not the equality  $ht(\alpha_1 \otimes \alpha_2) = ht(\alpha_1) + ht(\alpha_2)$  always holds. This equality is roughly equivalent to proving that the best way to approximate  $A_1 \otimes_{\min} A_2$  is to use algebras of the form  $M_{n_1}(\mathbb{C}) \otimes M_{n_2}(\mathbb{C})$ .

We should also point out that  $ht(\cdot)$  really does extend classical topological entropy (cf. [Vo, Prop. 4.8], [Br1, Prop. 1.4]).

**Proposition 2.4** *If  $\varphi : X \rightarrow X$  is a homeomorphism of the compact metric space  $X$  and  $\alpha \in \text{Aut}(C(X))$  is defined by  $\alpha(f) = f \circ \varphi^{-1}$  then  $ht(\alpha) = h_{\text{Top}}(\varphi)$ .*

One advantage of the definition given in [Br1] is the following result (cf. [Br1, Prop. 2.1]).

**Proposition 2.5 (Monotonicity)** *If  $A$  is exact,  $\alpha \in \text{Aut}(A)$  and  $A_0 \subset A$  is a  $C^*$ -subalgebra such that  $\alpha(A_0) = A_0$  then  $ht(\alpha|_{A_0}) \leq ht(\alpha)$ .*

In general, getting lower bounds for  $ht(\cdot)$  is the more difficult task and the previous proposition gives one strategy for doing so. Another way to get lower bounds for  $ht(\cdot)$  is to compute some of the current notions of measurable entropy.

**Proposition 2.6** (cf. [Vo, Prop. 4.6], [Ch2, Thm. 2.6.1]) *If  $A$  is a unital nuclear  $C^*$ -algebra,  $\alpha \in \text{Aut}(A)$  and  $\varphi$  is a state on  $A$  such that  $\varphi \circ \alpha = \varphi$  then*

$$h_\varphi(\alpha) \leq ht_\varphi(\alpha) \leq ht(\alpha),$$

where  $h_\varphi(\alpha)$  is defined in [CNT] and  $ht_\varphi(\alpha)$  is defined in [Ch2].

This result begs the question of whether or not we have a noncommutative analogue of the classical variational principle. It is known that in general  $ht(\alpha)$  is *not* the supremum over all invariant states of the [CNT] entropy  $h_\varphi(\alpha)$ . However, 2.6.2 in [Ch2] provides a variational principle for  $ht(\alpha)$  and  $ht_\varphi(\alpha)$  for certain shifts on UHF algebras. It is natural to ask whether or not this holds in general.

Another interesting open question (which would generalize a known result in the classical setting) is whether or not entropy decreases in quotients. Conceptually, it seems reasonable to conjecture that this should be true. However, it seems difficult to compare the two entropies without some nuclearity hypothesis. (Perhaps assume that either the ideal or the quotient is a nuclear  $C^*$ -algebra.)

The main new result of [Br1] provides a strong bridge between dynamics and geometry in  $C^*$ -algebras. It is our hope that in the future, the following result will be useful both in applications of dynamics to  $C^*$ -algebra theory and, conversely, applications of  $C^*$ -algebra theory to (classical) dynamics.

**Theorem 2.7** (cf. [Br1, Thm. 3.5]) *If  $A$  is a unital exact  $C^*$ -algebra,  $\alpha : G \rightarrow \text{Aut}(A)$  is a group homomorphism (with  $G$  discrete and abelian) taking  $g \mapsto \alpha_g$  and  $\lambda_g \in A \rtimes_\alpha G$  is the unitary implementing the automorphism  $\alpha_g \in \text{Aut}(A)$  then  $ht_A(\alpha_g) = ht_{A \rtimes_\alpha G}(Ad\lambda_g)$ .*

In particular, if  $\alpha \in \text{Aut}(A)$  and  $u \in A \rtimes_\alpha \mathbb{Z}$  is the implementing unitary then  $ht(\alpha) = ht(Ad u)$ , thus reducing most questions about topological entropy to the case of inner automorphisms.

The point of this theorem is that we may now investigate how the topological entropy of  $\alpha$  is related to the relative position of the implementing unitary  $u \in A \rtimes_\alpha \mathbb{Z}$ . For example, in [Br2] we showed that  $ht(\alpha) > 0$  has geometric consequences for any potential inductive limit decomposition of  $A \rtimes_\alpha \mathbb{Z}$  in terms of subhomogeneous algebras. Conversely, since  $C(X) \rtimes_\alpha \mathbb{Z}$  is known to have such an inductive limit decomposition for minimal Cantor systems (cf. [Pu]), we may hope to use the geometry of  $C(X) \rtimes_\alpha \mathbb{Z}$  to learn about dynamical properties of  $\varphi$ .

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