<table>
<thead>
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<th>Title</th>
<th>Effective base point freeness on normal surfaces (Free resolution of defining ideals of projective varieties)</th>
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</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kawachi, Takeshi</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1999), 1078: 86-92</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1999-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62666">http://hdl.handle.net/2433/62666</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Effective base point freeness on normal surfaces

1. Introduction

Let $M$ be a divisor on a normal variety $Y$. Our main aim is to get criteria which provide the base point freeness of the adjoint linear system $|K_Y + [M]|$ where $[M]$ is the round-up of $M$. For smooth manifolds, there are many good results in higher dimension. On the other hand, since singularity has much information, we would conclude the same result by a weaker condition. It is true in the two dimensional case, we introduce that worse singularity causes better base point freeness.

2. The invariant

Let $Y$ be a projective normal two dimensional variety over $\mathbb{C}$ (we will call "normal surface" for short), and $y$ be a fixed point on $Y$. Let $f : X \to Y$ be the blowing up at $y$ if $y$ is a smooth point, or the minimal resolution of $y$ if $y$ is singular.

Definition 1. (MRLT) Let $Y$, $y$ and $f$ be as above. Let $B$ be an effective $\mathbb{Q}$-divisor on $Y$. $(Y, B)$ is called minimal resolitional log terminal (MRLT) at $y$ if the following conditions are satisfied:
(1) the round-down $\lfloor B \rfloor = 0$,

(2) if we write $K_X + f^{-1}B = f^*(K_Y + B) - \Delta_B$ and $\Delta_B = \sum e_i E_i$ then all $e_i < 1$,

where $f^{-1}B$ means the strict transformation of $B$ by $f$. □

**Definition 2.** Let $Z$ be the fundamental cycle of $y$. We define $\delta_{B,y} = -(Z - \Delta_B)^2$. □

We set $\Delta = \Delta_0$, which is the case of $B = 0$; and also $\delta_y = \Delta_{0,y}$. Since $B$ is effective, we have $\Delta_B > \Delta$ and then $0 \leq \delta_{B,y} \leq \delta_y$ (cf. [F]). We have the following bound of $\delta_y$.

**Proposition 1.** [KM, Theorem 1]

(1) $\delta_y = 4$ if $y$ is a smooth point, and $\delta_y = 2$ if $y$ is a rational double point.

(2) $0 < \delta_y < 2$ if $Y$ is Kawamata log terminal at $y$.

Note that if $(Y, B)$ is MSLT at $y$ then $Y$ is Kawamata log terminal at $y$. Hence $\delta_{B,y}$ is also bounded if $(Y, B)$ is MRLT. Now we will take the above invariant a little bit smaller.

**Definition 3.**

$$\delta_{\min} = \min \{- (Z - \Delta_B + x)^2 \mid x \text{ is an effective } f-\text{exceptional divisor.}\}$$

$$\delta = \begin{cases} 
\delta_{\min}, & (Y, B) \text{ is an MRLT at } y \\
0, & \text{otherwise}
\end{cases}$$

$$\delta' = \begin{cases} 
1 - \max\{e_1, e_n\}, & y \text{ is of type } A_n, \\
\text{any positive number}, & y \text{ is of type } D_n, \\
0, & \text{otherwise}. \quad \square
\end{cases}$$

Note that if $y$ is of type $A_n$, the indices are taken in the standard way.

```
  O---O---O  ---  O---O
   E_1  E_2  E_3  ---  E_{n-1}  E_n
```
3. The main result

**Theorem 2.** Let $M$ be a nef and big Q-Weil divisor on $Y$, and $B = [M] - M$. Assume that $K_Y + [M]$ is Cartier. If $M^2 > \delta$ and $M \cdot C \geq \delta'$ for any curve $C$ on $Y$ passing through $y$, then $y$ is not a base point of $|K_Y + [M]|$.

Note that if $y$ is of type $D_n$ then the assumption $M \cdot C \geq \delta'$ is equivalent to assume $M \cdot C > 0$ by the definition of $\delta'$.

**Proof.** If $y$ is not an MRLT, the proof is well known. (cf. [KM, (2.1)]). So we assume that $y$ is an MRLT point.

Since the assertion is local, we may assume $Y - \{y\}$ is smooth.

First we take a good effective Q-divisor $D$ such that Q-linearly equivalent to $M$.

**Lemma 3.** There exists an effective Q-divisor $D$ on $Y$ such that $D \equiv M$ (numerically equivalent) and $f^*D > Z - \Delta_B + x$ where $x$ attains the minimum $\delta_{\text{min}}$.

**Proof.** Since $M^2 > \delta_{\text{min}}$, we have $(f^*M - (Z - \Delta_B + x))^2 > 0$ and $f^*M \cdot (f^*M - (Z - \Delta_B + x)) > 0$. Hence $f^*M - (Z - \Delta_B + x)$ is big, we can get an effective Q-divisor Q-linearly equivalent to $f^*M - (Z - \Delta_B + x)$. □

Let $D$ be an Q-divisor satisfying the above lemma. We set $D = \sum d_iC_i$, $B = \sum b_iC_i$, $D_i = f^{-1}C_i$, $f^*D = \sum d_iD_i + \sum d'_jE_j$, $f^*B = \sum b_iD_i + \sum b'_jE_j$. We choose the rational number $c$ as the following.

$$c = \min \left\{ \frac{1 - b_i}{d_i}, \frac{1 - e_j}{d'_j} \mid d_i > 0, D_i \cap f^{-1}(y) \neq \emptyset \text{ and } f(E_j) = \{y\} \right\}.$$ 

Since $(Y, B)$ is MRLT and the choice of $D$, we have $0 < c < 1.$
Let $R = f^*M - cf^*D$. Since $0 < c < 1$ and $D \equiv M$ is nef and big, $R$ is also nef and big. By a simple calculation, we have

$$[R] = f^*(K_Y + [M]) - K_X - [cf^*D + f^*B + \Delta] = R + \{cf^*D + f^*B + \Delta\},$$

where $\{\cdot\}$ means the fractional part. Hence we have

$$K_X + [R] = f^*(K_Y + M) - \sum [cd_i + b_i] D_i + \sum [cd'_j + e_j] E_j.$$  

We write $\sum [cd_i + b_i] D_i = A + N$ where all components of $A$ meet with $f^{-1}(y)$ and $N$ is disjoint from $f^{-1}(y)$. Let $E = \sum [cd'_j + e_j] E_j$. By the choice of $c$, both $A$ and $E$ are reduced or only one of them is zero. Let $A = D_1 + \cdots + D_t$.

**Lemma 4.** If $A \neq 0$ then $(Y, f_*A)$ is log canonical at $y$ and the dual graph is one of the followings.

1. ![](image1)

2. ![](image2)

3. ![](image3)

In the above lemma, we denote prime components of $E$ and $f_*A$ by $\bigcirc$ and $\bullet$ respectively. Note that only the case (1) is log terminal.

**Proof.** Because of $f^*(K_Y + f_*A) - K_X - A \leq E$, $(Y, f_*A)$ is log canonical at $y$. These are classified as in [A] and [K], they are only above 3 cases. \(\square\)

We divide the proof of the main theorem in two cases according to $E$.

Case 1: $E \neq 0$.

If $t > 0$ then $y$ is of type $A_n$ or $D_n$. Note that if $y$ is of type $E_n$ then $A$ must be 0.
Since $R$ is nef and big, each $D_i$ is integral in $R$ and $R \cdot D_i \geq \delta' > 0$, we have the following vanishing due to Kawamata-Viehweg.

$$H^1(X, K_X + [R] + A) = H^1(X, f^*(K_Y + [M]) - N - E) = 0.$$ 

Hence the morphism

$$H^0(X, f^*(K_Y + [M]) - N) \to H^0(E, (f^*(K_Y + [M]) - N)|_E)$$

is surjective.

Case 2: $E = 0$.

In this case, $(Y, f_* A)$ is log terminal of type $A_n$ at $y$ and $t = 1$. So we let $A = D_1$.

Hence the morphism

$$H^0(X, f^*(K_Y + [M]) - N) \to H^0(D_1, (f^*(K_Y + [M]) - N)|_{D_1})$$

is surjective. Since $(f^*(K_Y + [M]) - N)|_{D_1} = K_{D_1} + [R]|_{D_1}$, if $[R] \cdot D_1 > 1$ then there exists a section in $H^0(D_1, K_{D_1} + [R]|_{D_1})$ which does not vanish at $D_1 \cap f^{-1}(y)$ by [H]. Hence it is enough to show $[R] \cdot D_1 > 1$.

Note that $[R] \cdot D_1 \geq R \cdot D_1 + \sum(cd_j' + e_j)E_j \cdot D_1$ and $y \in \text{Supp } f_* D_1$, we have $R \cdot D_1 \geq (1 - c)\delta'$. By changing the indices we may assume $e_1 \leq e_n$. Hence $\delta' = 1 - e_n$.

If $D_1$ meets $E_n$ then the inequalities $f^* D > Z - \Delta_B$ and

$$[R] \cdot D_1 \geq (1 - c)(1 - e_n) + cd_n' + e_n = 1 + c(d_n' + e_n - 1)$$

imply $[R] \cdot D_1 > 1$.

So we assume that $D_1$ meets $E_1$.

Let $A = A(w_1, \ldots, w_n) = (-E_i \cdot E_j)_{ij}$ be the intersection matrix of the exceptional divisors of type $A_n$. Let $a(w_1, \ldots, w_n) = \det A(w_1, \ldots, w_n)$ be the determinant. We set
$a() = 1$ for convenience. Let $L_i$ be an irreducible curve on $Y$ such that $f^{-1}L_i \cdot E_i = 1$ and $f^{-1}L_i \cdot E_j = 0$ for all $j \neq i$. We set $f^*L_i = f^{-1}L_i + \sum c_{ij}E_j$.

By simple calculation of matrices, we have the following proposition.

**Proposition 5.** Let $\Delta = \sum a_j E_j$.

$$1 - a_i = \frac{a(w_1, \ldots, w_{i-1}) + a(w_{i+1}, \ldots, w_n)}{a(w_1, \ldots, w_n)},$$

$$c_{ij} = \frac{a(w_1, \ldots, w_{i-1})a(w_{j+1}, \ldots, w_n)}{a(w_1, \ldots, w_n)}, \text{ if } i \leq j, \quad c_{ij} = c_{ji}.$$  

Let $f^*C_1 = D_1 + \sum c_j E_j$. Let $y_{D,j} = d_j' - d_1 c_j$, the coefficients of $E_j$ arising from $D_i$'s except $D_1$. We also let $y_{B,j} = b_j' - b_1 c_j$ and $y_j = cy_{D,j} + y_{B,j}$. Since the minimality of $c$, we have $cd_1 + b_1 = 1$. Hence we have $cd_1' + b_1' = c_1 + y_1$. Therefore we have

$$[R] \cdot D_1 \geq (1 - c) \delta' + cd_1' + e_1 = (1 - c)(1 - e_n) + a_1 + c_1 + y_1.$$  

By Proposition 5, we have $a_1 + c_1 = 1/\alpha$, where $\alpha = \det A(w_1, \ldots, w_n)$. Since $E = 0$, we also have $y_1 \leq 1/\alpha$.

**Claim 6.**

$$(1 - c)(1 - e_n) > \frac{a(w_1, \ldots, w_{n-1})}{\alpha} \quad \text{and} \quad y_n \leq a(w_1, \ldots, w_{n-1})y_1.$$  

By this claim, we have $[R] \cdot D_1 > 1 + (a(w_1, \ldots, w_{n-1}) - 1)(1/\alpha - y_1)$. Since $a(w_1, \ldots, w_{n-1}) \geq 1$ and $y_1 < 1/\alpha$, we have $[R] \cdot D_1 > 1$.

**Proof of Claim 6.** By the choice of $D$, we have $d_n' > 1 - a_n - b_n'$. Hence

$$(d_n' - 1 + a_n + b_n') \frac{c}{1 - a_n} > 0 = \frac{cd_1 + b_1 - 1}{1 + a(w_1, \ldots, w_{n-1})},$$

since $cd_1 + b_1 = 1$. We set $\alpha' = a(w_1, \ldots, w_{n-1})$ for convenience. Then we have

$$\left( (d_n' - 1 + a_n + b_n') \frac{1}{1 - a_n} - \frac{d_1}{1 + \alpha'} \right) c > \frac{b_1 - 1}{1 + \alpha'}.$$  

Since $(1 - a_n)\alpha = 1 + \alpha'$ and $d_n' = d_1/\alpha + y_{D,n}$, the left-hand-side equals to
\[
\left( \frac{d'_n}{1-a_n} - 1 + \frac{b'_n}{1-a_n} - \frac{d_1}{1+\alpha'} \right) c = \left( \frac{y_{D,n}}{1-a_n} + \frac{b'_n}{1-a_n} - 1 \right) c.
\]

On the other hand, the right-hand-side equals to

\[
\frac{b_1 - 1}{1+\alpha'} = \frac{b_1 + \alpha y_{B,n}}{1+\alpha'} - \frac{1 + \alpha y_{B,n}}{1+\alpha'} = \frac{b'_n}{1-a_n} - 1 + \frac{\alpha' - \alpha y_{B,n}}{1+\alpha'}.
\]

Thus we have

\[
(1-c) \left( 1 - \frac{b'_n}{1-a_n} \right) > \frac{\alpha' - \alpha y_{B,n} - c y_{D,n}}{1-a_n}.
\]

The second assertion follows from Proposition 5 and the inequalities \(c_{11} > c_{12} > \cdots > c_{1n}\) and \(c_{n1} < c_{n2} < \cdots < c_{nn}\). □

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