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<th>On the integral closures of certain ideals generated by regular sequences (Free resolution of defining ideals of projective varieties)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1999), 1078: 111-115</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1999-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62663">http://hdl.handle.net/2433/62663</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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On the integral closures of certain ideals generated by regular sequences

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1 Introduction

The purpose of this report is to introduce a notion of equimultiplicity for filtrations in local rings. We will apply it’s theory for computation of the integral closures of certain ideals generated by regular sequences.

Throughout this report $A$ is a $d$-dimensional local ring with the maximal ideal $\mathfrak{m}$ and a family of ideals $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ is a filtration in $A$, which means (i) $F_n \supseteq F_{n+1}$ for all $n \in \mathbb{Z}$, (ii) $F_0 = A$, $F_1 \neq A$ and (iii) $F_m F_n \subseteq F_{m+n}$ for all $m, n \in \mathbb{Z}$. We can define the following graded algebras associated to a filtration $\mathcal{F}$.

$$R(\mathcal{F}) = \sum_{n \geq 0} F_n t^n \subseteq A[t],$$
$$R'(\mathcal{F}) = \sum_{n \in \mathbb{Z}} F_n t^n \subseteq A[t, t^{-1}]$$
$$G(\mathcal{F}) = R'(\mathcal{F})/t^{-1} R'(\mathcal{F}) = \oplus_{n \geq 0} F_n / F_{n+1},$$

where $t$ is an indeterminate. These algebras are respectively called the Rees algebra of $\mathcal{F}$, the extended Rees algebra of $\mathcal{F}$ and the associated graded ring of $\mathcal{F}$. We always assume that $R(\mathcal{F})$ is Noetherian and $\dim R(\mathcal{F}) = d + 1$.

2 The analytic spread of a filtration

We set $\ell(\mathcal{F}) = \dim A/\mathfrak{m} \otimes_A R(\mathcal{F})$ and call it the analytic spread of $\mathcal{F}$. It is easy to see that $\ell(\mathcal{F}) = \dim A/\mathfrak{m} \otimes_A G(\mathcal{F})$. We say that a system of elements $a_1, \cdots, a_r$ in $A$ is a reduction of $\mathcal{F}$, if the following condition $(\ast)$ is satisfied.

$$(\ast) \text{ There exist } m_i > 0 \text{ for all } 1 \leq i \leq r \text{ such that } a_i \in F_{m_i} \text{ and } F_n = \sum_{i=1}^r a_i F_{n-m_i} \text{ for all } n \gg 0.$$
This condition is equivalent to saying that we have a module-finite extension

\[ A[a_1t^{m_1}, \cdots, a_r t^{m_r}] \subseteq \mathbb{R}(\mathcal{F}) \]

of rings. If \( a_1, \cdots, a_r \) is a reduction of \( \mathcal{F} \), then obviously we have \( \ell(\mathcal{F}) \leq r \). We say that a reduction \( a_1, \cdots, a_r \) of \( \mathcal{F} \) is minimal, if \( \ell(\mathcal{F}) = r \). We always have a minimal reduction for any filtration \( \mathcal{F} \) (it is not necessary to assume that the residue field is infinite).

By the definition of filtration, we have \( \sqrt{F_n} = \sqrt{F_1} \) for all \( n \geq 1 \), and so \( \text{ht}_A F_n \) is constant for \( n \geq 1 \). We denote this number by \( \text{ht}_A \mathcal{F} \). Then the following inequality always holds:

\[ \text{ht}_A \mathcal{F} \leq \ell(\mathcal{F}) \leq \dim A. \]

We say that \( \mathcal{F} \) is equimultiple, if \( \text{ht}_A \mathcal{F} = \ell(\mathcal{F}) \). If \( \mathcal{F} \) is equimultiple and \( a_1, \cdots, a_r \) is a minimal reduction of \( \mathcal{F} \), the number \( m_i \) in (*) must coincide to

\[ \deg_{\tau} a_i := \max\{n \mid a_i \in F_n\} \]

for all \( 1 \leq i \leq r \).

**Example 2.1** Let \( \mathfrak{p} \) be a prime ideal in \( A \) such that \( \dim A/\mathfrak{p} = 1 \). Let \( F_n = \mathfrak{p}^{(n)} \) for all \( n \in \mathbb{Z} \), where \( \mathfrak{p}^{(n)} \) denotes the \( n \)-th symbolic power of \( \mathfrak{p} \). If \( \mathbb{R}(\mathcal{F}) \) is Noetherian, then \( \mathcal{F} \) is equimultiple.

**Proof.** Because \( \mathbb{R}(\mathcal{F}) \) is Noetherian, there exists a positive integer \( k \) such that \( \mathfrak{p}^{(kn)} = [\mathfrak{p}^{(k)}]^n \) for all \( n \in \mathbb{Z} \). This means the \( k \)-th Veronesean subring \( \mathbb{R}(\mathcal{F})^{(k)} = \sum_{n \geq 0} \mathfrak{p}^{(kn)} t^{kn} \) is isomorphic to \( \mathbb{R}(\mathfrak{p}^{(k)}) \) and depth \( A/[\mathfrak{p}^{(k)}]^n = 1 \) for all \( n \geq 1 \). Then the extension

\[ \mathbb{R}(\mathfrak{p}^{(k)}) \subseteq \mathbb{R}(\mathcal{F}) \]

is module-finite and \( \ell(\mathfrak{p}^{(k)}) = d-1 \) by Burch’s inequality. Let \( a_1, \cdots, a_{d-1} \) be a minimal reduction of \( \mathfrak{p}^{(k)} \). Then the extension

\[ A[a_1 t^k, \cdots, a_{d-1} t^k] \subseteq \mathbb{R}(\mathcal{F})^{(k)} \]

is module-finite, and so

\[ A[a_1, \cdots, a_{d-1}] \subseteq \mathbb{R}(\mathcal{F}) \]

is also module-finite.

**Example 2.2** Let \( J \) be an ideal in \( A \) generated by a subsystem of parameters \( a_1, \cdots, a_s \) for \( A \). Let \( \mathcal{F} \) be a filtration such that \( J^n \subseteq F_n \subseteq \overline{J^n} \) for all \( n \in \mathbb{Z} \). If \( \mathbb{R}(\mathcal{F}) \) is Noetherian, then \( \mathcal{F} \) is equimultiple and \( a_1, \cdots, a_s \) is a minimal reduction of \( \mathcal{F} \).
Proof. Obviously, $\text{ht}_A \mathcal{F} = s$. As $J^n \subseteq F_n$ for all $n \in \mathbb{Z}$, $R(\mathcal{F})$ contains $A[a_1t, \cdots, a_st]$. Moreover, as $F_n \subseteq \overline{J^n}$ for all $n \in \mathbb{Z}$, $R(\mathcal{F})$ is integral over $A[a_1t, \cdots, a_st]$. Because $R(\mathcal{F})$ is Noetherian, we see that the extension

$$A[a_1t, \cdots, a_st] \subseteq R(\mathcal{F})$$

is module-finite.

For a prime ideal $\mathfrak{p}$ in $A$ containing $F_1$, we set $\mathcal{F}_\mathfrak{p} = \{F_nA_{\mathfrak{p}}\}_{n \in \mathbb{Z}}$, which is a filtration in $A_{\mathfrak{p}}$. Obviously, $\ell(\mathcal{F}_\mathfrak{p}) \leq \ell(\mathcal{F})$ for any prime ideal $\mathfrak{p}$ in $A$ containing $F_1$.

3 Cohen-Macaulay property of the graded rings associated to equimultiple filtrations

Theorem 3.1 Let $A$ be a quasi-unmixed local ring. If $\mathcal{F}$ is equimultiple, then we have

$$a(G(\mathcal{F})) = \max\{a(G(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \text{Assh}_A A/F_1\}$$

Theorem 3.2 Let $A$ be a Cohen-Macaulay ring. Let $\mathcal{F}$ be an equimultiple filtration. We set $s = \text{ht}_A \mathcal{F}$. Then the following conditions are equivalent:

(1) $G(\mathcal{F})$ is a Cohen-Macaulay ring.

(2) $G(\mathcal{F}_\mathfrak{p})$ is Cohen-Macaulay for all $\mathfrak{p} \in \text{Assh}_A A/F_1$ and there exists a minimal reduction $a_1, \cdots, a_s$ of $\mathcal{F}$ such that $A/(a_1, \cdots, a_s) + F_n$ is Cohen-Macaulay for all $1 \leq n \leq a(G(\mathcal{F})) + \sum_{i=1}^s \deg_{\mathcal{F}} a_i$.

When this is the case, for any minimal reduction $b_1, \cdots, b_s$ of $\mathcal{F}$, $A/(b_1, \cdots, b_s) + F_n$ is Cohen-Macaulay for all $n \geq 1$ and

$$R(\mathcal{F}) = A[b_it^{\deg_{\mathcal{F}} b_i}]_{1 \leq i \leq s} \{F_nt^n\}_{1 \leq n \leq a(G(\mathcal{F})) + \sum_{i=1}^s \deg_{\mathcal{F}} b_i}.$$

Corollary 3.3 Let $A$ be a Cohen-Macaulay ring. Let $I$ be an equimultiple ideal. Then the following conditions are equivalent:

(1) $G(I)$ is a Cohen-Macaulay ring.

(2) $G(I_{\mathfrak{p}})$ is Cohen-Macaulay for all $\mathfrak{p} \in \text{Assh}_A A/I$ and there exists a minimal reduction $J$ of $I$ such that $A/J + I^n$ is Cohen-Macaulay for all $1 \leq n \leq r_J(I)$. 


4 Integral closures of certain ideals

Applying the results in section 3, we can prove the following assertions.

**Example 4.1** Let $A = k[[X, Y, Z]]$ be the formal power series ring over a field $k$. Suppose that the ideal generated by the maximal minors of the matrix

\[
\begin{pmatrix}
X^\alpha & Y^\beta & Z^\gamma \\
Y^\beta & Z^\gamma & X^\alpha
\end{pmatrix}
\]

is a prime ideal, where $\alpha, \beta, \gamma, \alpha', \beta'$ and $\gamma'$ are all positive integers. We put $a = Z^{\gamma + \gamma'} - X^\alpha Y^\beta$, $b = X^{\alpha + \alpha'} - Y^\beta Z^\gamma$ and $c = Y^{\beta + \beta'} - X^\alpha Z^\gamma$. Let $J = (a, b)A$. Then we have

\[
\overline{J^n} = J^{n-1} \cdot (a, b, \{X^iZ^jC | i, j \geq 0 \text{ and } i/\alpha' + j/\gamma' \geq 1\})A
\]

for all $n \geq 1$ and $\overline{R(J)}$ is a Cohen-Macaulay ring.

**Example 4.2** Let $A = k[[X, Y, Z, W]]$ be the formal power series ring over a field $k$. Let $\alpha, \beta$ and $\gamma$ be positive integers such that $0 < \alpha \leq \beta \leq \gamma$. We set

\[
a = X^{\alpha + \ell} - Y^\beta W, \quad b = Y^{\beta + m} - Z^\gamma W, \quad c = Z^{\gamma + 1} - X^\alpha W \quad \text{and} \quad d = W^3 - X^\ell Y^m Z,
\]

where $\ell = \gamma + \beta - 2\alpha + 1$ and $m = 2\gamma - \beta - \alpha + 1$. It is easy to see that $a, b, c$ is a regular sequence in $A$. Let $J = (a, b, c)A$. Then we have

\[
\overline{J} = J + (\{X^iY^jZ^k | i/\alpha + j/\beta + k/\gamma \geq 1\})A,
\]

\[
\overline{J^2} = \overline{J}^2 + (\{X^iY^jZ^k | i/2\alpha + j/2\beta + k/2\gamma \geq 1\})A \quad \text{and}
\]

\[
\overline{J^n} = \overline{J}^{n-2} \cdot \overline{J^2} \quad \text{for } n \geq 3.
\]

Moreover $\overline{R(J)}$ is a Cohen-Macaulay ring.

参考文献


