On the integral closures of certain ideals generated by regular sequences

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1 Introduction

The purpose of this report is to introduce a notion of equimultiplicity for filtrations in local rings. We will apply it's theory for computation of the integral closures of certain ideals generated by regular sequences.

Throughout this report $A$ is a $d$-dimensional local ring with the maximal ideal $\mathfrak{m}$ and a family of ideals $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ is a filtration in $A$, which means (i) $F_n \supseteq F_{n+1}$ for all $n \in \mathbb{Z}$, (ii) $F_0 = A$, $F_1 \neq A$ and (iii) $F_m F_n \subseteq F_{m+n}$ for all $m, n \in \mathbb{Z}$. We can define the following graded algebras associated to a filtration $\mathcal{F}$.

$$\begin{align*}
R(\mathcal{F}) &= \sum_{n \geq 0} F_n t^n \subseteq A[t], \\
R'(\mathcal{F}) &= \sum_{n \in \mathbb{Z}} F_n t^n \subseteq A[t, t^{-1}] \text{ and} \\
G(\mathcal{F}) &= R'(\mathcal{F})/t^{-1} R'(\mathcal{F}) = \oplus_{n \geq 0} F_n / F_{n+1},
\end{align*}$$

where $t$ is an indeterminate. These algebras are respectively called the Rees algebra of $\mathcal{F}$, the extended Rees algebra of $\mathcal{F}$ and the associated graded ring of $\mathcal{F}$. We always assume that $R(\mathcal{F})$ is Noetherian and $\dim R(\mathcal{F}) = d + 1$.

2 The analytic spread of a filtration

We set $\ell(\mathcal{F}) = \dim A/\mathfrak{m} \otimes_A R(\mathcal{F})$ and call it the analytic spread of $\mathcal{F}$. It is easy to see that $\ell(\mathcal{F}) = \dim A/\mathfrak{m} \otimes_A G(\mathcal{F})$. We say that a system of elements $a_1, \cdots, a_r$ in $A$ is a reduction of $\mathcal{F}$, if the following condition $(\ast)$ is satisfied.

$(\ast)$ There exist $m_i > 0$ for all $1 \leq i \leq r$ such that $a_i \in F_{m_i}$ and $F_n = \sum_{i=1}^r a_i F_{n-m_i}$ for all $n \gg 0$. 


This condition is equivalent to saying that we have a module-finite extension

\[ A[a_1 t^{m_1}, \ldots, a_r t^{m_r}] \subseteq \mathcal{R}(\mathcal{F}) \]

of rings. If \( a_1, \ldots, a_r \) is a reduction of \( \mathcal{F} \), then obviously we have \( \ell(\mathcal{F}) \leq r \). We say that a reduction \( a_1, \ldots, a_r \) of \( \mathcal{F} \) is minimal, if \( \ell(\mathcal{F}) = r \). We always have a minimal reduction for any filtration \( \mathcal{F} \) (It is not necessary to assume that the residue field is infinite).

By the definition of filtration, we have \( \sqrt{F_n} = \sqrt{F_1} \) for all \( n \geq 1 \), and so \( \text{ht}_A F_n \) is constant for \( n \geq 1 \). We denote this number by \( \text{ht}_A \mathcal{F} \). Then the following inequality always holds:

\[ \text{ht}_A \mathcal{F} \leq \ell(\mathcal{F}) \leq \dim A. \]

We say that \( \mathcal{F} \) is equimultiple, if \( \text{ht}_A \mathcal{F} = \ell(\mathcal{F}) \). If \( \mathcal{F} \) is equimultiple and \( a_1, \ldots, a_r \) is a minimal reduction of \( \mathcal{F} \), the number \( m_i \) in (*) must coincide to

\[ \deg_{\tau} a_i := \max \{ n \mid a_i \in F_n \} \]

for all \( 1 \leq i \leq r \).

**Example 2.1** Let \( \mathfrak{p} \) be a prime ideal in \( A \) such that \( \dim A/\mathfrak{p} = 1 \). Let \( F_n = \mathfrak{p}^{(n)} \) for all \( n \in \mathbb{Z} \), where \( \mathfrak{p}^{(n)} \) denotes the \( n \)-th symbolic power of \( \mathfrak{p} \). If \( \mathcal{R}(\mathcal{F}) \) is Noetherian, then \( \mathcal{F} \) is equimultiple.

**Proof.** Because \( \mathcal{R}(\mathcal{F}) \) is Noetherian, there exists a positive integer \( k \) such that \( \mathfrak{p}^{(kn)} = [\mathfrak{p}^{(k)}]^n \) for all \( n \in \mathbb{Z} \). This means the \( k \)-th Veronesean subring \( \mathcal{R}(\mathcal{F})^{(k)} = \sum_{n \geq 0} \mathfrak{p}^{(kn)} t^{kn} \) is isomorphic to \( \mathcal{R}(\mathfrak{p}^{(k)}) \) and depth \( A/\mathfrak{p}^{(k)} \) is constant for \( n \geq 1 \). Then the extension

\[ \mathcal{R}(\mathfrak{p}^{(k)}) \subseteq \mathcal{R}(\mathcal{F}) \]

is module-finite and \( \ell(\mathfrak{p}^{(k)}) = d - 1 \) by Burch's inequality. Let \( a_1, \ldots, a_{d-1} \) be a minimal reduction of \( \mathfrak{p}^{(k)} \). Then the extension

\[ A[a_1 t^k, \ldots, a_{d-1} t^k] \subseteq \mathcal{R}(\mathcal{F})^{(k)} \]

is module-finite, and so

\[ A[a_1, \ldots, a_{d-1}] \subseteq \mathcal{R}(\mathcal{F}) \]

is also module-finite.

**Example 2.2** Let \( J \) be an ideal in \( A \) generated by a subsystem of parameters \( a_1, \ldots, a_s \) for \( A \). Let \( \mathcal{F} \) be a filtration such that \( J^n \subseteq F_n \subseteq \overline{J^n} \) for all \( n \in \mathbb{Z} \). If \( \mathcal{R}(\mathcal{F}) \) is Noetherian, then \( \mathcal{F} \) is equimultiple and \( a_1, \ldots, a_s \) is a minimal reduction of \( \mathcal{F} \).
Proof. Obviously, \( \text{ht}_A \mathcal{F} = s \). As \( J^n \subseteq F_n \) for all \( n \in \mathbb{Z} \), \( R(\mathcal{F}) \) contains \( A[a_1 t, \cdots, a_s t] \). Moreover, as \( F_n \subseteq \overline{J^n} \) for all \( n \in \mathbb{Z} \), \( R(\mathcal{F}) \) is integral over \( A[a_1 t, \cdots, a_s t] \). Because \( R(\mathcal{F}) \) is Noetherian, we see that the extension

\[
A[a_1 t, \cdots, a_s t] \subseteq R(\mathcal{F})
\]

is module-finite.

For a prime ideal \( \mathfrak{p} \) in \( A \) containing \( F_1 \), we set \( \mathcal{F}_\mathfrak{p} = \{ F_n A_\mathfrak{p} \}_{n \in \mathbb{Z}} \), which is a filtration in \( A_\mathfrak{p} \). Obviously, \( \ell(\mathcal{F}_\mathfrak{p}) \leq \ell(\mathcal{F}) \) for any prime ideal \( \mathfrak{p} \) in \( A \) containing \( F_1 \).

3 Cohen-Macaulay property of the graded rings associated to equimultiple filtrations

**Theorem 3.1** Let \( A \) be a quasi-unmixed local ring. If \( \mathcal{F} \) is equimultiple, then we have

\[
a(G(\mathcal{F})) = \max\{a(G(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \text{Assh}_A A/F_1\}
\]

**Theorem 3.2** Let \( A \) be a Cohen-Macaulay ring. Let \( \mathcal{F} \) be an equimultiple filtration. We set \( s = \text{ht}_A \mathcal{F} \). Then the following conditions are equivalent:

1. \( G(\mathcal{F}) \) is a Cohen-Macaulay ring.
2. \( G(\mathcal{F}_\mathfrak{p}) \) is Cohen-Macaulay for all \( \mathfrak{p} \in \text{Assh}_A A/F_1 \) and there exists a minimal reduction \( a_1, \cdots, a_s \) of \( \mathcal{F} \) such that \( A/(a_1, \cdots, a_s) + F_n \) is Cohen-Macaulay for all \( 1 \leq n \leq a(G(\mathcal{F})) + \sum_{i=1}^s \deg_\mathcal{F} a_i \).

When this is the case, for any minimal reduction \( b_1, \cdots, b_s \) of \( \mathcal{F} \), \( A/(b_1, \cdots, b_s) + F_n \) is Cohen-Macaulay for all \( n \geq 1 \) and

\[
R(\mathcal{F}) = A[\{b^t \deg_\mathcal{F} b\}_{1 \leq i \leq s}, \{F^t \}_{1 \leq n \leq a(G(\mathcal{F})) + \sum_{i=1}^s \deg_\mathcal{F} b_i}].
\]

**Corollary 3.3** Let \( A \) be a Cohen-Macaulay ring. Let \( I \) be an equimultiple ideal. Then the following conditions are equivalent:

1. \( G(I) \) is a Cohen-Macaulay ring.
2. \( G(I_\mathfrak{p}) \) is Cohen-Macaulay for all \( \mathfrak{p} \in \text{Assh}_A A/I \) and there exists a minimal reduction \( J \) of \( I \) such that \( A/J + I^n \) is Cohen-Macaulay for all \( 1 \leq n \leq r_J(I) \).
4 Integral closures of certain ideals

Applying the results in section 3, we can prove the following assertions.

Example 4.1 Let \( A = k[[X, Y, Z]] \) be the formal power series ring over a field \( k \). Suppose that the ideal generated by the maximal minors of the matrix

\[
\begin{pmatrix}
X^\alpha & Y^{\beta'} & Z^{\gamma'} \\
Y^\beta & Z^{\gamma} & X^{\alpha'}
\end{pmatrix}
\]

is a prime ideal, where \( \alpha, \beta, \gamma, \alpha', \beta' \) and \( \gamma' \) are all positive integers. We put \( a = Z^{\gamma+\gamma'} - X^\alpha Y^{\beta'} \), \( b = X^{\alpha+\alpha'} - Y^\beta Z^{\gamma'} \) and \( c = Y^{\beta+\beta'} - X^\alpha Z^{\gamma} \). Let \( J = (a, b)A \). Then we have

\[
\overline{J^n} = J^{n-1} \cdot (a, b, \{X^i Z^j c \mid i, j \geq 0 \text{ and } i/\alpha' + j/\gamma' \geq 1\})A
\]

for all \( n \geq 1 \) and \( \overline{R(J)} \) is a Cohen-Macaulay ring.

Example 4.2 Let \( A = k[[X, Y, Z, W]] \) be the formal power series ring over a field \( k \). Let \( \alpha, \beta \) and \( \gamma \) be positive integers such that \( 0 < \alpha \leq \beta \leq \gamma \). We set

\[
a = X^{\alpha+\ell} - Y^\beta W, \quad b = Y^{\beta+m} - Z^\gamma W, \quad c = Z^{\gamma+1} - X^\alpha W \quad \text{and} \quad d = W^3 - X^\ell Y^m Z,
\]

where \( \ell = \gamma + \beta - 2\alpha + 1 \) and \( m = 2\gamma - \beta - \alpha + 1 \). It is easy to see that \( a, b, c \) is a regular sequence in \( A \). Let \( J = (a, b, c)A \). Then we have

\[
\overline{J} = J + (\{X^i Y^j Z^k d \mid i/\alpha + j/\beta + k/\gamma \geq 1\})A,
\]
\[
\overline{J^2} = \overline{J}^2 + (\{X^i Y^j Z^k d^2 \mid i/2\alpha + j/2\beta + k/2\gamma \geq 1\})A \quad \text{and}
\]
\[
\overline{J^n} = \overline{J}^{n-2} \cdot \overline{J^2} \quad \text{for } n \geq 3.
\]

Moreover \( \overline{R(J)} \) is a Cohen-Macaulay ring.

参考文献


