Sequences of Fuzzy Sets on $\mathbb{R}^n$

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Abstract

In this paper, we study the convergence of a sequence of fuzzy sets on $\mathbb{R}^n$ which in monotone w.r.t. a pseudo order $\preceq_K$ induced by a closed convex cone $K$. Our study is carried out by restricting the class of fuzzy sets into the subclass in which $\preceq_K$ becomes a partial order and a monotone convergence theorem is approved. This restricted subclass of fuzzy sets is created and characterized in the concept of a determining class.

Keywords: Pseudo-order, fuzzy max order, multidimensional fuzzy sets, monotone convergence theorem, determining class, rectangle-type fuzzy sets.

1. Introduction and notations

In our previous paper [3], we have introduced a pseudo order, $\preceq_K$, in the class of fuzzy sets, which is natural extension of fuzzy max order (cf. [2], [6]) in fuzzy numbers on $\mathbb{R}$ and induced by a closed convex cone $K$ in $\mathbb{R}^n$. For a lattice-structure of the fuzzy max order, see [1], [10]. Here, we study the convergence of a sequence of fuzzy sets on $\mathbb{R}^n$ which is monotone w.r.t. a pseudo order $\preceq_K$. Our study is done by restricting the class of fuzzy sets into the subclass in which $\preceq_K$ becomes a partial order and a monotone convergence theorem is approved. This restricted subclass of fuzzy sets is created and characterized in the concept of a determining class.

In the remainder of this section, we will give some notations and review a vector ordering of $\mathbb{R}^n$ by a convex cone. In Section 2, a pseudo order of fuzzy sets on $\mathbb{R}^n$ is reviewed referring to our previous paper [3]. In Section 3, we introduce a concept determining class and give a convergence theorem for convex compact subclass $\mathbb{R}^n$. These results are applied to obtain a monotone convergence theorem for fuzzy sets on $\mathbb{R}^n$ in Section 4.

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}^n$ an $n$-dimensional Euclidean space. We write fuzzy sets on $\mathbb{R}^n$ by their membership functions $\tilde{s} : \mathbb{R}^n \to [0, 1]$ (see Novák [5] and Zadeh [9]). The $\alpha$-cut ($\alpha \in [0, 1]$) of the fuzzy set $\tilde{s}$ on $\mathbb{R}^n$ is defined as

$$\tilde{s}_\alpha := \{ x \in \mathbb{R}^n | \tilde{s}(x) \geq \alpha \} \ (\alpha > 0) \quad \text{and} \quad \tilde{s}_0 := \text{cl}\{ x \in \mathbb{R}^n | \tilde{s}(x) > 0 \},$$

where cl denotes the closure of the set. A fuzzy set $\tilde{s}$ is called convex if

$$\tilde{s}(\lambda x + (1 - \lambda)y) \geq \tilde{s}(x) \land \tilde{s}(y) \quad x, y \in \mathbb{R}^n, \ \lambda \in [0, 1],$$

where $\land$ denotes the minimum of the set.
where $a \land b = \min\{a, b\}$. Note that $\tilde{s}$ is convex iff the $\alpha$-cut $\tilde{s}_\alpha$ is a convex set for all $\alpha \in [0, 1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all convex fuzzy sets whose membership functions $\tilde{s} : \mathbb{R}^n \rightarrow [0, 1]$ are upper-semicontinuous and normal ($\sup_{x \in \mathbb{R}^n} \tilde{s}(x) = 1$) and have a compact support. When the one-dimensional case $n = 1$, the fuzzy sets are called fuzzy numbers and $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy numbers.

Let $C(\mathbb{R}^n)$ be the set of all compact convex subsets of $\mathbb{R}^n$, and $C_r(\mathbb{R}^n)$ be the set of all rectangles in $\mathbb{R}^n$. For $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$, we have $\tilde{s}_\alpha \in C(\mathbb{R}^n)$ ($\alpha \in [0, 1]$). We write a rectangle in $C_r(\mathbb{R}^n)$ by

$$\text{[}x, y\text{]} = [x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_n, y_n]$$

for $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ with $x_i \leq y_i$ ($i = 1, 2, \cdots, n$). For the case of $n = 1$, $C(\mathbb{R}) = C_r(\mathbb{R})$ and it denotes the set of all bounded closed intervals. When $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$ satisfies $\tilde{s}_\alpha \in C_r(\mathbb{R}^n)$ for all $\alpha \in [0, 1]$, $\tilde{s}$ is called a rectangle-type. We denote by $\mathcal{F}_r(\mathbb{R}^n)$ the set of all rectangle-type fuzzy sets on $\mathbb{R}^n$. Obviously $\mathcal{F}_r(\mathbb{R}) = \mathcal{F}(\mathbb{R})$.

The definitions of addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ are as follows: For $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda \geq 0$,

\begin{equation}
(\tilde{m} + \tilde{n})(x) := \sup_{x_1, x_2 \in \mathbb{R}^n; x_1 + x_2 = x} \{\tilde{m}(x_1) \land \tilde{n}(x_2)\},
\end{equation}

\begin{equation}
(\lambda \tilde{m})(x) := \begin{cases}
\tilde{m}(x/\lambda) & \text{if } \lambda > 0 \\
I_{\{0\}}(x) & \text{if } \lambda = 0
\end{cases} (x \in \mathbb{R}^n),
\end{equation}

where $I_{\{1\}}(\cdot)$ is an indicator. By using set operations $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for any non-empty sets $A, B \subseteq \mathbb{R}^n$, the following holds immediately.

\begin{equation}
(\tilde{m} + \tilde{n})_\alpha := \tilde{m}_\alpha + \tilde{n}_\alpha \text{ and } (\lambda \tilde{m})_\alpha = \lambda \tilde{m}_\alpha (\alpha \in [0, 1]).
\end{equation}

Let $K$ be a non-empty cone of $\mathbb{R}^n$. Using this $K$, we can define a pseudo-order relation $\preceq_K$ on $\mathbb{R}^n$ by $x \preceq_K y$ iff $y - x \in K$. Let $\mathbb{R}^n_+$ be the subset of entrywise non-negative elements in $\mathbb{R}^n$. When $K = \mathbb{R}^n_+$, the order $\preceq_K$ will be denoted by $\preceq_n$ and $x \preceq_n y$ means that $x_i \leq y_i$ for all $i = 1, 2, \cdots, n$, where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$.

2. A pseudo-order on $\mathcal{F}(\mathbb{R}^n)$

In this section, we review a pseudo order introduced by [3]. Henceforth we assume that the convex cone $K \subseteq \mathbb{R}^n$ is given. A pseudo order $\preceq_K$ on $C(\mathbb{R}^n)$ is defined, whose idea is based on a set-relation treated in [4], as follows.

For $A, B \in C(\mathbb{R}^n)$, $A \preceq_K B$ means the following (C.a) and (C.b):

(C.a) For any $x \in A$, there exists $y \in B$ such that $x \preceq_K y$.

(C.b) For any $y \in B$, there exists $x \in A$ such that $x \preceq_K y$.

When $K = \mathbb{R}^n_+$, the relation $\preceq_K$ on $C(\mathbb{R}^n)$ will be written simply by $\preceq_n$ and for $[x, y], [x', y'] \in C_r(\mathbb{R}^n)$, $[x, y] \preceq_n [x', y']$ means $x \preceq_n x'$ and $y \preceq_n y'$.

Using a pseudo order $\preceq_K$ on $C(\mathbb{R}^n)$, a pseudo order $\preceq_K$ on $\mathcal{F}(\mathbb{R}^n)$ is defined as follows. For $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\tilde{s} \preceq_K \tilde{r}$ means the following (F.a) and (F.b):

(F.a) For any $x \in \mathbb{R}^n$, there exists $y \in \mathbb{R}^n$ such that $x \preceq_K y$.

(F.b) For any $y \in \mathbb{R}^n$, there exists $x \in \mathbb{R}^n$ such that $x \preceq_K y$. 

When $K = \mathbb{R}^n_+$, the relation $\preceq_K$ on $\mathcal{F}(\mathbb{R}^n)$ will be written simply by $\preceq_n$ and for $[\tilde{s}, \tilde{r}] = C_r(\mathbb{R}^n)$, $[\tilde{s}, \tilde{r}] \preceq_n [\tilde{s}', \tilde{r}']$ means $\tilde{s} \preceq_n \tilde{s}'$ and $\tilde{r} \preceq_n \tilde{r}'$.
(F.a) For any $x \in \mathbb{R}^n$, there exists $y \in \mathbb{R}^n$ such that $x \prec_K y$ and $\tilde{s}(x) \leq \tilde{r}(y)$.

(F.b) For any $y \in \mathbb{R}^n$, there exists $x \in \mathbb{R}^n$ such that $x \preceq_K y$ and $\tilde{s}(x) \geq \tilde{r}(y)$.

In [3], for $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n)$, it is shown that $\tilde{s} \preceq_K \tilde{r}$ if and only if $\tilde{s}_\alpha \preceq_K \tilde{r}_\alpha$ on $C(\mathbb{R}^n)$ for all $\alpha \in [0,1]$. Define the dual cone of a cone $K$ by

$$K^+ := \{ a \in \mathbb{R}^n | a \cdot x \geq 0 \text{ for all } x \in K \},$$

where $x \cdot y$ denotes the inner product on $\mathbb{R}^n$ for $x, y \in \mathbb{R}^n$. For a subset $A \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, we define

$$a \cdot A := \{ a \cdot x | x \in A \} (\subset \mathbb{R}).$$

The equation (2.1) means the projection of $A$ on the extended line of the vector $a$ if $a \cdot a = 1$. It is trivial that $a \cdot A \in C(\mathbb{R})$ if $A \in C(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$.

**Lemma 2.1** ([3]). Let $A, B \in C(\mathbb{R}^n)$. $A \preceq_K B$ on $C(\mathbb{R}^n)$ if and only if $a \cdot A \preceq_1 a \cdot B$ on $C(\mathbb{R})$ for all $a \in K^+$, where $\preceq_1$ is the natural order on $C(\mathbb{R})$.

For $a \in \mathbb{R}^n$ and $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$, we define a fuzzy number $a \cdot \tilde{s} \in \mathcal{F}(\mathbb{R})$ by

$$a \cdot \tilde{s}(x) := \sup_{\alpha \in [0,1]} \min \{ \alpha, 1_{a \cdot \tilde{s}_\alpha}(x) \}, \quad x \in \mathbb{R}.$$  

where $1_D(\cdot)$ is the classical indicator function of a closed interval $D \in C(\mathbb{R})$.

We define a partial relation $\preceq_M$ on $\mathcal{F}(\mathbb{R})$ as follows ([6]): For $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R})$, $\tilde{s} \preceq_M \tilde{r}$ means that $\tilde{s}_\alpha \preceq \tilde{r}_\alpha$ for all $\alpha \in [0,1]$.

The following theorem gives the correspondence between the pseudo-order $\preceq_K$ on $\mathcal{F}(\mathbb{R}^n)$ and the fuzzy max order $\preceq_M$ on $\mathcal{F}(\mathbb{R})$.

**Lemma 2.2** ([3]). For $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\tilde{s} \preceq_K \tilde{r}$ if and only if $a \cdot \tilde{s} \preceq_M a \cdot \tilde{r}$ for all $a \in K^+$.

Let $\rho_n$ be the Hausdorff metric on $C(\mathbb{R}^n)$, that is, for $A, B \in C(\mathbb{R}^n)$, $\rho_n(A, B) = \max_{a \in A} d(a, B) \vee \max_{b \in B} d(b, A)$, where $d$ is a metric in $\mathbb{R}^n$ and $d(x, Y) = \min_{y \in Y} d(x, y)$ for $x \in \mathbb{R}^n$ and $Y \in C(\mathbb{R}^n)$. It is well-known that $(C(\mathbb{R}^n), \rho_n)$ is a complete separable metric space. A sequence $\{ D_\ell \}_{\ell=1}^\infty \subset C(\mathbb{R}^n)$ converges to $D \in C(\mathbb{R}^n)$ w.r.t. $\rho_n$ if $\rho_n(D_\ell, D) \to 0$ as $\ell \to \infty$.

**Definition** (Convergence of fuzzy set, [8]). For $\{ \tilde{s}_\ell \}_{\ell=1}^\infty \subset \mathcal{F}(\mathbb{R}^n)$ and $\tilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\tilde{s}_\ell$ converges to $\tilde{r}$ w.r.t. $\rho_n$ if $\rho_n(\tilde{s}_\ell, \tilde{r}) \to 0$ as $\ell \to \infty$ except at most countable $\alpha \in [0,1]$.

In the sequel, the monotone convergence theorems for fuzzy sets are given under the concept of the above convergence.
3. Sequences in $C(\mathbb{R}^n)$

In this section, restricting $C(\mathbb{R}^n)$ into the subclass by use of the concept of determining class, we prove the monotone convergence theorem for $C(\mathbb{R}^n)$.

Let $\mathcal{L} \subset C(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$. Then we say that $\mathcal{L}$ is determined by $A$ if $a \cdot D = a \cdot F$ for $a \in A$ and $D, F \in \mathcal{L}$ implies $D = F$.

Note that $\mathcal{L}$ determined by some $A \subset \mathbb{R}^n$ is closed w.r.t. $\rho_n$. Obviously, $\mathcal{C}_r(\mathbb{R}^n)$ is determined by $\{e_1, e_2, \cdots, e_n\}$. Also, by the separation theorem, $C(\mathbb{R}^n)$ is determined $\mathbb{R}^n$.

**Theorem 3.1.** Let $K$ be a closed convex cone of $\mathbb{R}^n$. Suppose that $\mathcal{L} \subset C(\mathbb{R}^n)$ is determined by $K^+$. Then, the pseudo order $\lesssim_K$ is a partial one in the restricted class $\mathcal{L}$.

**Proof.** It suffices to show that $\lesssim_K$ is antisymmetric in $\mathcal{L}$. Let $D, F \in \mathcal{L}$ satisfy that $D \lesssim_K F$ and $F \lesssim_K D$. By Lemma 2.1, $aD \lesssim_1 aF$ and $aF \lesssim_1 aD$ for all $a \in K^+$. Since $\lesssim_1$ is a partial order, $aF = aD$ for all $a \in K^+$, which implies $F = D$ from the determining property of $K^+$. Q.E.D.

The sequence $\{D_\ell\}_{\ell=1}^\infty \subset C(\mathbb{R}^n)$ is said to be bounded w.r.t. $\lesssim_K$ if there exists $F, D \in C(\mathbb{R}^n)$ such that $F \lesssim_K D_\ell \lesssim_K D$ for all $\ell \geq 1$ and said to be monotone w.r.t. $\lesssim_K$ if $D_1 \lesssim_K D_2 \lesssim_K \cdots$. Then, as an application of Theorem 3.1, we have the following, whose proof is omitted.

**Theorem 3.2.** Let $K$ be a closed convex cone of $\mathbb{R}^n$ with $K^+ \cap (\mathbb{R}^n)^o \neq \emptyset$. Suppose that $\mathcal{L} \subset C(\mathbb{R}^n)$ is determined by $K^+$. Then, any sequence $\{D_\ell\}_{\ell=1}^\infty \subset \mathcal{L}$ which is monotone and bounded w.r.t. $\lesssim_K$ converges w.r.t. $\rho_n$, where $A^o$ is a set of inner points in $A$.

The following results are concerned with the scalarization method.

**Corollary 3.1.** Let $K = \{\lambda a | \lambda \geq 0\}$ for some $a \in \mathbb{R}^n_+$. Then, any sequence in $C(\mathbb{R}^n)$ with monotonicity and boundedness w.r.t. $\lesssim_K$ converges w.r.t. $\rho_n$.

**Corollary 3.2.** Any sequence in $\mathcal{C}_r(\mathbb{R}^n)$ with monotonicity and boundedness w.r.t. $\lesssim_n$ converges w.r.t. $\rho_n$.

4. Sequences in $\mathcal{F}(\mathbb{R}^n)$

In this section, applying the results in Section 3, we give the monotone convergence theorem in $\mathcal{F}(\mathbb{R}^n)$. Let $\tilde{\mathcal{L}} \subset \mathcal{F}(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$. Then we say that $\tilde{\mathcal{L}}$ is determined by $A$ if $a\tilde{s} = a\tilde{r}$ for all $a \in A$ and $\tilde{s}, \tilde{r} \in \tilde{\mathcal{L}}$ implies $\tilde{s} = \tilde{r}$.

Note that $\tilde{\mathcal{L}}$ determined by some $A \subset \mathbb{R}^n$ is closed in the convergence given in Definition 1. Applying Lemma 2.2, the same proof as Theorem 3.1 is useful in proving the following.

**Theorem 4.1.** Let $K$ be a closed convex cone of $\mathbb{R}^n$. Suppose that $\tilde{\mathcal{L}} \subset \mathcal{F}(\mathbb{R}^n)$ is
determined by $K^+$. Then, a pseudo order $\preceq_K$ is a partial order in the restricted class $\tilde{\mathcal{L}}$.

In order to get the convergence theorem, we need the concept of directionality given in [8]. Put the surface of the unit ball by $U := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$. Let $V \subset U$. Then, for $D, D' \in C(\mathbb{R}^n)$ with $D \subset D'$, we call $D'$ $V$-directional to $D$ (written by $D' \supset_V D$) if there exists a real $\lambda > 0, y \in D$ and $z \in D'$ such that

(i) $d(z, y) = \rho_n(D', D)$ and

(ii) $z - y = \lambda v$ for some $v \in V$.

**Definition 2.** Let $V \subset U \subset \mathbb{R}^n$. For $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$, $\tilde{s}$ is called $V$-directional if $\tilde{s}_\alpha \supset_V \tilde{s}_{\alpha'}$ for $0 \leq \alpha \leq \alpha' \leq 1$.

**Theorem 4.2.** Let $K$ be a closed convex cone of $\mathbb{R}^n$ with $K^+ \cap (\mathbb{R}^n)^+ \neq \emptyset$. Suppose that $\tilde{\mathcal{L}} \subset \mathcal{F}(\mathbb{R}^n)$ is determined by $K^+$. Then, any monotone and bounded sequence $\{\tilde{s}_\ell\}_{\ell=1}^\infty \subset \tilde{\mathcal{L}}$ with $\tilde{s}_\ell (\ell \geq 1)$ is $V$-directional for a finite set $V$ converges w.r.t. $\rho_n$.

**Corollary 4.1.** Let $K = \{\lambda a \mid \lambda \geq 0\}$ for some $a \in \mathbb{R}^n_+$. Then, any sequence $\{\tilde{s}_\ell\} \subset \mathcal{F}(\mathbb{R}^n)$ satisfying that it is monotone and bounded w.r.t. $\preceq_K$ and $\tilde{s}_\ell (\ell \geq 1)$ is $V$-directional for a finite set $V$ converges w.r.t. $\rho_n$.

**Corollary 4.2.** Any sequence in $\mathcal{F}_r(\mathbb{R}^n)$ with monotonicity and boundedness w.r.t. $\preceq_n$ converges w.r.t. $\rho_n$.

**References**


