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REMARKS ON LINKAGE CLASSES OF COHEN-MACAULAY MODULES

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INTRODUCTION

The present report is an excerpt from our paper [9], which is to appear in Journal of Pure and Applied Algebra.

$(R, \mathfrak{m}, k)$ を Gorenstein 完備局所環とし、考える加群は、全て R 上有限生成とする。イデアルのリンケージという概念は、Peskine-Szpiro [6]. によって導入された。われわれは、イデアルのリンケージに関する Herzog and Kühl [4] の結果をふまえて、この概念を余次元付きの Cohen-Macaulay 加群に拡張して考える。もう少し、詳しく述べると、R のイデアル I と J が、I と J の共通部分に含まれる正則列 $\underline{\lambda}$ を介して (代数的に) リンクしているというのは、$\mathrm{Hom}_{R/\underline{\lambda}R}(R/I, R/\underline{\lambda}R) \cong J/\underline{\lambda}R \cong \Omega^1_{R/\underline{\lambda}R}(R/J)$ が成り立つときを言いう。従って、R 加群 M と N に対しては、M と N 両方を零化する正則列 $\underline{\lambda}$ を取るとき、この二つの加群が $\underline{\lambda}$ を介してリンクしているということを次の様に定義することが自然であると思わる。則ち、$\mathrm{Hom}_{R/\underline{\lambda}R}(M, R/\underline{\lambda}R) \cong \Omega^1_{R/\underline{\lambda}R}(N)$、但し、$\Omega^1_{R/\underline{\lambda}R}$ は (first) syzygy 関手である。実際に、あとで Cohen-Macaulay 加群のリンケージの定義として、この定義を採用する (Definition (1.1) and (1.3)) 。

さて、イデアルのリンケージの場合には、Rao 対応と呼ばれる有用な理論があるが、これは、Cohen-Macaulay 加群のリンケージの場合には、Cohen-Macaulay 近似を用いて再構成することができる。ここで、Auslander - Buchweitz [1] に基づき、Cohen-Macaulay 近似について復習しておこう。

任意の有限生成 R 加群 M について、次のような完全列を構成することができる。

$$0 \longrightarrow \mathrm{Y}_R(M) \longrightarrow \mathrm{X}_R(M) \longrightarrow M \longrightarrow 0,$$

ここで、$\mathrm{X}_R(M)$ は極大 Cohen-Macaulay 加群であり、$\mathrm{Y}_R(M)$ は射影次元有限の加群である。とくに、X_R を有限生成 R 加群の安定的圏(stable category) から、

R 上の極大 Cohen-Macaulay 加群の安定的圏への関手とみなすことができ、この関手をCohen-Macaulay 近似関手と呼ぶ。

一方、このCohen-Macaulay 近似関手 X_R は同一の even リンケージクラスに属する加群に対しては一定であることが分かる (Corollary (1.6)) 。したがって、余次元 r の Cohen-Macaulay 加群の even リンケージクラスの代表元 M にそのCohen-Macaulay 近似 $X_R(M)$ を対応させることにより、余次元 r の Cohen-Macaulay 加群の even リンケージクラス全体の集合から、極大 Cohen-Macaulay 加群の安定的同型類全体の集合へ、写像 Φ_r を定義することができるが、この写像 Φ_r の性質を調べることが主たる目的である。

$r=1$ の場合には、Section 2 の冒頭で述べるように、もし R が整域であれば、 Φ_1 は全射である。また、Proposition (3.1) では、二つの Cohen-Macaulay 加群が Φ_1 により同一の像を持つための必要十分条件を与えた。$r=2$ の場合には、Theorem (2.2)で、R が 2 次元の整閉整域のとき、Φ_2 が全射であるための必要十分条件は、驚くべきことに、R が UFD であることを示した。最後の章では、応用として、 R が超曲面の場合をあつかい、あるクラスから取った 2 つの Cohen-Macaulay 加群については、同一の even リンケージクラスに属するかどうかを比較的容易に判定できることが分かった。

1. Linkage of modules and the map Φ_r

As in the introduction, we always assume that $(R, \mathfrak{m}, k)$ is a Gorenstein complete local ring of dimension d. We denote the category of finitely generated R-modules by R-mod and denote the category of maximal Cohen-Macaulay modules (resp. the category of Cohen-Macaulay modules of codimension r) as a full subcategory of R-mod by $\mathrm{CM}(R)$ (resp. $\mathrm{CM}^r(R)$). We also denote the stable category by $\underline{\mathrm{CM}}(R)$ (resp. $\underline{\mathrm{CM}}^r(R)$) that is defined in such a way that the objects are the same as that of $\mathrm{CM}(R)$ (resp. $\mathrm{CM}^r(R)$), while the morphisms from M to N are the elements of $\underline{\mathrm{Hom}}_R(M, N) = \mathrm{Hom}_R(M, N)/F(M, N)$ where $F(M, N)$ is the set of morphisms which factor through free R-modules.

First we recall the definition of Cohen-Macaulay approximations from the paper [1] of Auslander and Buchweitz. It is shown in [1] that for any $M \in R$-mod, there is an exact sequence

$$0 \longrightarrow \mathrm{Y}_R(M) \longrightarrow \mathrm{X}_R(M) \longrightarrow M \longrightarrow 0,$$

where $\mathrm{X}_R(M) \in \mathrm{CM}(R)$ and $\mathrm{Y}_R(M)$ is of finite projective dimension. Such a sequence is not unique, but $\mathrm{X}_R(M)$ is known to be unique up to free summand,

and hence it gives rise to the functor

$$\mathrm{X}_R : \underline{R\text{-mod}} \longrightarrow \underline{\mathrm{CM}}(R),$$

which we call the Cohen-Macaulay approximation functor.

Let us denote by D_R the R-dual functor $\mathrm{Hom}(\ \ , R)$. Note that D_R yields a duality on the category $\mathrm{CM}(R)$. Given an R-module M, we denote the ith syzygy module of M by $\Omega^i_R(M)$ for a non-negative integer i. We should notice that if $i \geqq d$, then Ω^i_R gives rise to the functor $\underline{R\text{-mod}} \to \underline{\mathrm{CM}}(R)$. If $M \in \underline{\mathrm{CM}}(R)$, then we can also consider $\Omega^i_R(M)$ even for a negative integer i, which is defined to be $\mathrm{D}_R(\Omega^{-i}_R(\mathrm{D}_R(M)))$. We call $\Omega^i_R(M)$ the $(-i)$th cosyzygy module of M if $i < 0$ and if $M \in \underline{\mathrm{CM}}(R)$. In such a way we get the functor $\Omega^i_R : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$ for any integer i. Note that the Cohen-Macaulay approximation functor X_R is just equal to the composite $\Omega^{-d}_R \circ \Omega^d_R$ as a functor from $\underline{R\text{-mod}}$ to $\underline{\mathrm{CM}}(R)$.

Definition 1.1 (Linkage functor L_R). We define the functor $\mathrm{L}_R : \underline{\mathrm{CM}}(R) \longrightarrow \underline{\mathrm{CM}}(R)$ by $\mathrm{L}_R = \mathrm{D}_R \circ \Omega^1_R$.

We should notice from the definition that $\mathrm{L}^2_R \cong \mathrm{id}_{\underline{\mathrm{CM}}(R)}$.

Lemma 1.2. *Let $M \in \underline{\mathrm{CM}}(R)$ and let $\underline{\lambda}$ be a regular sequence in $\mathfrak{m}$. Then we have the isomorphism $\mathrm{L}_{R/\underline{\lambda}R}(M/\underline{\lambda}M) \cong \mathrm{L}_R M \otimes_R R/\underline{\lambda}R$ in $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$.*

Let $\mathfrak{m} \supseteq \underline{\lambda} = \{\lambda_1, \lambda_2, \cdots, \lambda_r\}$ be a regular sequences of length r. We denote the stable category of maximal Cohen-Macaulay modules over $R/\underline{\lambda}R$ by $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$. We always consider the set of objects of $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$ as a subset of the set of objects of $\underline{\mathrm{CM}}^r(R)$. Note that for two modules M_1 and M_2 in $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$, $M_1 \cong M_2$ in the stable category $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$ if and only if M_1 is stably isomorphic to M_2 in $R/\underline{\lambda}R$-mod, that is, there is an isomorphism $M_1 \oplus F \cong M_2 \oplus G$ as $R/\underline{\lambda}R$-modules for some free $R/\underline{\lambda}R$-modules F and G.

Definition 1.3 (Linkage of Cohen-Macaulay modules). Let N_1, N_2 be two Cohen-Macaulay modules of codimension r. We assume that N_1 (resp. N_2) is a maximal Cohen-Macaulay module over $R/\underline{\lambda}R$ (resp. $R/\underline{\mu}R$) for some regular sequence $\underline{\lambda}$ (resp. $\underline{\mu}$). If there exists a module $N \in \underline{\mathrm{CM}}^r(R)$ that belongs to both $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$ and $\underline{\mathrm{CM}}(R/\underline{\mu}R)$ satisfying

$$N_1 \cong \mathrm{L}_{R/\underline{\lambda}R}(N) \text{ in } \underline{\mathrm{CM}}(R/\underline{\lambda}R) \quad \& \quad N_2 \cong \mathrm{L}_{R/\underline{\mu}R}(N) \text{ in } \underline{\mathrm{CM}}(R/\underline{\mu}R),$$

then we say N_1(resp. N_2) is linked to N through the regular sequence $\underline{\lambda}$ (resp. $\underline{\mu}$) and denote this by $N_1 \underset{\underline{\lambda}}{\sim} N$ (resp. $N_2 \underset{\underline{\mu}}{\sim} N$).

We also say in this case that N_1 and N_2 are doubly linked through $(\underline{\lambda}, \underline{\mu})$, and denote it by $N_1 \underset{\underline{\lambda}}{\sim} N \underset{\underline{\mu}}{\sim} N_2$, or simply $N_1 \underset{(\underline{\lambda},\, \underline{\mu})}{\sim} N_2$.

If there is a sequence of modules $N_1, N_2, \dots, N_s$ in $\underline{\mathrm{CM}}^r(R)$ such that N_i and N_{i+1} are doubly linked for $1 \leqq i < s$, then we say that N_1 and N_s are evenly linked.

Recalling the linkage of ideals from [6], we can see that the above definition agrees with it. Actually let $R \supseteq I,\ J$ be Cohen-Macaulay ideals of codimension r and take a regular sequence $\underline{\lambda} = \{\lambda_1, \lambda_2, \cdots, \lambda_r\}$ of length r contained in both I and J. Then I and J are linked through $\underline{\lambda}$ in the sense of [6] if and only if the Cohen-Macaulay modules R/I and R/J of codimension r are linked in the above sense (i.e. $R/I \underset{\underline{\lambda}}{\sim} R/J$).

Theorem 1.4. *For a given regular sequence $\underline{\lambda}$ of length r in $\mathfrak{m}$, the following diagram commutes:*

$$\begin{array}{ccc}
\underline{\mathrm{CM}}(R/\underline{\lambda}R) & \xrightarrow{\mathrm{X}_R} & \underline{\mathrm{CM}}(R) \\
{\scriptstyle \mathrm{L}_{R/\underline{\lambda}R}}\downarrow & & \downarrow{\scriptstyle \mathrm{L}_R \circ \Omega^r_R} \\
\underline{\mathrm{CM}}(R/\underline{\lambda}R) & \xrightarrow[\mathrm{X}_R]{} & \underline{\mathrm{CM}}(R).
\end{array}$$

Corollary 1.5. *Let $\{\underline{\lambda}, \underline{\mu}\} \subseteq \mathfrak{m}$ be a regular sequence of length $r+s$ where $\underline{\lambda}$ is of length r and $\underline{\mu}$ is of length s. Putting $R' = R/\underline{\lambda}R$ and $R'' = R/(\underline{\lambda}, \underline{\mu})R$, we have the following commutative diagram:*

$$\begin{array}{ccccc}
\underline{\mathrm{CM}}(R'') & \xrightarrow{\Omega^{s+r}_R} & \underline{\mathrm{CM}}(R) & = & \underline{\mathrm{CM}}(R) \\
\| & & & & \| \\
\underline{\mathrm{CM}}(R'') & \xrightarrow{\Omega^{s}_{R'}} & \underline{\mathrm{CM}}(R') & \xrightarrow{\Omega^{r}_R} & \underline{\mathrm{CM}}(R) \\
{\scriptstyle \mathrm{L}_{R''}}\downarrow & & {\scriptstyle \mathrm{L}_{R'}}\downarrow & & {\scriptstyle \mathrm{L}_{R}}\downarrow \\
\underline{\mathrm{CM}}(R'') & \xrightarrow{\mathrm{X}_{R'}} & \underline{\mathrm{CM}}(R') & \xrightarrow{\mathrm{X}_R} & \underline{\mathrm{CM}}(R) \\
\| & & & & \| \\
\underline{\mathrm{CM}}(R'') & \xrightarrow{\mathrm{X}_R} & \underline{\mathrm{CM}}(R) & = & \underline{\mathrm{CM}}(R).
\end{array}$$

Corollary 1.6. *Let N_1 and N_2 be modules in $\underline{\mathrm{CM}}^r(R)$. If N_1 and N_2 are doubly linked, then we have $\mathrm{X}_R(N_1) \cong \mathrm{X}_R(N_2)$ in $\underline{\mathrm{CM}}(R)$.*

It turns out from Corollary 1.6 that the Cohen-Macaulay approximation functor X_R yields a map from the set of even linkage classes in $\underline{\mathrm{CM}}^r(R)$ to the set of objects in $\underline{\mathrm{CM}}(R)$.

Definition 1.7. Let us denote by $\mathrm{B}_r(R)$ the set of even linkage classes of modules in $\underline{\mathrm{CM}}^r(R)$. Then we can define a map Φ_r from $\mathrm{B}_r(R)$ to the set of isomorphism classes of modules in $\underline{\mathrm{CM}}(R)$ by $[N] \longrightarrow \mathrm{X}_R(N)$.

2. A CONDITON MAKING THE MAP Φ_2 SURJECTIVE

If R is a local Gorenstein domain, then every Cohen-Macaulay module $M \in \mathrm{CM}(R)$ has a well-defined rank, say s, and a free module of rank s can be embedded in M:

$$0 \longrightarrow R^s \longrightarrow M \longrightarrow N \longrightarrow 0 \ \text{(exact)},$$

where one can easily see that $N \in \mathrm{CM}^1(R)$. Hence taking a nonzero divisor x that annihilates N, we see that $N \in \underline{\mathrm{CM}}(R/xR)$ and that $M \cong \mathrm{X}_R(N)$. In this way, if R is a domain, then any maximal Cohen-Macaulay module over R is in the image of X_R from $\underline{\mathrm{CM}}^1(R)$, hence Φ_1 is a surjective map.

This argument can be slightly generalized in the following way using the theorem of Bourbaki.

Lemma 2.1. *Let R be a normal Gorenstein domain and let $M \in \underline{\mathrm{CM}}(R)$. For any integer $j \geqq 1$, there is an ideal I of R such that $M \cong \Omega^{j+1}(R/I)$ in $\underline{\mathrm{CM}}(R)$.*

In this lemma the codimension of the module R/I is at most two. In the case that $R/I \in \underline{\mathrm{CM}}^r(R)$ and $\Omega^r(R/I) \cong M$, we see that $\mathrm{X}_R(R/I) \cong \Omega_R^{-r}(M)$, and hence $\Omega_R^{-r}(M)$ is the image of the even linkage class of R/I under the map Φ_r defined in (1.7).

As to the problem asking when a module in $\underline{\mathrm{CM}}(R)$ is the image of $\mathrm{B}_2(R)$ under the map Φ_2, we can show the following result.

Theorem 2.2. *Let R be a normal Gorenstein complete local ring of dimension 2. Then the following conditions are equivalent.*

(a) *R is a UFD.*

(b) *For any module $M \in \underline{\mathrm{CM}}(R)$, we can find an R-module L of finite length (hence a CM module of codimension 2) such that $M \cong \Omega_R^2(L)$ in $\underline{\mathrm{CM}}(R)$.*

(c) *The map Φ_2 is surjective onto the set of isomorphism classes of modules in $\underline{\mathrm{CM}}(R)$.*

Example 2.3. Let k be a field and set $R = k[[\ x, y, z\ ]]/(x^2 - yz)$ that is a normal Gorenstein domain of dimension 2. Now let p be the ideal of R generated by $\{x, y\}$. It is easily verified that p is a prime ideal of height one and $\mathrm{Cl}(R)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by $c(p)$. It is also known that p is a unique indecomposable nonfree maximal Cohen-Macaulay module over R. By the proof of the above theorem, p is not in the image of Φ_2 . On the other hand we can easily verify that $\Omega_R^2(k) \cong p \oplus p$. Therefore we conclude that the image of Φ_2 is just the set of classes of modules that are isomorphic to the direct sum of even number of copies of p.

3. Linkage of CM modules of codimension 1

We have defined a map Φ_r in Definition (1.7) for any $r \geqq 1$. In the case $r = 1$, the following proposition shows the condition for two classes in $B_1(R)$ to have the same image under Φ_1.

Proposition 3.1. *Let λ, μ be regular elements in $\mathfrak{m}$ and put $\xi = \lambda\mu$. And let N_1 (resp. N_2) be a module in $\underline{\mathrm{CM}}(R/\lambda R)$ (resp. $\underline{\mathrm{CM}}(R/\mu R)$). Then the following two conditions are equivalent.*

(a) $\mathrm{X}_R(N_1) \cong \mathrm{X}_R(N_2)$ *in* $\underline{\mathrm{CM}}(R)$

(b) *There exists a module $N \in \underline{\mathrm{CM}}(R/\xi^2 R)$ that contains N_2 as a submodule such that $\mathrm{pd}_R(N/N_2) < \infty$ and $N_1 \underset{(\xi,\ \xi^2)}{\sim} N$.*

4. Linkage of CM modules over hypersurface rings

In this section we consider the following three hypersurface rings :

$$R = k[[\ \underline{x}\ ]]/(f)$$
$$R^\sharp = k[[\ \underline{x}, y\ ]]/(f + y^2)$$
$$R^{\sharp\sharp} = k[[\ \underline{x}, y, z\ ]]/(f + y^2 + z^2) \cong k[[\ \underline{x}, u, v\ ]]/(f + uv),$$

where $\underline{x} = \{x_1, \cdots, x_{d-1}\}, y, z$ are $d+1$ variables over an algebraically closed field k of characteristic 0 where $d \geqq 2$, and $u = y + \sqrt{-1}\ z$, $v = y - \sqrt{-1}\ z$ and f is a non zero element in $k[[\ \underline{x}\ ]]$.

Note that $\{y\}$ (resp. $\{y,z\}$) is a regular sequence on $R^\sharp$ (resp. $R^{\sharp\sharp}$) and that $R \cong R^\sharp/yR^\sharp \cong R^{\sharp\sharp}/(y,z)R^{\sharp\sharp}$. Therefore, an object in $\underline{\mathrm{CM}}(R)$ can be naturally regarded as an object in $\underline{\mathrm{CM}}^1(R^\sharp)$ and $\underline{\mathrm{CM}}^2(R^{\sharp\sharp})$.

Let (φ_M, ψ_M) be a matrix factorization for $M \in \underline{\mathrm{CM}}(R)$, which is, by definition, a pair of two square matrices with entries in $k[[\underline{x}]]$ satisfying $\varphi_M \circ \psi_M = \psi_M \circ \varphi_M = f \cdot 1$ and Coker $\varphi_M \cong M$. Recalling Knörror's periodicity theorem from [5], the functor Lif : $\underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R^{\sharp\sharp})$ defined by $M \mapsto \mathrm{Coker}\begin{pmatrix} \varphi_M & u \cdot 1 \\ -v \cdot 1 & \psi_M \end{pmatrix}$ gives the category equivalence. See [8, Chapter 12] for more details.

Also recall that $\Omega^1_R M \cong \mathrm{Coker}\ \psi_M$ and $\mathrm{D}_R M \cong \mathrm{Coker}\ {}^t\varphi_M$, and hence that $\Omega^2_R M \cong M$ and $\mathrm{L}_R M \cong \mathrm{Coker}\ {}^t\psi_M$. These observations show the following

Proposition 4.1. *The following diagram is commutative:*

$$\begin{array}{ccc} \underline{\mathrm{CM}}(R) & \xrightarrow{\mathrm{Lif}} & \underline{\mathrm{CM}}(R^{\sharp\sharp}) \\ {\scriptstyle \mathrm{L}_R \circ \Omega^1_R}\downarrow & & \downarrow{\scriptstyle \mathrm{L}_{R^{\sharp\sharp}}} \\ \underline{\mathrm{CM}}(R) & \xrightarrow{\mathrm{Lif}} & \underline{\mathrm{CM}}(R^{\sharp\sharp}). \end{array}$$

Lemma 4.2. $\Omega^2_{R^{\sharp\sharp}} M \cong \mathrm{Lif}(M \oplus \Omega^1_R M) \cong \Omega^n_{R^{\sharp\sharp}} M$ *for* $M \in \underline{\mathrm{CM}}(R)$ *and for any integer* $n \geqq 2$.

Lemma 4.3. $\Omega^1_{R^\sharp} M \simeq \Omega^n_{R^\sharp} M \cong \Omega^1_{R^\sharp}\Omega^1_R M$ *for* $M \in \underline{\mathrm{CM}}(R)$ *and any integer* $n \geq 1$.

Proposition 4.4. *The following conditions are equivalent for* M_1 *and* M_2 *in* $\underline{\mathrm{CM}}(R)$.

(a) $M_1 \oplus \Omega^1_R M_1 \cong M_2 \oplus \Omega^1_R M_2$ *in* $\underline{\mathrm{CM}}(R)$.
(b) $\mathrm{X}_{R^\sharp}(M_1) \cong \mathrm{X}_{R^\sharp}(M_2)$ *in* $\underline{\mathrm{CM}}(R^\sharp)$.
(c) $\mathrm{X}_{R^{\sharp\sharp}}(M_1) \cong \mathrm{X}_{R^{\sharp\sharp}}(M_2)$ *in* $\underline{\mathrm{CM}}(R^{\sharp\sharp})$.

Corollary 4.5. *Let* M_1 *and* M_2 *be in* $\underline{\mathrm{CM}}(R)$. *Suppose that they belong to the same even linkage class in* $\underline{\mathrm{CM}}(R^\sharp)$ *or in* $\underline{\mathrm{CM}}(R^{\sharp\sharp})$. *Then we have* $M_1 \oplus \Omega_R M_1 \cong M_2 \oplus \Omega_R M_2$. *Furthermore if we assume that both modules are indecomposable, then we must have either* $M_1 \cong M_2$ *or* $M_1 \cong \Omega^1_R M_2$.

Example 4.6. Using this corollary we are sometimes able to find the condition for given modules to belong to the same even linkage class.

For the simplest example, let $R^\sharp = k[[x,y]]/(x^n + y^2)$ and $R = k[[x]]/(x^n)$. Take an integer r as $n = 2r$ or $n = 2r+1$. It is easy to see that the set of classes of indecomposable modules in $\underline{\mathrm{CM}}(R)$ is $\{R/(x^i) \mid 1 \leqq i < n\}$.

Then we can claim that the modules $R/(x^i)$ for $1 \leqq i \leqq r$ define r distinct even linkage classes in $\underline{\mathrm{CM}}(R^\sharp)$.

In fact, if $R/(x^i)$ and $R/(x^j)$ belong to the same even linkage class in $\underline{\mathrm{CM}}^1(R^\sharp)$, then, since $\Omega_R(R/(x^i)) \cong R/(x^{n-i})$, it follows from the corollary that $R/(x^i) \cong R/(x^j)$ or $R/(x^i) \cong R/(x^{n-j})$, but since we assumed $1 \leqq i, j \leqq r$, we must have $i = j$.

Note that $R/(x^i)$ and $R/(x^{n-i})$ belong to the same even linkage class in $\underline{\mathrm{CM}}^1(R^\sharp)$ for $1 \leqq i \leqq r$. This is just a result of computation as follows ;

$$R/(x^i) \underset{x^{i+1}y}{\sim} R^\sharp/(x^{i+1}y, x^{n+1}) \underset{x^{n+1}}{\sim} R/(x^{n-i}).$$

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