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Basic sequences of torsion free graded modules

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Let $R := k[x_1, \ldots, x_r]$ denote a polynomial ring in r indeterminates over an infinite field k of arbitrary characteristic, $\mathfrak{m} := (x_1, \ldots, x_r)$ its maximal ideal, and E a finitely generated graded R-module. All modules treated here are graded.

Let us recall first the main results of [7].

Theorem 1 ([7, Corollary 2.6, Theorem 2.11]). *There exist a finitely generated graded $k[x_i, \ldots, x_r]$-submodule $E^{[i]} \subset E$ and a finitely generated graded free $k[x_i, \ldots, x_r]$-submodule $E^{\langle i \rangle} \subset E$ for each $i = 1, \ldots, r+1$ such that*

(1.1) $$E^{[1]} = E, \ E^{[r+1]} = E^{\langle r+1 \rangle},$$

(1.2) $$E^{[i]} = E^{\langle i \rangle} \oplus E^{[i+1]} \text{ as } k[x_{i+1}, \ldots, x_r]\text{-module, and}$$

(1.3) $$x_i E^{[i+1]} \subset (x_{i+1}, \ldots, x_r) E^{\langle i \rangle} \oplus E^{[i+1]}$$

for all $i = 1, \ldots, r$, if and only if

(1.4) $$(x_r, \ldots, x_{i+1})E :_E x_i \subset (x_r, \ldots, x_{i+1})E :_E \langle \mathfrak{m} \rangle$$

for all $i = 1, \ldots, r$, where

$$Z :_E \langle \mathfrak{m} \rangle = \{ \ e \in E \mid \mathfrak{m}^t e \subset Z \text{ for some } t \in \mathbf{N} \ \}.$$

If the submodules as above exist, denoting homogeneous free bases of $E^{\langle i \rangle}$ by e_l^i $(1 \leq l \leq m_i)$, we call $W := \{ \ e_l^i \mid 1 \leq i \leq r+1, \ 1 \leq l \leq m_i \ \}$ a *weak Weierstrass basis* of E with respect to $x_1, \ldots, x_r$.

Remark 2. (1) If the condition (1.4) is satisfied, we say that $x_r, \ldots, x_1$ form a filter-regular sequence with respect to E (see [19, Appendix:Definition 1]).

(2) For fixed $x_1, \ldots, x_r$, the structures of $E^{\langle i \rangle}$ and $E^{[i]}$ are uniquely determined up to isomorphism over $k[x_i, \ldots, x_r]$ for each $i = 1, \ldots, r+1$ by the conditions (1.1), (1.2) and (1.3).

(3) By (1.2),

(2.1) $$E^{[i]} = E^{\langle i \rangle} \oplus E^{\langle i+1 \rangle} \oplus \cdots \oplus E^{[r+1]}$$

as k-vector space.

Remark 3. (1) If E satisfies $l_R(H^i_{\mathfrak{m}}(E)) < \infty$ for all $i < r$, then the sequence $x_1, \dots, x_r$ is always filter-regular with respect to E (see [19, Appendix:Proposition 16]).

(2) If the variables $x_1, \dots, x_r$ are chosen sufficiently generally regarding E, then $x_1, \dots, x_r$ form a filter-regular sequence with respect to E.

Example 4. Let $E = R/(x_1, \dots, x_{j-1})$ and let

$$\begin{aligned} &E^{\langle 1\rangle} = \cdots = E^{\langle j-1\rangle} = 0, \quad E^{\langle j+1\rangle} = \cdots = E^{\langle r+1\rangle} = 0, \\ &E^{\langle j\rangle} = k[x_j, \dots, x_r]e^j_1 \quad \text{with} \quad e^j_1 = 1, \\ &E^{[i]} = 0 \ (j+1 \le i \le r+1), \quad E^{[i]} = E \ (1 \le i \le j). \end{aligned}$$

Then these submodules satisfy (1.1) – (1.3).

Proof. It is clear that $E = R \cdot 1 = k[x_j, \dots, x_r] \cdot 1$, since $x_i = 0$ in E for $i = 1, \dots, j-1$. Hence (1.1) and (1.2) hold. Verification of (1.3) is easy. □

Example 5. Let $r := 3$ and $E := \operatorname{Syz}^R_2(R/\mathfrak{m})$. Then E is the image of the matrix

$$\begin{pmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{pmatrix}.$$

Denote the first column by e^1_1, the second by e^1_2, and the third by e^2_1. Set

$$\begin{aligned} &E^{\langle 1\rangle} = Re^1_1 \oplus Re^1_2, \quad E^{\langle 2\rangle} = k[x_2, x_3]e^2_1, \quad E^{\langle 3\rangle} = E^{\langle 4\rangle} = 0, \\ &E^{[1]} = E, \quad E^{[2]} = E^{\langle 2\rangle}, \quad E^{[3]} = E^{[4]} = 0. \end{aligned}$$

Then these submodules satisfy (1.1) – (1.3).

Proof. The condition (1.2) follows from e.g. [7, Lemma 1.1]. Since $x_3e^1_1 - x_2e^1_2 + x_1e^2_1 = 0$, we find $x_1e^2_1 = -x_3e^1_1 + x_2e^1_2 \in (x_2, x_3)E^{\langle 1\rangle}$. Hence (1.3) holds. This implies on the other hand that $E = E^{[1]}$ by [7, Lemma 2.7]. Thus (1.1) – (1.3) are satisfied. □

Example 6. Let $r := 2$ and $E := (x^2_1, x^2_2) \subset R = k[x_1, x_2]$. Let further

$$\begin{aligned} &e^1_1 = x^2_1, \ e^2_1 = x^2_2, \ e^2_2 = x_1x^2_2, \\ &E^{\langle 1\rangle} = Re^1_1, \quad E^{\langle 2\rangle} = k[x_2]e^2_1 \oplus k[x_2]e^2_2, \quad E^{\langle 3\rangle} = 0, \\ &E^{[1]} = E, \quad E^{[2]} = E^{\langle 2\rangle}, \quad E^{[3]} = 0. \end{aligned}$$

Then these submodules satisfy (1.1) – (1.3).

Proof. It is easy to verify (1.1) and (1.2). Since $x_1e^2_1 = x_1x^2_2 = 1 \cdot e^2_2 \in E^{[2]}$ and $x_1e^2_2 = x^2_1x^2_2 = x^2_2e^1_1 \in (x_2)E^{\langle 1\rangle}$, the condition (1.3) holds, too. □

Example 7. Let $r := 4$ and $E := (x_1^2, x_1x_2, x_2^2, x_1x_3 - x_2x_4) \subset R = k[x_1, x_2, x_3, x_4]$. Let further

$$e_1^1 = x_1^2,\ e_1^2 = x_1x_2,\ e_2^2 = x_2^2,\ e_1^3 = x_1x_3 - x_2x_4,$$
$$E^{\langle 1\rangle} = Re_1^1,\quad E^{\langle 2\rangle} = k[x_2, x_3, x_4]e_1^2 \oplus k[x_2, x_3, x_4]e_2^2,\quad E^{\langle 3\rangle} = k[x_3, x_4]e_1^3,$$
$$E^{[1]} = E,\quad E^{[2]} = E^{\langle 2\rangle} \oplus E^{\langle 3\rangle},\quad E^{[3]} = E^{\langle 3\rangle},\quad E^{\langle 4\rangle} = E^{[4]} = 0,\quad E^{\langle 5\rangle} = E^{[5]} = 0.$$

Then these submodules satisfy (1.1) – (1.3).

Proof. It is easy to verify (1.1) and (1.2). Since $x_1e_1^2 = x_2e_1^1 \in (x_2)E^{\langle 1\rangle}$, $x_1e_2^2 = x_2e_1^2 \in E^{[2]}$, $x_1e_1^3 = x_3e_1^1 - x_4e_1^2 \in (x_3)E^{\langle 1\rangle} \oplus E^{[2]}$, and $x_2e_1^3 = x_3e_1^2 - x_4e_2^2 \in (x_3, x_4)E^{\langle 2\rangle}$ the condition (1.3) holds, too. □

Definition 8. Choose a sufficiently general set of variables $x_1, \ldots, x_r$. Let $\{\ e_l^i \mid 1 \le i \le r+1,\ 1 \le l \le m_i\ \}$ be a weak Weierstrass basis of E with respect to $x_1, \ldots, x_r$. We define the *basic sequence* $B_R(E)$ of E to be the sequence $(\bar{n}^1; \bar{n}^2; \cdots; \bar{n}^{r+1})$ made of the nondecreasing sequences of integers $\bar{n}^i$ $(1 \le i \le r+1)$ such that $\bar{n}^i = (\deg(e_1^i), \ldots, \deg(e_{m_i}^i))$ up to permutation. In case $E^{\langle i\rangle} = 0$, then $m_i = 0$ and $\bar{n}^i = \emptyset$.

Remark 9. (1) $\operatorname{depth}_{\mathfrak{m}}(E) = r + 1 - \max\{\ i \mid E^{\langle i\rangle} \ne 0\ \}$.

(2) $\bar{n}^1 = (n_1^1, \ldots, n_{m_1}^1)$ with $m_1 = \operatorname{rank}_R(E)$

(3) If $E \subset R$, then $m_1 = 1$ and $n_1^1 = \min\{\ d \mid [E]_d \ne 0\ \}$. Further $n_1^1 = m_2$ if $\operatorname{ht}(E) \ge 2$.

The notion of basic sequence was established first in [2] for the case of homogeneous ideals defining space curves and was applied to the study of arithmetically Buchsbaum curves in $\mathbf{P}^3$. It was extended to homogeneous ideals in polynomial rings of arbitrary number of variables in [6]. Later, for the purpose of generalizing a result on the structure of homogeneous ideals defining graded Buchsbaum rings obtained in [6], we further extended it to arbitrary finitely generated graded modules over polynomial rings.

One of the most difficult and important problems concerning basic sequence is to give a characterization of those of the homogeneous prime ideals. We have only few results so far in this direction, among which we think the following generalization of Gruson-Peskine's connectedness theorem (see [16]) a good one.

Theorem 10 (cf. [3, Corollary 1.2]). *Let $I \subset R$ be a homogeneous prime ideal of height larger than or equal to two and let $(\bar{n}^1; \cdots; \bar{n}^{r+1})$ its basic sequence. Then $n_l^2 \le n_{l+1}^2 \le n_l^2 + 1$ for all $l = 1, \ldots, m_2 - 1$ (note that $m_2 = n_1^1$).*

Proof. You will find a proof for the case $r = 4$ in [3, Section 1]. Reading it carefully, you will be convinced that the assertion is true for arbitrary k and $r \ge 2$. See [14] for another method. □

There are a lot of other applications of basic sequences to the study homogeneous ideals defining curves in $\mathbf{P}^3$. For them, see [2] – [5].

We pass on to the next topic we are most interested in now. Let $p \geq 2$ and let M be a finitely generated torsion-free graded R-module with no free direct summand satisfying $\operatorname{Ext}^i_R(M,R) = 0$ for $i = 1, \ldots, p-1$. Let further $\mathfrak{I}(M,p)$ be the set of all homogeneous ideals I in R of height p fitting into exact sequences of the form

$$0 \longrightarrow S_{p-1} \longrightarrow S_{p-2} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 \oplus M \longrightarrow I(c) \longrightarrow 0.$$

where c is an integer and S_i $(0 \leq i \leq p-1)$ are finitely generated graded free R-modules. By this sequence one obtains

$$H^{i-1}_{\mathfrak{m}}(R/I)(c) \cong H^i_{\mathfrak{m}}(M) \quad \text{for} \quad i = 1, \ldots, \dim(R/I) = r - p.$$

Since $H^i_{\mathfrak{m}}(M) = 0$ for $i = r-p+1, \ldots, r-1$ and $i = 0$ by local duality, considering the local cohomologies of R/I is the same thing as considering those of M.

Problem 11. Describe $B_R(I)$ for all $I \in \mathfrak{I}(M,p)$.

Our first result is this.

Theorem 12 ([9, Theorem 3]). *For all M and p as above, the set $\mathfrak{I}(M,p)$ is not empty.*

Though it may not be explicit in [2], the argument of [2] and [6] on the structure of homogeneous ideals defining graded Buchsbaum rings was based wholly on the comparison of $I^{[p]}$ and $M^{[p]}$. This comparison theorem can be generalized in the following form.

Theorem 13 ([8, Theorem 2.3]). *Let $I \in \mathfrak{I}(M,p)$. If the variables $x_1, \ldots, x_r$ satisfy* (1.4) *for both I and M then $I^{[p]} \cong C \oplus M^{[p]}(-c)$ as $k[x_p, \ldots, x_r]$-module with a finitely generated graded free $k[x_p, \ldots, x_r]$-module C.*

Corollary 14 ([8, Corollary 2.4]). *Let $I \in \mathfrak{I}(M,p)$, $B_R(I) = (\bar{n}^1; \bar{n}^2; \cdots; \bar{n}^{r+1})$, and $B_R(M) = (\bar{\gamma}^1; \bar{\gamma}^2; \cdots; \bar{\gamma}^{r+1})$. Then we have*

$$\begin{cases} \bar{n}^p = (\bar{w}', \bar{\gamma}^p + c) & \text{up to permutation and} \\ \bar{n}^i = \bar{\gamma}^i + c & \text{for } i = p+1, \ldots, r+1 \end{cases}$$

with a suitable sequence of integers $\bar{w}'$, where $\bar{\nu} + c = (\nu_1 + c, \ldots, \nu_l + c)$ for a sequence $\bar{\nu} = (\nu_1, \ldots, \nu_l)$.

We have some results obtained as applications of the above theorem. They are formulated mainly in terms of C and $\bar{w}'$.

- An answer to Problem 11 for the case $p = 2$ (see [10] and [11]).

- An answer to Problem 11 for the case M is Buchsbaum (see [6, Sections 5 and 6]).
- Almost complete description of $B_R(I)$ for I defining graded Buchsbaum integral domains of codimension two (see [5] and [6, Section 7]).
- Some description of $B_R(I)$ for I defining graded integral domains of codimension two (see [17] and [11]).

Finally we explain the computational aspect of (weak) Weierstrass bases and basic sequences in the case where E is a submodule of a graded free module, namely, the case where E is torsion-free. If $\mathrm{rank}_R(E) = 1$, then essentially E is a homogeneous ideal and weak Weierstrass bases can be obtained with the use of generalized Weierstrass preparation theorem due to Hironaka or Grauert (see [18] and [15]). In fact, for generic coordinates, Gröbner bases with respect to reverse lexicographic order form Weierstrass bases if $\mathrm{char}(k) = 0$ and $\mathrm{rank}_R(E) = 1$. We will make this point clear for the general torsion-free case below.

Let $\bar{a} = (a_1, \ldots, a_s)$ be a sequence of integers. Suppose $E \subset R(-\bar{a}) := \bigoplus_{i=1}^{s} R(-a_i)$. Denote by v_i the free base ${}^t(0, \ldots, 0, \overset{i}{\check{1}}, 0, \ldots, 0) \in R(-\bar{a})$ of degree a_i for each $i = 1, \ldots, s$. Let $v = \sum_{i=1}^{s} f_i v_i$ be an element of $R(-\bar{a})$. We define the degree of v to be $\max\{\ \deg(f_i) + a_i \mid 1 \le i \le s\ \}$ and denote it by $\deg_{\bar{a}}(v)$. If every f_i is homogeneous and there is an integer b such that $\deg(f_i) + a_i = b$ for all i with $f_i \neq 0$, then we say that v is homogeneous of degree b. An element of $R(-\bar{a})$ of the form $f v_i$ $(1 \le i \le s)$ with a monomial $f \in R$ in $x_1, \ldots, x_r$ will be called a monomial of $R(-\bar{a})$ in $x_1, \ldots, x_r$. An element of $R(-\bar{a})$ is homogeneous if and only if it is the linear combination of monomials of the same degree over k.

Let $<$ denote the reverse lexicographic order on the monomials of R in $x_1, \ldots, x_r$. We denote by the same symbol $<$ the term order on the monomials of $R(-\bar{a})$ in $x_1, \ldots, x_r$ such that

$$f v_i < g v_j \quad \text{if and only if} \quad \begin{cases} f < g & \text{or} \\ f = g & \text{and} \quad i > j \end{cases}$$

(cf. [13, Definition 3.5.2]). Taking the gradation determined by $\deg_{\bar{a}}(\ )$ into account, we further consider the term order $<_{\bar{a}}$ on the monomials of $R(-\bar{a})$ in $x_1, \ldots, x_r$ such that

$$f v_i <_{\bar{a}} g v_j \quad \text{if and only if} \quad \begin{cases} \deg_{\bar{a}}(f v_i) < \deg_{\bar{a}}(g v_j) & \text{or} \\ \deg_{\bar{a}}(f v_i) = \deg_{\bar{a}}(g v_j) & \text{and} \quad f v_i < g v_j. \end{cases}$$

The initial term of $v \in R(-\bar{a})$ with respect to $<_{\bar{a}}$ will be denoted by $\mathrm{in}_{\bar{a}}(v)$.

As at the beginning the base field k is infinite of arbitrary characteristic.

Theorem 15 ([12]). *Assume that $x_1, \ldots, x_r$ are sufficiently general. Then there exists a set $W = \{ e^i_l \mid 1 \leq i \leq r,\ 1 \leq l \leq m_i \}$ of homogeneous generators of E such that $e^i_l \neq 0$ for all i, l which satisfies the following conditions.*

(15.1) $E = \bigoplus_{i=1}^{r} E^{\langle i \rangle}$ *as k-module with* $E^{\langle i \rangle} := \bigoplus_{l=1}^{m_i} k[x_i, \ldots, x_r]e^i_l,$

(15.2) $x_{i'}e^j_l \in (x_{i'+1}, \ldots, x_r)\left(\bigoplus_{l=1}^{m_{i'}} k[x_{i'+1}, \ldots, x_r]e^{i'}_l\right) \oplus \left(\bigoplus_{i=i'+1}^{r} E^{\langle i \rangle}\right)$

for every triple i', j, l such that $1 \leq i' < j \leq r,\ 1 \leq l \leq m_j$,

(15.3) *the coefficient of* $\mathrm{in}_{\bar{a}}(e^i_l)$ *is one for all i, l,*

(15.4) $\mathrm{in}_{\bar{a}}(e^1_l) \in k[x_1]v_j$ for some $j = 1, \ldots, s$,

(15.5) $\mathrm{in}_{\bar{a}}(e^i_l) \in k[x_1, \ldots, x_i]x_i v_j$ for some $j = 1, \ldots, s$

for all $i = 2, \ldots, r$,

(15.6) $\sum_{i=1}^{r}\sum_{l=1}^{m_i} g^i_l \mathrm{in}_{\bar{a}}(e^i_l) = 0$ with $g^i_l \in k[x_i, \ldots, x_r]$ $(1 \leq i \leq r,\ 1 \leq l \leq m_i)$

if and only if $g^i_l = 0$ for all i, l,

(15.7) $e^i_l - \mathrm{in}_{\bar{a}}(e^i_l) \notin \sum_{j=1}^{r}\sum_{l'=1}^{m_j} k[x_j, \ldots, x_r]\mathrm{in}_{\bar{a}}(e^j_{l'})$ *for all* i, l,

where $\bigoplus_{l=1}^{m}(\ )$ means that the sum $\sum_{l=1}^{m}(\ )$ is direct and we understand $\bigoplus_{l=1}^{m}(\ ) = 0$ if $m = 0$.

Corollary 16 ([12]). *Let $E^{\langle i \rangle}$ $(1 \leq i \leq r)$ be as in the above theorem and $E^{\langle r+1 \rangle} := 0$. Let further $E^{[i]}$ be the subsets of E defined by the formula* (2.1) *for each $i = 1, \ldots, r+1$. Then $E^{\langle i \rangle}$ $(1 \leq i \leq r+1)$ and $E^{[i]}$ $(1 \leq i \leq r+1)$ satisfy the conditions* (1.1) – (1.3).

Definition 17 ([12]). We call the system W of generators of E stated in Theorem 15 a *perfect Weierstrass basis* of E with respect to $x_1, \ldots, x_r$.

Proposition 18 ([12]). *Let W be a perfect Weierstrass basis of E with respect to $x_1, \ldots, x_r$. Then the members of W form a Gröbner basis of E with respect to the term order $<_{\bar{a}}$. In particular, the basic sequence of E is a sequence consisting of the degrees of the members of a generic Gröbner basis with respect to the term order $<_{\bar{a}}$.*

Remark 19. See [6, Section 3] for free resolutions starting with Weierstrass bases.

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