Introduction

In this article we survey [Te] and [Fr-Te].

Alexander duality theorem plays an important role in the study on a minimal free resolution of Stanley-Reisner rings. (See [Br-He2], [Te-Hi1], [Te-Hi2], for example.) In particular, Eagon and Reiner used Alexander dual complexes and proved the following interesting theorem:

**Theorem 0.1 ([Ea-Re, Theorem 3]).** Let $k$ be a field. and let $\Delta$ be a simplicial complex and $\Delta^*$ its Alexander dual complex. Then $k[\Delta]$ has a linear resolution if and only if $k[\Delta^*]$ is Cohen-Macaulay.

The above result is a starting point of this article. We generalize it in the following way.

**Theorem 0.2.** Let $k$ be a field. Let $\Delta$ be a $(d-1)$-dimensional complex on the vertex set $[n]$. Suppose $d \leq n - 2$. Then

\[
\text{reg } I_\Delta - \text{indeg } I_\Delta = \dim k[\Delta^*] - \text{depth } k[\Delta^*].
\]

Note that Theorem 0.2 corresponds to Theorem 0.1 in the case that either side of the equality is 0.

Using the Auslander-Buchsbaum formula, we have the following corollary:

**Corollary 0.3.** Let $k$ be a field. Let $\Delta$ be a $(d-1)$-dimensional complex on the vertex set $[n]$. Suppose $d \leq n - 2$. Then

\[
\text{reg } I_\Delta = \text{pd } k[\Delta^*].
\]
Here, we use $\text{indeg } I_{\Delta} = \text{embdim } k[\Delta^*] - \dim k[\Delta^*]$.

It is an interesting problem to estimate regularity of homogeneous ideals. Upper bounds of regularity are studied very actively in algebraic geometry and commutative algebra, that seems to be motivated by Eisenbud-Goto Conjecture. See, for example, [Kw] and [Mi-Vo]. Here we focus on monomial ideals. We give two kind of inequalities as an application of Alexander duality.

**Theorem 0.4** ([Ho-Tr, Theorem 1.1], [Fr-Te, Theorem 3.8]). Let $I$ be a monomial ideal in the polynomial $A = k[x_1, x_2, \ldots, x_n]$ over a field $k$. Assume $\text{codim } A/I \geq 2$. Then we have

$$\text{reg } I \leq \text{arith-deg } I.$$ 

Theorem 0.4 was first proved by Hoa and Trung. After that, Frübis-Krüger and the author proved it independently using Alexander duality.

**Theorem 0.5** (Monomial version of Eisenbud-Goto Conjecture). Let $k$ be a field and let $\Delta$ be a pure simplicial complex connected in codimension 1. Then we have

$$\text{reg } I_{\Delta} \leq \deg I_{\Delta} - \text{codim } k[\Delta] + 1.$$ 

As another application, we give some upper bound for the multiplicities of homogeneous $k$-algebras. In [Ba-Mu] and [He-Sr], among other things, the following inequality is proved:

**Theorem 0.6** ([Ba-Mu, Proposition 3.6], [He-Sr, Corollary 3.8]). Let $k$ be a field and let $R = k[x_1, x_2, \ldots, x_n]/I$ be a homogeneous $k$-algebra of codimension $h_1$. Then

$$e(R) \leq \binom{\text{reg } I + h_1 - 1}{h_1}.$$ 

We improve it as follows:

**Theorem 0.7.** Let $k$ be a field and let $R = k[x_1, x_2, \ldots, x_n]/I$ be a homogeneous $k$-algebra of codimension $h_1 \geq 2$. Then

$$e(R) \leq \binom{\text{reg } I + h_1 - 1}{h_1} - \binom{\text{reg } I - \text{indeg } I + h_1 - 1}{h_1}.$$
§1. Preliminaries

We first fix notation. Let $\mathbb{N}$ (resp. $\mathbb{Z}$) denote the set of nonnegative integers (resp. integers). Let $|S|$ denote the cardinality of a set $S$.

We recall some notation on simplicial complexes and Stanley-Reisner rings according to [St]. We refer the reader to, e.g., [Br-He], [Hi], [Ho] and [St] for the detailed information about combinatorial and algebraic background.

A simplicial complex $\Delta$ on the vertex set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ such that (i) $\{i\} \in \Delta$ for every $1 \leq i \leq n$ and (ii) $F \in \Delta$, $G \subseteq F \Rightarrow G \in \Delta$. Each element $F$ of $\Delta$ is called a face of $\Delta$. We call $F \in \Delta$ an $i$-face if $|F| = i + 1$. We set $d = \max\{|F| \mid F \in \Delta\}$ and define the dimension of $\Delta$ to be $\dim \Delta = d - 1$. We call a maximal face a facet. We say that $\Delta$ is pure if every facet has the same cardinality. When $\Delta$ is pure, we call $\Delta$ connected in codimension 1, if for every two facets $F$ and $G$, there is a sequence of facets $F = F_0, F_1, \ldots, F_p = G$ such that $|F_i \cap F_{i+1}| = |F_i| - 1$ for $0 \leq i \leq p - 1$.

Let $f_i = f_i(\Delta)$, $0 \leq i \leq d - 1$, denote the number of $i$-faces in $\Delta$. We define $f_{-1} = 1$. We call $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ the $f$-vector of $\Delta$. Define the $h$-vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of $\Delta$ by

$$
\sum_{i=0}^{d} f_{i-1} (t - 1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.
$$

If $F$ is a face of $\Delta$, then we define a subcomplex $\text{link}_\Delta F$ as follows:

$$\text{link}_\Delta F = \{G \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta\}.$$ 

Let $\tilde{H}_i(\Delta; k)$ denote the $i$-th reduced simplicial homology group of $\Delta$ with the coefficient field $k$.

Let $A = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in $n$-variables over a field $k$. Define $I_\Delta$ to be the ideal of $A$ which is generated by square-free monomials $x_{i_1}x_{i_2} \cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, with $\{i_1, i_2, \ldots, i_r\} \not\in \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the Stanley-Reisner ring of $\Delta$ over $k$.

**Theorem 1.1** (Hochster's formula on the local cohomology modules
(cf. [St, Theorem 4.1]).

\[ F(H^i_m(k[\Delta]), t) = \sum_{F \in \Delta} \dim_k \tilde{H}_{i-|F|-1}(\text{link}_\Delta F; k) \left( \frac{t^{-1}}{1 - t^{-1}} \right)^{|F|}. \]

where \( H^i_m(k[\Delta]) \) denote the \( i \)-th local cohomology module of \( k[\Delta] \) with respect to the graded maximal ideal \( m \).

Let \( A \) be the polynomial ring \( k[x_1, x_2, \ldots, x_n] \) for a field \( k \). Let \( M \) be a finitely generated graded \( A \)-module and let

\[ 0 \rightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{0,j}(M)} \rightarrow \ldots \rightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{i,j}(M)} \rightarrow M \rightarrow 0 \]

be a graded minimal free resolution of \( M \) over \( A \). We call \( \beta_i(M) = \sum_{j \in \mathbb{Z}} \beta_{i,j}(M) \) the \( i \)-th Betti number of \( M \) over \( A \). We sometimes denote \( \beta^A_i(M) \) for \( \beta_i(M) \) to emphasize the base ring \( A \). We define a Castelnuovo-Mumford regularity \( \text{reg} \) of \( M \) by

\[ \text{reg} \, M = \max \{ j - i \mid \beta_{i,j}(M) \neq 0 \}. \]

We define an initial degree \( \text{indeg} \) of \( M \) by

\[ \text{indeg} \, M = \min \{ i \mid M_i \neq 0 \} = \min \{ j \mid \beta_{0,j}(M) \neq 0 \}. \]

**Theorem 1.2** (Hochster’s formula on the Betti numbers[Ho, Theorem 5.1]).

\[ \beta_{i,j}(k[\Delta]) = \sum_{F \subset [n], |F|=j} \dim_k \tilde{H}_{j-i-1}(\Delta_F; k), \]

where

\[ \Delta_F = \{ G \in \Delta \mid G \subset F \}. \]

Finally we quote some result on Gröbner bases we use later. See [Ei, Chapter 15] for complete explanation.

Let \( A \) be the polynomial ring \( k[x_1, x_2, \ldots, x_n] \) for an infinite field \( k \). Let \( I \) be a homogeneous ideal in \( A \). We denote \( \text{Gin} \,(I) \) to be a generic initial ideal of \( I \) with respect to the reverse lexicographic order. It is well known that \( e(A/\text{Gin} \,(I)) = e(A/I) \).

Further we have:

**Theorem 1.3** ([Ba-St]).

1. \( \text{depth} \, A/\text{Gin} \,(I) = \text{depth} \, A/I. \)
2. \( \text{reg} \, \text{Gin} \,(I) = \text{reg} \, I. \)
§2. Alexander duality and some generalization of the Eagon-Reiner theorem

First we recall the definition of Alexander dual complexes.

Definition (cf. [Ea-Re]). For a simplicial complex $\Delta$ on the vertex set $[n]$, we define an Alexander dual complex $\Delta^*$ as follows:

$$\Delta^* = \{ F \subset [n] : [n] \setminus F \not\in \Delta \}.$$ 

If $\dim \Delta \leq n - 3$, then $\Delta^*$ is also a simplicial complex on the vertex set $[n]$.

In the rest of the paper we always assume $\dim k[\Delta] = d$ and $\dim k[\Delta^*] = d^*$ for a fixed field $k$.

Now we give some generalization of the Eagon-Reiner theorem.

**Theorem 2.1.** Let $\Delta$ be a $(d - 1)$-dimensional complex on the vertex set $[n]$. Suppose $d \leq n - 2$. Then

$$\text{reg } I_\Delta - \text{indeg } I_\Delta = \dim k[\Delta^*] - \text{depth } k[\Delta^*].$$

**Proof.** Put $\text{depth } k[\Delta^*] = p^*$. By Hochster's formula on the local cohomology modules, we have

$$F(H_m^l(k[\Delta^*]), t) = \sum_{F \in \Delta^*} \dim_k \tilde{H}_{l-[F]-1}(\text{link}_{\Delta^*} F; k) \left( \frac{t^{-1}}{1-t^{-1}} \right)^{|F|}.$$ 

Hence if $l < p^*$, then $\tilde{H}_{l-[F]-1}(\text{link}_{\Delta^*} F; k) = (0)$ for all $F \in \Delta^*$. By the proof in [Ea-Re, Proposition 1], we have $\tilde{H}_{n-l-2}(\Delta_F; k) = (0)$ for all $F \subset [n]$. By Hochster's formula on the Betti numbers this means that $\beta_{i,i+n-l-1}(k[\Delta]) = 0$ for $i \geq 1$. Hence

$$\beta_{i,i+n}(I_\Delta) = \beta_{i,i+n-1}(I_\Delta) = \cdots = \beta_{i,i+n-p^*+1}(I_\Delta) = 0$$

for $i \geq 0$. Similarly, since $\tilde{H}_{n-p^*-2}(\Delta_{[n]\setminus F}; k) \cong \tilde{H}_{p^*-[F]-1}(\text{link}_{\Delta^*} F; k) \neq (0)$ for some $F \in \Delta$, we have $\beta_{i,i+n-p^*}(I_\Delta) \neq 0$ for some $i \geq 0$. Hence $\text{reg } I_\Delta = n - p^*$. By the definition of the Alexander dual complex we have $\text{indeg } I_\Delta = n - d^*$. Therefore, we have $\text{reg } I_\Delta - \text{indeg } I_\Delta = d^* - p^*$. Q.E.D.
§3. On upper bounds for regularity of monomial ideals

In this section we give some upper bounds for regularity of monomial ideals.

**Theorem 3.1** ([Fr-Te, Theorem 3.1]). Let $k$ be a field and let $\Delta$ be a simplicial complex. Assume $\text{codim} k[\Delta] \geq 2$. Then we have

$$\text{reg } I_\Delta \leq \text{arith-deg } I_\Delta.$$

See, for example, [Ba-Mu] for the definition of arithmetic degree of an ideal $I$. Here we just remark that arithmetic degree $\text{arith-deg } I_\Delta$ of a square-free monomial ideal $I_\Delta$ is the number of the facets in $\Delta$.

**Proof.** Taylor resolution guarantees $\text{pd } k[\Delta^*] \leq \beta_0(I_\Delta^*)$. Then we have

$$\text{reg } I_\Delta = \text{pd } k[\Delta^*] \leq \beta_0(I_\Delta^*) = \text{arith-deg } I_\Delta$$

by Corollary 0.3. Q.E.D.

By combinatorial argument on standard pairs, which are introduced by [St-Tr-Vo], we can show:

**Theorem 3.2** ([Fr-Te, Corollary 3.6]). Let $I$ be a monomial ideal of a polynomial ring. Put $I^{\text{pol}}$ be the polarization of $I$. Then we have

$$\text{reg } I = \text{reg } I^{\text{pol}}.$$

See, for example, [St-Vo] for the definition and basic properties of the polarization of monomial ideals.

Combining Theorem 3.1 and 3.2, we have:

**Theorem 3.3** ([Ho-Tr, Theorem 1.1], [Fr-Te, Theorem 3.8]). Let $I$ be a monomial ideal in the polynomial $A = k[x_1, x_2, \ldots, x_n]$ over a field $k$. Assume $\text{codim} A/I \geq 2$. Then we have

$$\text{reg } I \leq \text{arith-deg } I.$$
Next, we will prove a certain conjecture of Eisenbud (see [Ei-Po]), which is a monomial version of Eisenbud-Goto Conjecture (see [Ei-Go]).

**Theorem 3.4.** Let $k$ be a field and let $\Delta$ be a pure simplicial complex connected in codimension 1. Then we have

$$\text{reg} I_\Delta \leq \deg k[\Delta] - \text{codim} k[\Delta] + 1.$$  

We give a sketch of a proof, which is simplified by suggestions of Eisenbud.

**Sketch of proof.** Let $V$ be the vertex set of $\Delta$. Put $\#(V) = n$ and $\dim k[\Delta] = d$. We prove the theorem by induction on the number $f_{d-1}$ of facets in $\Delta$.

First if $\text{codim} k[\Delta] \leq 1$, then $k[\Delta]$ is a hypersurface. In this case the theorem is clear.

Suppose $\text{codim} k[\Delta] \geq 2$ and $f_{d-1} \geq 2$. Then there exists a facet $\sigma \in \Delta$ such that

$$\Delta' := \Delta \setminus \{\tau \in \Delta \mid \text{For any facet } \rho(\neq \sigma) \in \Delta; \ \tau \not\subset \rho\}$$

is pure and connected in codimension 1. Denote by $V'$ the vertex set of $\Delta'$. We prove the theorem by induction on the number of facets in $\Delta'$.

**Case(i)** $V \neq V'$. Put $V \setminus V' = \{v\}$. For $W \subset V$ with $v \not\in W$ we have $\Delta_W \cong \Delta'_W$. On the other hand, for $W \subset V$ with $v \in W$, We have $\tilde{H}_i(\Delta_W; k) \cong \tilde{H}_i(\Delta'_W \setminus \{v\}; k)$ for $i \geq 1$. Since

$$\text{reg} I_\Delta = \max\{i + 2 \mid \tilde{H}_i(\Delta_W; k) \neq 0 \text{ for some } W \subset V\},$$

we have

$$\text{reg} I_\Delta = \text{reg} I_{\Delta'},$$

$$\leq f'_{d-1} - (n - 1 - d) + 1,$$

$$= f_{d-1} - (n - d) + 1.$$  

**Case(ii)** $V = V'$. We have $\text{reg} I_\Delta = \text{pd} k[\Delta^*]$ by Corollary 0.3. If we prove $\text{pd} k[\Delta^*] \leq \text{pd} k[\Delta^*(\Delta')^*] + 1$, we have

$$\text{reg} I_\Delta \leq \text{reg} I_{\Delta'} + 1,$$

$$\leq f'_{d-1} - (n - d) + 2,$$

$$= f_{d-1} - (n - d) + 1.$$
Then we have only to prove
\[ \text{pd } k[\Delta^*] \leq \text{pd } k[(\Delta')^*] + 1. \]

Put \( k[\Delta^*] = k[(\Delta')^*]/(m) \), where \( m = \Pi_{x_i \in V \setminus \sigma x_i} \). If we show that
\[ \text{pd } k[(\Delta')^*] \geq \text{pd } (I_{(\Delta')^*} + (m))/I_{(\Delta')^*}, \]
then the mapping cone guarantees that
\[ \text{pd } k[\Delta^*] \leq \text{pd } k[(\Delta')^*] + 1 \]
by [E, Exercise A.3.30]. But now we have
\[
(I_{(\Delta')^*} + (m))/I_{(\Delta')^*} \cong (m)/(m) \cap I_{(\Delta')^*} \
\cong (m)/(m_1, \ldots, m_t) \cap (m, m_1) \
\cong (m)/(\text{lcm}(m, m_1), \ldots, \text{lcm}(m, m_t)) \
\cong A/(m_1', \ldots, m_t') \otimes_A (m),
\]
where \( I_{(\Delta')^*} = (m_1, \ldots, m_t) \), \( m_i' = \text{lcm}(m, m_i)/m \), and \( A = k[x_i | x_i \in V] \).

Hence, we have only to show
\[ \text{pd } k[(\Delta')^*] \geq \text{pd } A/(m_1', \ldots, m_t'). \]

Now we have \( k[(\Delta')^*] \cong A_m/(m_1', \ldots, m_t')A_m \). Hence we have
\[ \text{pd } k[(\Delta')^*] \geq \text{pd } k[(\Delta')^*]_m = \text{pd } A_m/(m_1', \ldots, m_t')A_m = \text{pd } A/(m_1', \ldots, m_t'). \]

Q.E.D.

§4. On upper bounds for multiplicities

In this section we give some upper bound for the multiplicities of homogeneous \( k \)-algebras.

First we prove the following lemma:

**LEMMA 4.1.**

\[ e(k[\Delta]) = \beta_{1,h_1}(k[\Delta^*]). \]

**Proof.** We have
\[ h_0(\Delta) + h_1(\Delta)(1 - t) + \cdots + h_d(\Delta)(1 - t)^d = \frac{1 - (1 - t)^{n - d^*}}{t^{n - d}} (h_0(\Delta^*) + h_1(\Delta^*)t + \cdots + h_d(\Delta^*)t^{d^*}). \]
Since \( \text{indeg} \ I_{\Delta^*} = n - d = h_1 \), we have

\[
\begin{align*}
\beta_{1,n-d}(k[\Delta^*]) \\
= & \quad (\text{the coefficient of } t^{n-d} \text{ in } - (1 - t)^{n-d}(h_0(\Delta^*) + h_1(\Delta^*)t + \cdots + h_d(\Delta^*)t^d)) \\
= & \quad (\text{the coefficient of } t^{n-d} \text{ in the numerator in (2)}) \\
= & \lim_{t \to 0}(h_0(\Delta) + h_1(\Delta)(1-t) + \cdots + h_d(\Delta)(1-t)^d) \\
= & \quad e(k[\Delta]).
\end{align*}
\]

Q.E.D.

**Theorem 4.2.** Let \( R = A/I \) be a homogeneous \( k \)-algebra of codimension \( h_1 \geq 2 \). Then

\[
e(R) \leq \left( \frac{\text{reg } I + h_1 - 1}{h_1} \right) - \left( \frac{\text{reg } I - \text{indeg } I + h_1 - 1}{h_1} \right).
\]

**Proof.** We may assume \( |k| = \infty \). By Theorem 1.3, we have \( \text{reg Gin}(I) = \text{reg } I \) and \( h(A/I) = h(A/\text{Gin}(I)) \). Considering the polarization, we obtain a Stanley-Reisner ring \( k[\Delta] = B/I_\Delta \) with \( e(A/I) = e(k[\Delta]) \) and \( \text{reg } I = \text{reg } I_\Delta \). Put \( p^* = \text{depth } k[\Delta^*] \). By Theorem 2.1, we have \( d^* - p^* = \text{reg } I - (n - d^*) \), where \( n = \text{embdim } k[\Delta^*] \). Hence \( \text{reg } I = n - p^* \).

Let \( y_1, y_2, \ldots, y_{p^*} \) be a regular sequence in \( k[\Delta^*]_1 \), and let \( z_1, z_2, \ldots, z_{d^*-p^*} \in (k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*}))_1 \) be a system of parameters of \( k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*}) \).

We have \( k[z_1, z_2, \ldots, z_{d^*-p^*} \subset k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*}) \). Since \( k[z_1, z_2, \ldots, z_{d^*-p^*}] \) is isomorphic to the polynomial ring with \( d^* - p^* \) variables, we have \( \dim_k(k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*}))_{h_1} \geq (d^*-p^*-h_1-1) \). By Lemma 4.1, we have

\[
\begin{align*}
e(k[\Delta]) \\
= & \quad \beta_{1,h_1}(k[\Delta^*]) \\
= & \quad \beta_{1,h_1}^B(y_1, y_2, \ldots, y_{p^*})(k[\Delta^*/(y_1, y_2, \ldots, y_{p^*})) \\
= & \quad \dim_k(B/(y_1, y_2, \ldots, y_{p^*}))_{h_1} - \dim_k(k[\Delta^*/(y_1, y_2, \ldots, y_{p^*}))_{h_1} \\
\leq & \quad \left( \frac{n - p^* + h_1 - 1}{h_1} \right) - \left( \frac{d^*-p^*+h_1-1}{h_1} \right).
\end{align*}
\]

Q.E.D.

**References**

[Ba-Mu] D. Bayer and D. Mumford, *What can be computed in algebraic geometry*, in "Computational Algebraic Geometry and Commutative


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