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Kyoto University
Remarks on High Linear Syzygy

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In this note we explain some properties that follow from a high linear syzygy. We consider the \( r \)-th, \((r-1)\)-st, and \((r-2)\)-nd linear syzygies over a polynomial ring in \( r \) variables. The most interesting, and the only nontrivial, case is the \((r-2)\)-nd linear syzygy which produces skew-symmetric matrices that are helpful in understanding certain geometric situations.

Let \( S = K[x_1, \cdots, x_r] = \oplus_{d \geq 0} S_d \) be a polynomial ring over a field \( K \) with the usual \( \mathbb{N} \)-grading. Let \( M = \oplus_{d \geq t} M_d \) be a finitely generated graded \( S \)-module. As usual, \( M(n) \) denotes the same module \( M \) with its degrees shifted to the left by \( n \) units, i.e., \( M(n)_d := M_{d+n} \). Let \( F_\bullet \) denote the minimal graded free resolution of \( M \) over \( S \), i.e.,

\[
F_\bullet : 0 \to F_r \to F_{r-1} \to \cdots \to F_p \to \cdots \to F_0 \to 0,
\]

where \( F_p = \oplus_{q \in \mathbb{Z}} S(-q-p)^{b_{p,q}(M)} \).

The reason for the extra degree shift of \(-p\) in the \( p \)-th free module \( F_p \) is because the entries of the maps in the minimal resolution are all of positive degrees. We say that \( M \) has a \( q \)-linear \( p \)-th syzygy if the graded betti number \( b_{p,q}(M) \neq 0 \). When \( q = 0 \) we drop \( 0 \)- to call it a linear \( p \)-th syzygy. The most important result concerning the
linear syzygy is the vanishing theorem of Green ([G, Theorem 3.a.1]) which asserts that if $M$ has a linear $p$-th syzygy, then $\dim M_0 \geq p$ under certain conditions, which are satisfied in geometric situations. Some progress in finding more precise algebraic conditions affecting the linear syzygies were made in [EK1] and [EK2], but much more remains a mystery.

$Tor$-modules of the graded modules are also graded and can be computed in the usual way using $M(n) \otimes_S N(q) \cong (M \otimes_S N)(n + q)$. Let $K$ denote the graded $S$-module $S/S_+$, where $S_+ := \oplus_{d>0} S_d$ is the unique homogeneous maximal ideal. We note that $K$ is a graded module concentrated in degree 0. Using $F \otimes_S K$, we compute

$$Tor^S_p(M, K) = \oplus_{q \in \mathbb{Z}} K(-q - p)^{b_{p,q}(M)},$$

which implies that

$$b_{p,q}(M) = \dim_K Tor^S_p(M, K)_{q+p}.$$

We may also compute $Tor^S_p(M, K)$ using the Koszul resolution $G_\bullet$ of $K$, where

$$G_\bullet : 0 \to S(-r) \to S(-r+1) \to \cdots \to S(-p) \to \cdots \to S \to 0.$$

Using $M \otimes_S G_\bullet$, we again compute

$$Tor^S_p(M, K) = \text{homology} (M(-p - 1)_{(p+1)} \to M(-p)_{(p)} \to M(-p + 1)_{(p-1)}),$$

and hence

$$(*) \quad Tor^S_p(M, K)_{q+p} = \text{homology} (M_{q-1}^{(p+1)} \to M_q^{(p)} \to M_{q+1}^{(p-1)}).$$

Since the differential maps in the Koszul complex is given by the natural maps between the wedge products, it is customary to write the right-hand-side of $(*)$ above as:

$$(**) \quad \text{homology} (\bigwedge S_1 \otimes_K M_{q-1} \xrightarrow{d_{p+1}} \bigwedge S_1 \otimes_K M_q \xrightarrow{d_p} \bigwedge S_1 \otimes_K M_{q+1}).$$

We remark that $\mathcal{K}_{p,q}(M)$ was the notation for $Tor^S_p(M, K)_{q+p}$ Green used in [G] in his systematic study of the relationship between the graded resolution and the geometry of projective algebraic variety.
Let \( \{x_1, \ldots, x_r\} \) be a basis of \( S_1 \). To simplify notation we write \( x_{i_1}^* \cdots x_{i_r}^* \) to denote the wedge product of \( \{x_1, \ldots, x_r\} = \{x_{i_1}, \ldots, x_{i_r}\} \). We first consider some trivial cases.

\( p = r \). Suppose that \( M \) has a \( q \)-linear \( r \)-th syzygy. Since \( \wedge^{r+1} S_1 = 0 \), this syzygy corresponds to a nonzero element \( a \in M_q \) in the kernel of \( d_r \). Since
\[
d_r(x_1 \wedge \cdots \wedge x_r \otimes a) = \sum_{1 \leq i \leq r} x_i^* \otimes (-1)^i x_i a,
\]
x_i a = 0, for all \( 1 \leq i \leq r \). Hence \( a \) is a nonzero element of degree \( q \) that is killed by \( S_+ \). The converse is equally trivial for us to state:

\( M \) has a \( q \)-linear \( r \)-th syzygy if and only if \( (\text{Soc } M)_q \neq 0 \).

\( p = r-1 \). Suppose now that \( M \) has a \( q \)-linear \( (r-1) \)-st syzygy. By \((**))\) above this syzygy is determined by an element in the kernel of \( d_{r-1} \) that is not in the image of \( d_r \). Let \( a_{ij} \), \( 1 \leq i \leq r \), be elements of \( M_q \) such that \( \Sigma_{1 \leq i \leq r} x_i^* \otimes a_{ij} \) is in the kernel of \( d_{r-1} \). Using
\[
d_{r-1}\left( \sum_{1 \leq i \leq r} x_i^* \otimes a_i \right) = \sum_{1 \leq i < j \leq r} x_{ij}^* \otimes \pm(x_j a_i - x_i a_j),
\]
We can easily check the validity of the following statement:

\( M \) has a \( q \)-linear \( (r-1) \)-st syzygy if and only if there is a \( 2 \times r \) matrix
\[
\begin{pmatrix}
x_1 & \cdots & x_r \\
a_1 & \cdots & a_r
\end{pmatrix}
\]
such that

i) \( a_i \in M_q \), for all \( 1 \leq i \leq r \),

ii) all of its \( 2 \times 2 \) minors are 0, and

iii) there is no element \( a \in M_{q-1} \) such that \( a_i = (-1)^i x_i a \) for all \( 1 \leq i \leq r \).

We now consider the main case.

\( p = r-2 \). Let \( M \) has a \( q \)-linear \( (r-2) \)-nd syzygy. As before, we can find elements \( a_{ij} \), \( 1 \leq i < j \leq r \), of \( M_q \) such that \( \Sigma_{1 \leq i < j \leq r} x_{ij}^* \otimes a_{ij} \) is in the kernel of \( d_{r-2} \). Since
\[
d_{r-2}\left( \sum_{1 \leq i < j \leq r} x_{ij}^* \otimes a_{ij} \right) = \sum_{1 \leq i < j < k \leq r} x_{ijk}^* \otimes \pm(x_i a_{jk} - x_j a_{ik} + x_k a_{ij}),
\]
\[ x_i a_{jk} - x_j a_{ik} + x_k a_{ij} = 0, \text{ for all } 1 \leq i < j < k \leq r. \] Since these are nothing other than \(4 \times 4\) pfaffians of \(Q\) below involving the first row and column, we have the following characterization:

\(M\) has a \(q\)-linear \((r-2)\)-nd syzygy if and only if there is a \((r+1) \times (r+1)\) skew symmetric matrix

\[
Q = \begin{pmatrix}
0 & x_1 & \cdots & \cdots & x_r \\
-x_1 & 0 & \cdots & a_{ij} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-x_r & \vdots & \cdots & 0
\end{pmatrix}
\]

such that

i) the first row spans \(S_1\),

ii) \(a_{ij} \in M_q\) for \(1 \leq i < j \leq r\),

iii) each \(4 \times 4\) pfaffian of \(Q\) involving the first row and column is zero, and

iv) there are no elements \(a_i \in M_{q-1}\) such that \(a_{ij} = \pm(x_i a_j - x_j a_i)\) for all \(i < j\).

We consider two geometric situations where all, not just the ones involving the first row and column, \(4 \times 4\) pfaffians are zero. To consider general pfaffians the products of elements in \(M\) have to be defined. The first situation deals with the homogeneous coordinate ring of a set of points in, or more generally, a 0-dimensional subscheme of, \(\mathbb{P}^{r-1}\), and the second deals with the canonical image of a nonsingular projective curve. We assume that the field \(K\) is algebraically closed in the rest of this note.

\(X\) is a set of points. Let \(X\) be a 0-dimensional subscheme of \(\mathbb{P}^{r-1}\) in "general" position. Our discussion of this case is not rigorous because we use "general" to mean the argument below works. Let \(S\) be the homogeneous coordinate ring of \(\mathbb{P}^{r-1}\), and \(I\) the saturated ideal defining \(X\). Suppose that \(S/I\) has a 1-linear \((r-2)\)-nd syzygy. Then we may view \(Q\) in (1) above as a matrix of linear forms of \(S\). The following trick expresses any \(4 \times 4\) pfaffian of \(Q\) in terms of those involving the first row and column: for \(1 \leq i < j < k < l \leq r\),

\[ x_i(a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk}) \]
\begin{align*}
= a_{ij}(x_{i}a_{ik} - x_{k}a_{il} + x_{i}a_{kl}) - a_{ik}(x_{l}a_{ij} - x_{j}a_{il} + x_{i}a_{jl}) \\
+ a_{il}(x_{k}a_{ij} - x_{j}a_{ik} + x_{i}a_{jk}) \in I.
\end{align*}

(2)

Since $S/I$ is a 1-dimensional Cohen-Macaulay ring, we may assume that each $x_{i}$ is a nonzero divisor on $S/I$, and hence the $4 \times 4$ pfaffian $a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk}$ determined by $i < j < k < l$ is in $I$.

Since the vector space spanned by the entries of $Q$ is of dimension $r$, the following result forces $Q$ to have a generalized zero, i.e., one can produce a 0 off the diagonal after performing suitable (symmetric) row and column operations on $Q$.

Lemma ([KS, Lemma 1.5]). Let $T$ be a $v \times v$ skew symmetric matrix of linear forms. If $\dim T < 2v - 3$, then $T$ has a generalized zero.

We may, after a suitable row and column operations, put $Q$ in the form

\[
\begin{pmatrix}
0 & 0 & | & A \\
0 & 0 & | & - \\
- & - & | & -A^t \\
&A & & \star
\end{pmatrix},
\]

where $A$ is a $2 \times (r - 1)$ matrix of linear forms. We assume that the points in $X$ are in "general" position so that if $A$ is not 1-generic, i.e., one can produce a 0 after performing suitable row and column operations, then the whole column of $A$ containing zero is zero. Since the $2 \times 2$ minors of $A$ are $4 \times 4$ pfaffians of $Q$, this assumption is satisfied when $I$ doesn’t contain too many rank 2 quadrics, e.g., when $X$ contains at least $2r - 1$ reduced points in linearly general position because $I$ can’t contain a product of linear forms in this case. Under this assumption we may put $Q$ in the form

\[
\begin{pmatrix}
0 & | & A \\
- & & - \\
-A^t & & \star
\end{pmatrix},
\]

where $A$ is a $m \times n$ 1-generic matrix. Since $m + n = r + 1$ and $\dim A = r$, a result of Eisenbud ([E,Theorem 5.1]) implies that $2 \times 2$ minors of $A$ define a rational
normal curve. This argument provides a reason for one, more involved, direction of the following result of Green ([G, Theorem 3.c.6]).

Theorem (Strong Castelnuovo Lemma). Let $X$ be a set of points in $\mathbb{P}^{r-1}$ in general position. Then $X$ lies on a rational normal curve if and only if $S/I$ has a 1-linear $(r-2)$-nd syzygy.

We remark here that Yanagawa used the same result of Eisenbud in proving his Generalized Castelnuovo’s Lemma ([Y, Theorem 2.1]).

$X$ is a nonsingular projective curve. We sketch the argument given in [KS] to prove a result of Green and Lazarsfeld ([GL]) on normal generation of line bundles. Let $X$ be a nonsingular projective curve in $\mathbb{P}^{r-1}$. Let $L$ be a very ample line bundle on $X$. Write $r = h^0(L)$, the dimension of $H^0(X, L)$, and $S = \text{Sym} \ H^0(X, L)$, the symmetric algebra. For a line bundle $F$ on $X$, let $M(F)$ denote the graded $S$-module $\bigoplus_{n \in \mathbb{Z}} \text{Sym}^n H^0(X, F \otimes L^n)$. There is a natural map $\varphi : S \rightarrow M(\mathcal{O})$ whose kernel is the ideal $I$ of the image of the morphism $f$ defined by $L$. $L$ is said to be normally generated if $f(X)$ is a normal subvariety of $\mathbb{P}^{r-1}$, or equivalently, the map $\varphi$ is onto. In terms of the graded betti numbers, this condition is equivalent to $b_{0,q}(M(\mathcal{O})) = 0$, for all $q > 0$. (In fact, for all $q \geq 2$ because $\varphi$ is onto in degree 1.) To obtain a $(r-2)$-nd syzygy we apply the following result of Green.

Duality Theorem ([G] or [EKS]). Let $\omega$ denote the canonical bundle on $X$. For any line bundle $\mathcal{F}$ on $X$,

$$b_{p,q}(M(\mathcal{F})) = b_{r-2-p,r-q}(M(\mathcal{F}^{-1} \omega)).$$

Suppose that $L$ is not normally generated. Since $b_{0,q}(M(\mathcal{O})) \neq 0$ for some $q \geq 2$, $b_{r-2-r,q}(M(\omega)) \neq 0$ by the Duality Theorem. We now assume that $(\text{Cliff}(X)$ will be defined below.)

$$\deg L \geq 2g + 1 - \text{Cliff}(X). \quad (3)$$

This implies that $H^0(X, L^n \omega) = 0$ for all $n \leq -2$ and $h^1(\mathcal{L}) := \dim H^1(X, \mathcal{L}) \leq 1$ (see [GL] or [KS]). Hence $b_{r-2,r-2}(M(\omega))$ is the only nonzero graded betti numbers
for $q \geq 2$, and $M(\omega)$ has a $(0)$-linear $(r-2)$-nd syzygy. As in the previous case we get a skew symmetric matrix $Q$ in (1), where $a_{ij}$ are sections of the canonical bundle. Since $X$ is irreducible, the similar argument as in (2) shows that all $4 \times 4$ pfaffians of $Q$ are zero when viewed as elements either in $H^0(\mathcal{L}_\omega)$ or $H^0(\omega^2)$. If $h^1(\mathcal{L})=0$, we take $B$ to be the $r \times r$ skew symmetric submatrix of $Q$ without the first row and the first column. If $h^1(\mathcal{L})=1$, we let $B=Q$.

Since $X$ is irreducible, the similar argument as in (2) shows that all $4 \times 4$ pfaffians of $Q$ are zero when viewed as elements either in $H^0(\mathcal{L}_\omega)$ or $H^0(\omega^2)$.

If $h^1(\mathcal{L})=0$, we take $B$ to be the $r \times r$ skew symmetric submatrix of $Q$ without the first row and the first column.

Let $\mathcal{F} := \text{Im}(A : \mathcal{O}^m \to \omega^n)$. Since all $2 \times 2$ minors vanish on the canonical image of $X$, $\mathcal{F}$ is a rank one subsheaf of $\omega^n$, and hence a line bundle because $X$ is nonsingular. Since the rows of a $1$-generic matrix are linearly independent $h^0(\mathcal{F}) \geq m$. It can further be shown ([KS, Claim 2]) that $h^1(\mathcal{F}) \geq n$. We now recall the definition of the Clifford index of $X$:

$$\text{Cliff}(X) := \inf \{ g + 1 - (h^0(\mathcal{G}) + h^1(\mathcal{G})) : \mathcal{G} \text{ is a line bundle with } h^0(\mathcal{G}), h^1(\mathcal{G}) \geq 2 \}.$$ 

Our discussion on $\mathcal{F}$ above shows that

$$\text{Cliff}(X) \leq g + 1 - (h^0(\mathcal{F}) + h^1(\mathcal{F})) \leq g + 1 - (h^0(\mathcal{L}) + h^1(\mathcal{L})).$$

Applying Riemann-Roch Theorem, $h^0(\mathcal{L}) = \deg \mathcal{L} - g + 1 + h^1(\mathcal{L})$, we get

$$\deg \mathcal{L} \leq 2g - 2h^1(\mathcal{L}) - \text{Cliff}(X),$$
which contradicts the assumption on the degree of $\mathcal{L}$ in (3). We have thus proved the following result of Green and Lazarsfeld ([GL]):

Theorem. Let $\mathcal{L}$ be a very ample line bundle on a nonsingular projective curve $X$ of genus $g$. If $\deg \mathcal{L} \geq 2g + 1 - 2h^1(\mathcal{L}) - \text{Cliff}(X)$, then $\mathcal{L}$ is normally generated.

References


