ON COMPLEX MANIFOLDS POLARIZED
BY AN AMPLE LINE BUNDLE
OF SECTIONAL GENUS $q(X) + 2$

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Let $X$ be a smooth projective variety defined over the complex number field, and let $L$ be a line bundle over $X$. Then $(X, L)$ is called a polarized (resp. quasi-polarized) manifold if $L$ is ample (resp. nef and big). For such pair $(X, L)$, the delta genus $\Delta(L)$ and the sectional genus $g(L)$ are defined by the following formula:

$$
\Delta(L) := n + L^n - h^0(L),
$$

$$
g(L) := 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1},
$$

where $h^0(L) = \dim H^0(L)$, and $K_X$ is the canonical divisor of $X$.

In this report, we will state some recent results about the sectional genus of quasi-polarized manifolds, and we will propose some conjectures and problems.

The following results are known for the fundamental properties of the sectional genus;

(1) The value of $g(L)$ is a non-negative integer. (Fujita [Fj1], Ionescu [I])

(2) There exists a classification of polarized manifolds $(X, L)$ with the sectional genus $g(L) \leq 2$. (Fujita [Fj1], [Fj2], Ionescu [I], Beltrametti-Lanteri-Palleschi [BLP], e.t.c.)
Let \((X, L)\) be a polarized manifold with \(\dim X = n\). For any fixed \(n\) and \(g(L)\), there are only finitely many deformation types of polarized manifolds except scrolls over smooth curves. (For the definition of deformation types of polarized manifolds, see Chapter II, §13 in [Fj4].)

Here we give the definition of a scroll over a variety.

**Definition.** Let \((X, L)\) be a quasi-polarized manifold with \(\dim X = N\), and let \(Y\) be a projective variety with \(\dim Y = m\) and \(N > m \geq 1\). Then \((X, L)\) is called a scroll over \(Y\) if there exists a surjective morphism \(\pi : X \rightarrow Y\) such that any fiber \(F\) of \(\pi\) is isomorphic to \(\mathbb{P}^{N-m}\) and \(L_{F} \cong \mathcal{O}(1)\).

Here we consider (3) above. By (3), if \((X, L)\) is not a scroll over a smooth curve, then the topological invariant of \(X\) is expected to be bounded by using some invariant of \(L\). Here we mainly consider the irregularity \(q(X) := \dim H^{1}(\mathcal{O}_{X})\) of \(X\). If \((X, L)\) is a scroll over a smooth curve, then we can easily get that \(g(L) = q(X)\).

So by considering the fact (3) above, Fujita propose the following conjecture;

**Conjecture 1.** ([Fj3]) Let \((X, L)\) be a quasi-polarized manifold. Then \(g(L) \geq q(X)\).

For the time being, this conjecture is true if \((X, L)\) is one of the following;

1. \(n = 2, \kappa(X) \leq 1\) ([Fk2]),
2. \(n = 2, \kappa(X) = 2, h^{0}(L) \geq 1\) ([Fk2]),
3. \(n = 3, h^{0}(L) \geq 2\) ([Fk7]),
4. \(\kappa(X) = 0, 1, L^{n} \geq 2\) ([Fk3]),
5. \(\dim \text{Bs} |L| \leq 1\) (For the case in which \(\dim \text{Bs} |L| \leq 0\), see [Fk6], and for the case in which \(\dim \text{Bs} |L| = 1\), this result is unpublished).

So our first goal is to prove that this conjecture is true if \(n = 2, \kappa(X) = 2,\) and \(h^{0}(L) = 0\).

Here we give some comments about Conjecture 1. First, for a polarized surface \((X, L)\) with \(\kappa(X) \geq 0\) we explain a relation between the value of \(g(L) - q(X)\) and the type of divisor \(D \in |L|\). Let
(X, L) be a quasi-polarized surface with \( \kappa(X) \geq 0 \) and \( h^0(L) > 0 \). Let \( D \) be an effective divisor on \( X \) which is linearly equivalent to \( L \). Let \( D = \sum a_i C_i \), where \( C_i \) is an irreducible reduced curve and \( a_i > 0 \) for each \( i \). We take a birational morphism \( \mu : X^\alpha \rightarrow X \) such that \( C_i^* \cap C_j^* \cap C_3^* = \phi \) for any distinct three irreducible components \( C_i^* \), \( C_j^* \) and \( C_3^* \) of \( \mu^*(D) \), and if two irreducible curves \( C_i^* \) and \( C_j^* \) of \( \mu^*(D) \) intersect at \( x \), then the intersection number \( i(C_i^*, C_j^*; x) = 1 \), where \( i(C_i^*, C_j^*; x) \) is the intersection number of \( C_i^* \) and \( C_j^* \) at \( x \in C_i^* \cap C_j^* \). Let \( \mu_i : X_i \rightarrow X_{i-1} \) be one point blowing up such that \( \mu = \mu_1 \circ \mu_2 \circ \cdots \circ \mu_t \) and let \( D_{\text{red}} = B_0 \). Let \( (\mu_i^*(B_{i-1}))_{\text{red}} = B_i \) and \( B_i = \mu_i^*(B_{i-1}) - b_i E_i \), where \( E_i \) is a \((-1)\)-curve such that \( \mu_i(E_i) = \text{point} \). Then \( b_i \geq 1 \). Let \( D^\beta = \sum C_{\beta,i} \) and \( \mu^\gamma : X^\gamma \rightarrow X^\beta \) be a resolution of singular points \( S = \bigcup \text{Sing}(C_{\beta,i}) \).

Let \( \mu_x : X_x^{\beta,i,t_x} \rightarrow X_x^{\beta,i,t_x-1} \rightarrow \cdots \rightarrow X_x^{\beta,i,0} \) be a resolution of singularity at \( x \in \text{Sing}(C_{\beta,i}) \), where \( \mu_x^k : X_x^{\beta,i,k} \rightarrow X_x^{\beta,i,k-1} \) is one point blowing up. Let \( (\mu_x^k)^*(D_{\beta,k-1}) = D_{\beta,k} + m(k,x)E^k \), where \( E^k \) is \((-1)\)-curve of \( \mu_x^k \) such that \( \mu_x^k(E^k) = \text{point} \) and \( D_{\beta,k} \) is the strict transform of \( D_{\beta,k-1} \) for each \( k \).

By using the above notation, we get the following result:

**Theorem 1.** ([Fk5]) Let \( (X, L) \) be a polarized surface. Assume that \( \kappa(X) \geq 0 \) and \( h^0(L) > 0 \). Let \( D \in |L| \) be an effective divisor which is linearly equivalent to \( L \). Then the following inequality holds;

\[
g(D) \geq q(X) + \sum_{x_j \in S} \sum_{k=1}^{t_x} \frac{m(k,x_j)(m(k,x_j) - 1)}{2} + \sum_{i=1}^{t} b_i(b_i - 1).\]

**Proof.** See [Fk5].

By this theorem, if the value of \( g(L) - q(X) \) is small, then the singularities of \( \text{Supp} \, D \) is simple.
Next we consider the dimension of the global section of the adjoint bundle $K_X + (n-1)L$. The value of $g(L) - q(X)$ is thought to control the value of $h^0(K_X + (n-1)L)$. In [Fk6], the author proposed the following conjecture;

**Conjecture 2.** ([Fk6]) Let $(X, L)$ be a quasi-polarized manifold with $\dim X = n$. Then the following inequality holds;

$$h^0(K_X + (n-1)L) \geq g(L) - q(X).$$

For the time being, this conjecture is true if $(X, L)$ is one of the following;

1. $\dim \text{Bs} |L| \leq 0$,
2. $\dim X = 2$.

If Conjecture 1 is true, then the following natural problem arises;

**Problem 1.** For small non-negative integer $m$, give a classification of quasi-polarized manifolds $(X, L)$ with $m = g(L) - q(X)$.

If $m = 0$ and $n = 2$, then this problem relates to the problem of blowing up of polarized surfaces. Let $S$ be a smooth projective surface and let $L$ be an ample line bundle on $X$. Let $p_1, \ldots, p_r$ be points on $S$ in a general position, and let $\pi : \tilde{S} \to S$ be blowing ups at $p_1, \ldots, p_r$. Let $a_1, \ldots, a_r$ be positive integers and $\tilde{L} := \pi^*L - \sum_j a_j E_j$, where $E_j := \pi^{-1}(p_j)$. Then it is difficult to check that $\tilde{L}$ is ample. For the case where $a_1 = \cdots = a_r = 1$, Yokoyama proved the following theorem;

**Theorem 2.** (Yokoyama) Assume that $a_1 = \cdots = a_r = 1$ and $|L|$ has an irreducible reduced curve. If $g(L) > q(X)$, then $\tilde{L}$ is ample.

When we use this theorem, it is important to know the classification of polarized surfaces $(S, L)$ with $g(L) = q(S)$.

**Remark.** If $\kappa(X) \geq 0$ and an irreducible reduced curve $C \in |L|$ has a singularity, then $\tilde{L}$ is ample because $g(L) > q(S)$ in this case (see [Fk1] and [Fk2]).
Here we consider the classification of quasi-polarized manifolds $(X, L)$ with small value $m = g(L) - q(X)$.

First we study the case in which $X$ is a surface. Then the following facts are known;

(2-0-1) A classification of quasi-polarized surfaces $(X, L)$ with $\kappa(X) \leq 1$ and $g(L) = q(X)$ ([Fk2]).

(2-0-2) A classification of quasi-polarized surfaces $(X, L)$ with $\kappa(X) = 2$, $h^0(L) \geq 1$, and $g(L) = q(X)$ ([Fk1], [Fk8]).

(2-1) A classification of polarized surfaces $(X, L)$ with $\kappa(X) \geq 0$, $h^0(L) \geq 1$, and $g(L) = q(X)$ ([Fk5]).

Next we consider the case in which $\dim X = 3$.

(3-0) A classification of polarized 3-folds $(X, L)$ with $h^0(L) \geq 3$ and $g(L) = q(X)$ ([Fk7]). In this case $(X, L)$ is one of the following two types;

(3-0-1) Polarized 3-folds $(X, L)$ with $\Delta(L) = 0$. (This was classified by Fujita. See [Fj4].)

(3-0-2) A scroll over a smooth curve.

(3-1) A classification of polarized 3-folds $(X, L)$ with $h^0(L) \geq 4$ and $g(L) = q(X) + 1$ ([Fk4]). Then $(X, L)$ a Del Pezzo 3-fold.

By considering (3-0) and (3-1), in [Fk4] and [Fk7] the author proposed the following conjecture;

**Conjecture 3.** ([Fk4], [Fk7] .) Let $(X, L)$ be a polarized manifold with $n = \dim X \geq 4$.

(n-0) Assume that $g(L) = q(X)$ and $h^0(L) \geq n$. Then $(X, L)$ is a polarized manifold with $\Delta(L) = 0$ or a scroll over a smooth curve.

(n-1) Assume that $g(L) = q(X) + 1$ and $h^0(L) \geq n + 1$. Then $(X, L)$ is a Del Pezzo manifold.

By considering (3-0) and (3-1) above, we expect that we can classify polarized 3-folds $(X, L)$ with $g(L) = q(X) + 2$ and $h^0(L) \geq 5$. The following result is one of the main theorems of the author’s talk.
Main Theorem 1. Let \((X, L)\) be a polarized 3-fold. Assume that \(h^0(L) \geq 5\) and \(g(L) = q(X) + 2\). Then \((X, L)\) is one of the following:

1. A hyperquadric fibration over \(\mathbb{P}^1\).
2. A scroll over a smooth surface \(S\) with \(q(S) = 0\).

Remark. In each case, the irregularity of \(X\) is zero. Hence we get \(g(L) = 2\). Therefore we obtain an explicit classification of \((X, L)\). (See [Fj2].)

Proof of Main Theorem 1. Here we get a sketch of proof of the Main Theorem 1. First assume that \(K_X + 2L\) is not nef. Then \((X, L)\) is one of the following types:

1. \((\mathbb{P}^3, O(1))\),
2. \((\mathbb{Q}^3, O(1))\),
3. scroll over a smooth curve.

But in these cases, we obtain \(g(L) = q(X)\) and this is a contradiction by hypothesis.

So we may assume that \(K_X + 2L\) is nef. Let \((X', L')\) be the first reduction of \((X, L)\). (Let \(X\) be a smooth projective variety with \(\text{dim } X = n\) and let \(L\) be an ample line bundle \(L\) on \(X\). Then we call that \((X', L')\) is the first reduction of \((X, L)\) if there exist a smooth projective variety \(X'\), an ample line bundle \(L'\) on \(X'\), and a birational morphism \(\pi : X \to X'\) such that \(\pi\) is a blowing up at a finite set on \(X'\), \(K_X + (n - 1)L = \pi^*(K_{X'} + (n - 1)L')\), and \(K_{X'} + (n - 1)L'\) is ample.)

We remark that \(L^n \leq (L')^n\) in this case.

Here we use the following Theorem, which is very important for the proof of Main Theorem.

Theorem A. Let \((X, L)\) be a polarized 3-fold with \(g(L) = q(X) + m\), \(h^0(L) \geq m + 3\), and \(q(X) \geq m - 1\), where \(m\) is a non-negative integer. Assume that \(K_X + L\) is nef. Then \(L^3 \leq 2m\).

By using Theorem A and the theory of \(\Delta\)-genus, we can prove the following Claim.
Claim B. $K_{X'} + L'$ is not nef.

Proof of Claim B. Assume that $K_{X'} + L'$ is nef.
If $q(X) \geq 1$, then by Theorem A, we get that $L^3 \leq (L')^3 \leq 4$.
If $q(X) = 0$, then $L^3 \leq (L')^3 \leq 2$ since $K_{X'} + L'$ is nef.
We set $t = 4 - L^3$. Then $t = 0, 1, 2$ or 3.
Since $h^0(L) \geq 5$, we get
\[
\Delta(L) = 3 + L^3 - h^0(L) = 7 - t - h^0(L) \leq 2 - t.
\]
If $t > 0$, then $\Delta(L) \leq 1$. By using the theory of $\Delta$-genus, we can easily get a contradiction.
So we assume $t = 0$. If $h^0(L) \geq 6$, then we get $\Delta(L) \leq 1$ and by using the same method as above, we get a contradiction.
If $h^0(L) = 5$, then $\Delta(L) \leq 2$. Here we also use the $\Delta$-genus theory, we also get a contradiction.
Therefore $K_{X'} + L'$ is not nef. By adjunction theory, polarized manifolds $(X, L)$ such that $K_{X'} + L'$ is not nef is classified.

(1) $K_X \sim -2L$, that is, $(X, L)$ is a Del Pezzo manifold.
(2) A hyperquadric fibration over a smooth curve.
(3) A scroll over a smooth surface.
(4) Let $(X', L')$ be the first reduction of $(X, L)$.
(4-1) $(X', L') = (\mathbb{Q}^3, \mathcal{O}(2))$,
(4-2) $(X', L') = (\mathbb{P}^3, \mathcal{O}(3))$,
(4-3) $X'$ is a $\mathbb{P}^2$-bundle over a smooth curve $C$ with $(F', L'|_{F'}) = (\mathbb{P}^2, \mathcal{O}(2))$ for any fiber $F'$ of it.

In the end we check these cases in detail, and we obtain the result.

Next we consider the case where dim $X \geq 3$. In particular, we mainly consider the case in which Bs $|L| = \emptyset$. Then we get the following results; Let $(X, L)$ be a polarized manifold such that Bs $|L| = \emptyset$.

(f-0) If $g(L) = q(X)$, then $\Delta(L) = 0$ or $(X, L)$ is a scroll over a smooth curve.
(f-1) If $g(L) = q(X) + 1$, then $(X, L)$ is a Del Pezzo manifold.
By using the method of Main Theorem 1, we get a classification of polarized manifolds $(X, L)$ with $n = \dim X \geq 3$, $Bs|L| = \emptyset$, and $g(L) = q(X) + 2$.

**Main Theorem 2.** ([Fk9]) Let $(X, L)$ be a polarized manifold with $\dim X = n \geq 3$. Assume that $Bs|L| = \emptyset$ and $g(L) = q(X) + 2$. Then $(X, L)$ is one of the following type:

(1) $X$ is a double covering of $\mathbb{P}^n$ with branch locus being a smooth hypersurface of degree 6, and $L$ is the pull back of $O_{\mathbb{P}^n}(1)$,

(2) $(X, L)$ is a scroll over a smooth surface $Y$. Let $\mathcal{E}$ be a locally free sheaf of rank two on $Y$ such that $(X, L) \cong (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$. Then $(Y, \mathcal{E})$ is either

2-1) $Y \cong \mathbb{P}_\alpha \times \mathbb{P}_\beta$ and $\mathcal{E} \cong [H_\alpha + 2H_\beta] \oplus [H_\alpha + H_\beta]$, where $H_\alpha$ (resp. $H_\beta$) is the ample generator of $\text{Pic}(\mathbb{P}_\alpha)$ (resp. $\text{Pic}(\mathbb{P}_\beta)$).

2-2) $Y$ is the blowing up of $\mathbb{P}^2$ at a point and $\mathcal{E} \cong [2H - E]^\oplus 2$, where $H$ is the pull back of $O_{\mathbb{P}^2}(1)$ and $E$ is the exceptional divisor,

2-3) $Y \cong \mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is a rank two vector bundle over an elliptic curve $C$ with $c_1(\mathcal{F}) = 1$ and $\mathcal{E} = H(\mathcal{F}) \otimes p^*(\mathcal{G})$, where $p : Y \to C$ is the bundle projection and $\mathcal{G}$ is any rank two vector bundle on $C$ defined by a non splitting exact sequence

$$0 \to O_C \to \mathcal{G} \to O_C(x) \to 0,$$

where $x \in C$.

(3) There is a fibration $f : X \to C$ over a smooth curve $C$ with $g(C) \leq 1$ such that every fiber $F$ of $f$ is a hyperquadric in $\mathbb{P}^n$ and $L_F = O(1)$. Then $\mathcal{E} := f_*(O(L))$ is a locally free sheaf of rank $n + 1$ on $C$, $X \in [2H(\mathcal{E}) + \pi^*(B)]$ on $\mathbb{P}(\mathcal{E})$ for some line bundle $B$ on $C$, and $L = H(\mathcal{E})|_X$, where $\pi$ is the projection $\mathbb{P}(\mathcal{E}) \to C$, and $H(\mathcal{E})$ is the tautological line bundle on $\mathbb{P}(\mathcal{E})$. We put $d = L^n$, $e = c_1(\mathcal{E})$, and $b = \deg B$.

3-1) If $g(C) = 1$, then we have $n = 3$, $d = 6$, $e = 4$, $b = -2$, and $\mathcal{E}$ is ample.
If \( g(C) = 0 \), then we have \( 3 \leq d \leq 9 \), \( e = d - 3 \), \( b = 6 - d \), and their lists are table 2 in [FI].

In the end, we propose a problem which is induced from Main Theorem 1.

**Problem 2.** Classify \( n \)-dimensional polarized manifolds with \( g(L) = q(X) + m \) and \( h^0(L) \geq n + m \).

If \( \mathrm{Bs} |L| = \emptyset \), \( n \geq 3 \), and \( m \geq 0 \), then we can get a classification of these polarized manifolds. We will report this in a future paper.

**REFERENCES**


[Fk1] Fukuma, Y., *On polarized surfaces \((X, L)\) with \( h^0(L) > 0 \), \( k(X) = 2 \), and \( g(L) = q(X) \)*, Trans. Amer. Math. Soc. **348** (1996), 4185–4197.


[Fk4] Fukuma, Y., *On polarized 3-folds \((X, L)\) with \( g(L) = q(X) + 1 \) and \( h^0(L) \geq 4 \)*, Ark. Mat. **35** (1997), 299–311.

[Fk5] Fukuma, Y., *On polarized surfaces \((X, L)\) with \( h^0(L) > 0 \), \( k(X) \geq 0 \) and \( g(L) = q(X) + 1 \)*, Geom. Dedicata **69** (1998), 189–206.


