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A note on Hilbert-Kunz multiplicity

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1 Introduction

This is a joint work with Prof. Kei-ichi Watanabe in Nihon University; see [WY].

Throughout this talk, let $(A, m, k)$ be a Noetherian local ring of characteristic $p > 0$. Put $d := \dim A \geq 1$. Let $\hat{A}$ denote the $m$-adic completion of $A$, and let $\text{Ass}(A)$ (resp. $\text{Min}(A)$) denote the associated prime ideals (resp. minimal prime ideals) of $A$. Moreover, unless specified, let $I$ denote an $m$-primary ideal of $A$ and $M$ a finite $A$-module.

First, we recall the notion of Hilbert-Kunz multiplicity which was defined by Kunz [Kul]; see also Monsky [Mo], Huneke [Hu].

**Definition 1.1** The Hilbert-Kunz multiplicity $e_{HK}(I, M)$ of $M$ with respect to $I$ is defined as follows:

$$e_{HK}(I, M) := \lim_{q \to \infty} \frac{\lambda_A(M/I^qM)}{q^d},$$

where $q = p^e$ and $I^q = (a^q | a \in I)A$. For simplicity, we put $e_{HK}(I) := e_{HK}(I, A)$ and $e_{HK}(A) := e_{HK}(m)$.

The following question is fundamental but still open.

**Question 1.2** Is $e_{HK}(I)$ always a rational number?

**Known Results.**

(1.3.1) Let $e(I)$ be the multiplicity of $A$ with respect to $I$. Then we have the following inequalities:

$$\frac{e(I)}{d!} \leq e_{HK}(I) \leq e(I).$$

(1.3.2) $e_{HK}(I) \geq e_{HK}(A) \geq 1$.

(1.3.3) Put $\text{Assh}(A) = \{P \in \text{Spec}(A) | \dim A/P = d\}$. Then

$$e_{HK}(I, M) = \sum_{P \in \text{Assh}(A)} e_{HK}(I, A/P) \cdot l_{AP}(M_P).$$

For example, if $A$ is a local domain and $B$ is a torsion free $A$-module of rank $r$, then $e_{HK}(I, B) = r \cdot e_{HK}(A)$. 
(1.3.4) (Kunz [Ku2]) For any prime ideal $P \in \text{Spec}(A)$ such that height $P + \dim A/P = \dim A$, we have

$$e_{HK}(A_P) \leq e_{HK}(A).$$

(1.3.5) If $A$ is a regular local ring, then $e_{HK}(I) = \lambda_A(A/I)$.

(1.3.6) If $I$ is a parameter ideal, then $e_{HK}(I) = e(I)$.

(1.3.7) We recall the notion of tight closure. An element $x \in A$ is said to be in the tight closure $I^*$ of $I$ if there exists an element $c \in A^0$ such that for all large $q = p^e$, $cx^q \in I^{[q]}$, where $A^0 := A \setminus \cup\{P | P \in \text{Min}(A)\}$.

Let $I, J$ be $m$-primary ideals such that $I \subseteq J$. Then if $I^* = J^*$, then $e_{HK}(I) = e_{HK}(J)$. Furthermore, if, in addition, $\hat{A}$ is equidimensional and reduced, then the converse is also true.

(1.3.8) ([WY] or [BCP]) Let $(A, m) \subseteq (B, n)$ be a module-finite extension of local domains. Then

$$e_{HK}(I, A) = \frac{[B:n:A/m]}{[Q(B):Q(A)]} \cdot e_{HK}(IB, B),$$

where $Q(A)$ denotes the fraction field of $A$.

**Question 1.4** If $\text{pd}_A A/I < \infty$, then does the same formula as that in (1.3.5) hold?

**Background and Questions.**

In general, there is an example such that $e_{HK}(I) = e(I)$; for instance, let $q$ be a minimal reduction of $m$. If $q^* = m$, then we have $e_{HK}(m) = e_{HK}(q) = e(q) = e(m)$. However, we have no example such that $\frac{e(I)}{d!} = e_{HK}(I)$. On the other hand, if $A = k[[X_1, \ldots, X_d]](r)$, then

$$e_{HK}(A) = \frac{1}{r} \left( \begin{array}{c} d + r - 1 \\ r - 1 \end{array} \right) \text{ and } e(A) = r^{d-1}.$$ 

Thus if we tend $r$ to $\infty$, then the limit $\frac{e_{HK}(A)}{e(A)}$ tends to $\frac{1}{d!}$. So we consider the following question.

**Question 1.5** Is there a constant number $\alpha > 0$ depending on $d = \dim A$ alone such that

$$e_{HK}(I) \geq \frac{e(I)}{d!} + \alpha?$$

On the other hand, in [WY], we proved the following theorem.

**Theorem 1.6** [WY, Theorem (1.5)] If $A$ is an unmixed (i.e. $\text{Ass}(\hat{A}) = \text{Assh}(\hat{A})$) local ring with $e_{HK}(A) = 1$, then it is regular.
In the above theorem, we cannot remove the assumption that $A$ is "unmixed". For instance, if $e(A) = 1$, then $e_{HK}(A) = 1$. We now consider the case of Cohen-Macaulay local rings. Then the following question is a natural extension of the above theorem.

**Question 1.7** If $A$ is a Cohen-Macaulay local ring with $e_{HK}(A) < 2$, then is it F-regular?

The following conjecture is related to the above questions.

**Conjecture 1.8** Let $A$ be a quasi-unmixed (i.e. Min($\hat{A}$) = Assh($\hat{A}$)) local ring. Then $e_{HK}(I) \geq \lambda(A/I^s)$ for any $m$-primary ideal $I$.

Further, if $A$ is a Cohen-Macaulay local ring then $e_{HK}(I) \geq \lambda(A/I)$ for any $m$-primary ideal $I$.

## 2 A positive answer to Question 1

Throughout this section, let $A$ be a Noetherian local ring with $\dim A = 2$ and suppose that $k = A/m$ is infinite. The following theorem is a main result in this section.

**Theorem 2.1** (cf. [WY, Section 5]) Suppose $\dim A = 2$. Then for any $m$-primary ideal $I$, we have

$$e_{HK}(I) \geq \frac{e(I) + 1}{2} \left( > \frac{e(I)}{2} \right).$$

First, we consider the case of Cohen-Macaulay local rings. Now suppose that $A$ is Cohen-Macaulay. Let $I$ be an $m$-primary ideal and $J$ its minimal reduction, that is, $J = (a, b)$ is a parameter ideal of $A$ and $I^{n+1} = JI^n$ for some $n \geq 1$.

**Lemma 2.2** Suppose that $A$ is Cohen-Macaulay, $1 \leq s < 2$ and $q = p^s$. We define $I^x = I^{\lfloor x \rfloor}$ for any positive real number $x$. Then we have

1. $\lambda_A(A/I^{(s-1)q}) = \frac{e(I)}{2} (s - 1)^2 q^2 + o(q^2)$, where $f(q) = o(q^2)$ means $\lim_{q \to \infty} \frac{f(q)}{q^2} = 0$.
2. $\lambda_A \left( \frac{I^q + J^q}{J^q} \right) = \frac{e(I)}{2} (2 - s)^2 q^2 + o(q^2)$.

**Proof.** Put $n = [(s - 1)q]$ and $\epsilon = (s - 1)q - n$.

1. $\lambda_A(A/I^{(s-1)q}) = \lambda_A(A/I^n) = \frac{e(I)}{2} n^2 + f(n)$, where $\lim_{n \to \infty} \frac{f(n)}{n^2} = 0$.

Thus we get

$$\lambda_A(A/I^{(s-1)q}) = \frac{e(I)}{2} (s - 1)^2 q^2 + o(q^2) = \frac{e(I)}{2} (s - 1)^2 q^2 + o(q^2).$$

2. $\lambda_A \left( \frac{I^q + J^q}{J^q} \right) \leq \lambda_A \left( \frac{J^q + J^q}{J^q} \right) + \lambda_A \left( \frac{I^q}{J^q} \right).$
First, we estimate the second term. Since $e(I) = e(J)$, we have

$$\lambda_A(I^{sq}/J^{sq}) = \lambda_A(A/J^{sq}) - \lambda_A(A/I^{sq}) = o(q^2).$$

Next, we estimate the first term.

$$\lambda_A\left(\frac{J^{sq} + J^{[q]}}{J^{[q]}}\right) \leq \sum_{l=n}^{2q} \{(x, y) \in \mathbb{Z}^2 | 0 \leq x, y \leq q-1, x+y = l\} \times \lambda_A(A/J) + o(q^2)$$

$$= \frac{1}{2} (2q - sq)^2 \cdot e(I) + o(q^2). \quad \text{Q.E.D.}$$

**Lemma 2.3** Suppose that $A$ is Cohen-Macaulay. Let $I$ be an $m$-primary ideal of $A$ and $J$ a minimal reduction of $I$. If $I/J$ is generated by $r$ elements (i.e. $r \geq \mu_A(I) - 2$), then we have

$$\lambda_A(I^{[q]}/J^{[q]}) \leq \frac{r}{2(r+1)} e(I) \cdot q^2 + o(q^2).$$

Moreover, if $J^* \subseteq I$ and $I/J^*$ is generated by $r$ elements, the same result holds.

**Proof.** Let $s$ be any real number such that $1 \leq s < 2$. Then

$$\lambda_A\left(\frac{I^{[q]}}{J^{[q]}}\right) \geq \lambda_A\left(\frac{I^{[q]} + J^{[q]}}{J^{[q]}}\right) + \lambda_A\left(\frac{J^{[q]} + I^{[q]}}{J^{[q]}}\right) =: (E1) + (E2).$$

Since we can write as $I = Au_1 + \cdots Au_r + J$, we get

$$(E1) \leq \sum_{i=1}^{r} \lambda_A\left(\frac{u_i^q A + J^{[q]} + I^{[q]}}{J^{[q]}}\right) = \sum_{i=1}^{r} \lambda_A\left(\frac{A}{(J^{[q]} + I^{[q]}) : u_i^q}\right)$$

$$\leq r \cdot \lambda_A\left(\frac{A}{I^{(s-1)q}}\right) = r \cdot \frac{e(I)}{2} (s-1)^2 q^2 + o(q^2) \quad \text{by (2.2).}$$

On the other hand, by (2.2) again, $(E2) = \frac{e(I)}{2} (2-s)^2 q^2 + o(q^2)$. Thus

$$\lambda_A\left(\frac{J^{[q]}}{J^{[q]}}\right) \leq \frac{e(I)}{2} q^2 \left\{(r+1)s^2 - 2(r+2)s + (r+4)\right\} + o(q^2).$$

Put $s = \frac{r+2}{r+1}$, and we get the required inequality.

Further, the last statement follows from the fact $\lambda_A(A/J^{[q]}) = \lambda_A(A/(J^*)^{[q]}) + o(q^2)$.

**Q.E.D.**

Next proposition easily follows from the above lemma.

**Proposition 2.4** Suppose that $A$ is Cohen-Macaulay. Let $I$ be an $m$-primary ideal of $A$ and $J$ a minimal reduction of $I$. If $I/J$ is generated by $r$ elements then we have

$$e_{HK}(I) \geq \frac{r+2}{2(r+1)} \cdot e(I).$$

Moreover, if $J^* \subseteq I$ and $I/J^*$ is generated by $r$ elements (i.e. $r \geq \mu_A(I/J^*) = \lambda_A(I/J^* + Im)$), the same result holds.
We now give a proof of Theorem (2.1). First, we suppose that $A$ is Cohen-Macaulay and let $J$ be a minimal reduction of $m$. Since

$$e(I) - 1 = \lambda_A(m/J) = \lambda_A(I/J) + \lambda_A(m/I) \geq \lambda_A(I/J + I)m + \lambda_A(m/I),$$

we have $e(I) - 1 \geq \lambda_A(m/I) \geq \mu_A(I/J)$. By virtue of Proposition (2.4), we get

$$e_{HK}(I) \geq \frac{r + 2}{2(r + 1)} \cdot e(I) \geq \frac{e(I) + 1}{2},$$

where $r = e(I) - 1 - \lambda_A(m/I)$.

We remark that Equality $e_{HK}(I) = (e(I) + 1)/2$ implies $I = m$.

Next, we consider about general local rings. Since $e_{HK}(I) = e_{HK}(I\hat{A})$ and $e(I) = e(I\hat{A})$, we may assume that $A$ is complete. Moreover, since

$$e(I) = \sum_{P \in Assh(A)} e(I, A/P) \cdot \lambda_A(A_P),$$

we may assume that $A$ is a complete local domain. Let $B$ be the integral closure of $A$ in its fraction field. Then $B$ is a complete normal local domain and a finite $A$-module; thus it is a two-dimensional Cohen-Macaulay local ring. Let $n$ be an unique maximal ideal of $B$ and put $t = [B/n : A/m]$. Then we have

$$e_{HK}(I) = t \cdot e_{HK}(IB, B), \quad e(I) = t \cdot e_{HK}(IB, B).$$

Thus by the argument in the Cohen-Macaulay case, we get

$$e_{HK}(I) = t \cdot e_{HK}(IB, B) \geq t \cdot \frac{e_{HK}(IB, B) + 1}{2} \geq \frac{e_{HK}(I) + 1}{2}.$$ 

**Corollary 2.5** If $A$ is a non-Cohen-Macaulay, unmixed local ring (with $\dim A = 2$), then

$$e_{HK}(I, A) > \frac{e(I) + 1}{2}$$

for any $m$-primary ideal $I$ of $A$.

**Proof.** By the above proof, we may assume that $A$ is a complete local domain. With the same notation as in the proof of Theorem, $B$ is a torsion free $A$-module. If $\mu_A(B) = 1$, then $B \cong A$; this contradicts the assumption that $A$ is not Cohen-Macaulay. Thus $\lambda_A(B/mB) = \mu_A(B) \geq 2$.

When $t := [B/n : A/m] = 1$, since $\lambda_B(B/mB) = \lambda_A(B/mB) \geq 2$, we have $IB \subseteq mB \subseteq n$. Hence

$$e_{HK}(I) = e_{HK}(IB, B) > \frac{e(IB) + 1}{2} = \frac{e(I) + 1}{2}.$$ 

On the other hand, when $t \geq 2$, we have

$$e_{HK}(I) \geq \frac{e(I) + t}{2} > \frac{e(I) + 1}{2}. \quad \text{Q.E.D.}$$
Corollary 2.6 Let $A$ be a local ring with $\dim A = 2$. Then

(1) When $e(A) = 1$, we have $e_{HK}(A) = 1$.

(2) When $e(A) \geq 2$, we have $e_{HK}(A) \geq \frac{3}{2}$.

3 Local rings with small Hilbert-Kunz multiplicity

In this section, we consider Question (1.7) in case of local rings with $\dim A = 2$. In order to state the main theorem, we recall the notion of $F$-regular rings. A local ring $A$ is said to be $F$-regular (resp. $F$-rational) if $I^* = I$ for every ideal (resp. parameter ideal) $I$ of $A$. We are now ready to state the main theorem, which is a slight generalization of Theorem (5.4) in [WY].

Theorem 3.1 (cf. [WY, Theorem (5.4)]) Let $A$ be an unmixed local ring with $\dim A = 2$ and suppose $k = \overline{k}$. Then

(1) $1 < e_{HK}(A) < 2$ if and only if $\hat{A}$ is an F-rational double point, that is, $\hat{A} \cong k[[X, Y, Z]]/(f)$, where $f$ is given by the list below (3.2).

(2) $e_{HK}(A) = 2$ if and only if $A$ satisfies either one of the following conditions:

(a) $A$ is not $F$-regular with $e(A) = 2$.
(b) $\hat{A} \cong k[[X^3, X^2Y, XY^2, Y^3]]$.

Corollary 3.2 Let $A$ be an unmixed local ring with $\dim A = 2$. If $e_{HK}(A) < 2$, then $\hat{A}$ is isomorphic to the completion of the ring $k[X, Y]^G$ where $G$ is a finite subgroup of $SL_2(k)$.

In particular, $A$ is a module-finite subring of $k[[X, Y]]$ and $e_{HK}(A) = 2 - \frac{1}{|G|}$.

In fact, $|G|$ is given by the following table.

| type  | $f$                      | $|G|$          | $n$     | $p$     |
|-------|--------------------------|---------------|---------|---------|
| $(A_n)$ | $f = xy + z^{n+1}$       | $n + 1$       | $n \geq 1$ |
| $(D_n)$ | $f = x^2 + yz^2 + y^{n-1}$ | $4(n - 2)$   | $n \geq 4$, $p \geq 3$ |
| $(E_6)$ | $f = x^2 + y^3 + z^4$    | $24$          | $p \geq 3$ |
| $(E_7)$ | $f = x^2 + y^3 + yz^3$   | $48$          | $p \geq 5$ |
| $(E_8)$ | $f = x^2 + y^3 + z^5$    | $120$         | $p \geq 7$ |

From now on, let $A$ be an unmixed local ring with $\dim A = 2$. In order to prove the above theorem, we give several lemmas.

Lemma 3.3 If $1 < e_{HK}(A) < 2$, then $\hat{A}$ is an integral domain with $e(\hat{A}) = 2$ and $\hat{A}_P$ is regular for any prime ideal $P \neq m\hat{A}$.
Proof. We may assume that $A$ is complete. First, we observe that $e(A) = 2$. Actually, it follows from Theorem (2.1).

Next, we show that $A$ is a local domain with isolated singularity. For any prime ideal $P \neq \mathfrak{m}$, we have $e_{HK}(A_P) \leq e_{HK}(A) < 2$. Since $e_{HK}(A_P)$ must be a positive integer, we have $e_{HK}(A_P) = 1$. Hence $A_P$ is regular.

On the other hand, $\# \text{Ass}(A) = \# \text{Assh}(A) = 1$. Actually, if $\# \text{Assh}(A) \geq 2$,

$$2 > e_{HK}(A) = \sum_{P \in \text{Assh}(A)} e_{HK}(A_P) \cdot \lambda_{A_P}(A_P) \geq \# \text{Assh}(A) \geq 2,$$

gives a contradiction. Hence $\# \text{Ass}(A) = 1$. Therefore $A$ is a local domain. \quad Q.E.D.

Corollary 3.4 Let $A$ be a Cohen-Macaulay local ring with $e(A) = 2$ and suppose that $\hat{A}$ is reduced. Then

(1) If $A$ is $F$-regular, then $e_{HK}(A) < 2$.

(2) If $A$ is not $F$-regular, then $e_{HK}(A) = 2$.

Proof. Let $\mathfrak{q}$ be a minimal reduction of $\mathfrak{m}$. Since $A$ is Cohen-Macaulay, we have $\lambda_{A}(A/\mathfrak{q}) = e(A) = 2$; thus $\mathfrak{q}^* = \mathfrak{q}$ or $\mathfrak{q}^* = \mathfrak{m}$, because $\mathfrak{q} \subseteq \mathfrak{q}^* \subseteq \mathfrak{m}$.

When $\mathfrak{q}^* = \mathfrak{q}$, since $A$ is Gorenstein, $A$ must be $F$-regular. Moreover, since $\mathfrak{m} \neq \mathfrak{q}^*$ and $\hat{A}$ is reduced, we get

$$e_{HK}(A) := e_{HK}(\mathfrak{m}) < e_{HK}(\mathfrak{q}^*) = e_{HK}(\mathfrak{q}) = e(\mathfrak{q}) = 2.$$ 

On the other hand, when $\mathfrak{q}^* = \mathfrak{m}$, $A$ is not $F$-regular and $e_{HK}(A) = e_{HK}(\mathfrak{q}) = 2$. \quad Q.E.D.

We now give an outline of the proof of Theorem (3.1). Let $A$ be an unmixed local ring with $\dim A = 2$ and suppose $k = \overline{k}$.

Step 1. When $A$ is a complete Cohen-Macaulay local ring with $e_{HK}(A) < 2$, it is an $F$-rational double point.

Proof. In fact, by Lemma (3.3), $A$ is a complete local domain with $e(A) = 2$. Thus Corollary (3.4) implies that $A$ is $F$-regular. Then $A$ is given by the list in Corollary (3.2).

Step 2. If $A$ is unmixed local ring with $e_{HK}(A) < 2$, then $\hat{A}$ is $F$-regular.

Proof. We may assume that $A$ is complete. By Lemma (3.3), $A$ is a complete local domain with $e(A) = 2$. Let $B$ the integral closure of $A$ in its fraction field. Then $\lambda_{A}(B/A) < \infty$ and $B$ is a local domain and is a module-finite extension of $A$. Let $\mathfrak{n}$ be an unique maximal ideal of $B$. In order to show that $A$ is $F$-regular it is enough to show $A = B$, for $B$ is Cohen-Macaulay. As $A/\mathfrak{m} \cong B/\mathfrak{n}$, we get

$$2 > e_{HK}(A) = e_{HK}(\mathfrak{m}, B) \geq e_{HK}(\mathfrak{n}, B) =: e_{HK}(B).$$

According to Step 1, $B$ is $F$-regular with $e_{HK}(B) = 2 - \frac{1}{|G|}$ and is a module-finite subring of $C = k[[X, Y]]$ such that $|G| = [Q(C) : Q(B)]$. \quad |G|
Now suppose $A \neq B$. Then $H^1_m(A) \cong B/A \neq 0$ and thus $A$ is not Cohen-Macaulay. Further, as $\mu_A(B) \geq 2$, we have $m.B \subseteq n$. Moreover, since both $B$ and $C$ are F-regular rings, we obtain that $I.C \cap B = I$ for any ideal $I$ of $B$. In particular, we have $m.C \subseteq n$. Hence we get
\[
e_{HK}(A) - e_{HK}(B) = \frac{1}{|G|} \lambda_A(C/m.C) - \frac{1}{|G|} \lambda_A(C/n.C)
= \frac{1}{|G|} \lambda_A(n.C/m.C) \geq \frac{1}{|G|}.
\]

Thus\[
e_{HK}(A) \geq e_{HK}(B) + \frac{1}{|G|} \geq \frac{e(A) + 1}{2}.
\]

Thus we conclude that $A = B$ as required. 

**Step 3.** Let $A$ be a complete Cohen-Macaulay local ring. Then $e_{HK}(A) = 2$ if and only if $A$ is not F-regular with $e(A) = 2$ or $A \cong k[[X^3, X^2Y, XY^2, Y^3]]$.

**Proof.** If part is easy. But only if part is hard. See [WY, Section5] for details. 

**Step 4.** Suppose that $A$ is unmixed but not Cohen-Macaulay. Then $e_{HK}(A) = 2$ if and only if $e(A) = 2$.

**Proof.** If part: If $e(A) = 2$, then $e_{HK}(A) \leq 2$. If $e_{HK}(A) < 2$, then $A$ is Cohen-Macaulay by Step 2. However, this contradicts the assumption. Hence $e_{HK}(A) = 2$.

Only if part follows from Corollary (2.5). 

**Q.E.D.**

In the final of this section, we give the following problem.

**Problem 3.5** Let $A$ be an unmixed local ring with dim $A = 2$. Characterize the ring $A$ which satisfies $e_{HK}(A) = \frac{e(A) + 1}{2}$.

In fact, if $A = k[[X, Y]](e)$ then $e(A) = e$ and $e_{HK}(A) = \frac{e + 1}{2}$. Further, the proof of the above theorem implies that if $e_{HK}(A) = \frac{e(A) + 1}{2}$ and $e(A) \leq 3$ then $A \cong k[[X, Y]]^{e(A)}$. Moreover, the following proposition gives a partial answer to this problem.

**Proposition 3.6** If $A$ is an unmixed local ring with $e_{HK}(A) = \frac{e(A) + 1}{2}$, then it is F-rational.

**Proof.** By Cor (2.5), $A$ is Cohen-Macaulay. Then we show that $A$ has a minimal multiplicity, that is, $\text{emb}(A) = e(A) + \text{dim } A - 1$. Let $q$ be a minimal reduction of $m$. Then since\[
e(A) - 1 = \lambda_A(m/q) \geq \lambda_A(m/q + m^2) = \mu_A(m/q).
\]

If $e(A) - 1 > \mu_A(m/q) =: r_0$, then
\[
e_{HK}(A) \geq \frac{r_0 + 2}{2(r_0 + 1)} \cdot e(A) > \frac{e(A) + 1}{2};
\]

If $e(A) - 1 = \mu_A(m/q)$, then
\[
e_{HK}(A) = \frac{e(A) + 1}{2};
\]

If $e(A) - 1 = \mu_A(m/q) - 1$, then
\[
e_{HK}(A) = \frac{e(A) + 1}{2}. 
\]
see the proof of Theorem (2.1) for detail. Thus \( e(A) - 1 = \mu_A(m/q) \). It follows that \( m^2 \subseteq q \); thus \( A \) has a minimal multiplicity.

We will show that \( A \) is F-rational. Suppose not. Then \( q^* \neq q \). Since \( m^2 \subseteq q \subseteq q^* \), we have \( r_1 := \mu_A(m/q^*) < \mu_A(m/q) = r_0 \). Thus by virtue of (2.4), we get

\[
e_{HK}(A) \geq \frac{r_1 + 2}{2(r_1 + 1)} \cdot e(A) > \frac{r_0 + 2}{2(r_0 + 1)} \cdot e(A) = \frac{e(A) + 1}{2}.
\]

This contradicts the assumption. Hence we conclude that \( A \) is F-rational.  \( Q.E.D. \)

### 4 Extended Rees Rings.

In this section, we consider the following question.

**Question 4.1** Let \( A \) be a local ring and \( F = \{F_n\} \) a filtration of \( A \). Then does \( e_{HK}(A) \leq e_{HK}(G_F(A)) \) always hold? Further, when does equality hold?

In order to state our result, we recall the definition of Rees ring, extended Rees ring and the associated graded ring.

Let \( A \) be a local ring of \( A \) with \( d := \dim A \geq 1 \). Then \( F = \{F_n\}_{n \in \mathbb{Z}} \) is said to be a filtration of \( A \) if the following conditions are satisfied:

(a) \( F_i \) is an ideal of \( A \) such that \( F_i \supseteq F_{i+1} \) for each \( i \).

(b) \( F_i = A \) for each \( i \leq 0 \) and \( m \supseteq F_1 \).

(c) \( F_i F_j \subseteq F_{i+j} \) for each \( i, j \).

For a given filtration \( F = \{F_n\}_{n \in \mathbb{Z}} \) of \( A \), we define

\[
R := R_F(A) := \bigoplus_{n=0}^{\infty} F_n t^n.
\]

\[
S := R'_F(A) := \bigoplus_{n \in \mathbb{Z}} F_n t^n.
\]

\[
G := G_F(A) := \bigoplus_{n=0}^{\infty} F_n/F_{n+1} \cong S/t^{-1} S \cong R/R(1).
\]

\( R_F(A) \) (resp. \( R'_F(A), G_F(A) \)) is said to be the Rees (resp. the extended Rees, the associated graded) ring with respect to a filtration \( F \) of \( A \).

Then our main result in this section is the following theorem.

**Theorem 4.2** Let \( A \) be any local ring with \( d := \dim A > 0 \) and let \( F = \{F_n\}_{n \in \mathbb{Z}} \) be a filtration of \( A \). Suppose that \( R_F(A) \) is a Noetherian ring with \( \dim R_F(A) = d + 1 \). Then for any \( m \)-primary ideal \( I \) of \( A \) such that \( F_1 \subseteq I \subseteq m \), we have

(1) \( e_{HK}(I, A) \leq e_{HK}(N, S) \), where \( N = (t^{-1}, I, S_+) \).
(2) If $F_1$ is an $m$-primary ideal, then $e_{HK}(N, S) \leq e_{HK}(G)$.

In particular, if $F_1$ is an $m$-primary ideal, then

$$e_{HK}(A) \leq e_{HK}(S) \leq e_{HK}(G).$$

**Question 4.3** In the above theorem, when does equality hold? How about $e_{HK}(A) \leq e_{HK}(R_F(A))$?

**Example 4.4** Let $A = k[[X, Y]]$ and $I = (X^m, Y^n)$, where $m \geq n \geq 1$. Then

1. $e(R(I)) = n + 1$.
2. $e_{HK}(R(I)) = n + 1 - \frac{n(3m - 1)}{3m^2}$.
3. $e(R'(I)) = n + 2$ (if $n \geq 2$), $= 2$ (otherwise).
4. $e_{HK}(R'(I)) = n + 2 - \frac{n}{m} - \frac{1}{n}$.

**References**


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