<table>
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<th>Title</th>
<th>Sequences of Fuzzy Sets on $\mathbb{R}^n$ (Decision Theory in Mathematical Modelling)</th>
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<tr>
<td>Author(s)</td>
<td>Kurano, Masami; Yasuda, Masami; Nakagami, Jun-ichi; Yoshida, Yuji</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1999), 1079: 233-237</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1999-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62675">http://hdl.handle.net/2433/62675</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Sequences of Fuzzy Sets on $\mathbb{R}^n$

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Abstract

In this paper, we study the convergence of a sequence of fuzzy sets on $\mathbb{R}^n$ which in monotone w.r.t. a pseudo order $\preceq_K$ induced by a closed convex cone $K$. Our study is carried out by restricting the class of fuzzy sets into the subclass in which $\preceq_K$ becomes a partial order and a monotone convergence theorem is approved. This restricted subclass of fuzzy sets is created and characterized in the concept of a determining class.

Keywords: Pseudo-order, fuzzy max order, multidimensional fuzzy sets, monotone convergence theorem, determining class, rectangle-type fuzzy sets.

1. Introduction and notations

In our previous paper [3], we have introduced a pseudo order, $\preceq_K$, in the class of fuzzy sets, which is natural extension of fuzzy max order (cf. [2], [6]) in fuzzy numbers on $\mathbb{R}$ and induced by a closed convex cone $K$ in $\mathbb{R}^n$. For a lattice-structure of the fuzzy max order, see [1], [10]. Here, we study the convergence of a sequence of fuzzy sets on $\mathbb{R}^n$ which is monotone w.r.t. a pseudo order $\preceq_K$. Our study is done by restricting the class of fuzzy sets into the subclass in which $\preceq_K$ becomes a partial order and a monotone convergence theorem is approved. This restricted subclass of fuzzy sets is created and characterized in the concept of a determining class.

In the remainder of this section, we will give some notations and review a vector ordering of $\mathbb{R}^n$ by a convex cone. In Section 2, a pseudo order of fuzzy sets on $\mathbb{R}^n$ is reviewed referring to our previous paper [3]. In Section 3, we introduce a concept determining class and give a convergence theorem for convex compact subclass $\mathbb{R}^n$. These results are applied to obtain a monotone convergence theorem for fuzzy sets on $\mathbb{R}^n$ in Section 4.

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}^n$ an n-dimensional Euclidean space. We write fuzzy sets on $\mathbb{R}^n$ by their membership functions $\tilde{s}: \mathbb{R}^n \to [0,1]$ (see Novák [5] and Zadeh [9]). The $\alpha$-cut ($\alpha \in [0,1]$) of the fuzzy set $\tilde{s}$ on $\mathbb{R}^n$ is defined as

$$\tilde{s}_\alpha := \{ x \in \mathbb{R}^n | \tilde{s}(x) \geq \alpha \} \ (\alpha > 0) \quad \text{and} \quad \tilde{s}_0 := \text{cl}\{ x \in \mathbb{R}^n | \tilde{s}(x) > 0 \},$$

where cl denotes the closure of the set. A fuzzy set $\tilde{s}$ is called convex if

$$\tilde{s}(\lambda x + (1 - \lambda)y) \geq \tilde{s}(x) \wedge \tilde{s}(y) \quad x, y \in \mathbb{R}^n, \lambda \in [0,1],$$

Let $\tilde{s}$ be a fuzzy set on $\mathbb{R}^n$. A number $\beta \in [0,1]$ is called the membership of $\tilde{s}$ in $\mathbb{R}^n$ if $\tilde{s}(x) \geq \beta$ for all $x \in \mathbb{R}^n$. The membership of $\tilde{s}$ in $\mathbb{R}^n$ is denoted by $\text{mem}(\tilde{s}) = \beta$.

Let $\tilde{s}_0$ be a fuzzy set on $\mathbb{R}^n$. A number $\beta \in [0,1]$ is called the membership of $\tilde{s}_0$ in $\mathbb{R}^n$ if $\tilde{s}_0(x) \geq \beta$ for all $x \in \mathbb{R}^n$. The membership of $\tilde{s}_0$ in $\mathbb{R}^n$ is denoted by $\text{mem}\left(\tilde{s}_0\right) = \beta$.
where \( a \land b = \min\{a, b\} \). Note that \( \tilde{s} \) is convex iff the \( \alpha \)-cut \( \tilde{s}_\alpha \) is a convex set for all \( \alpha \in [0, 1] \). Let \( \mathcal{F}(\mathbb{R}^n) \) be the set of all convex fuzzy sets whose membership functions \( \tilde{s} : \mathbb{R}^n \to [0, 1] \) are upper-semicontinuous and normal (\( \sup_{x \in \mathbb{R}^n} \tilde{s}(x) = 1 \)) and have a compact support. When the one-dimensional case \( n = 1 \), the fuzzy sets are called fuzzy numbers and \( \mathcal{F}(\mathbb{R}) \) denotes the set of all fuzzy numbers.

Let \( \mathcal{C}(\mathbb{R}^n) \) be the set of all compact convex subsets of \( \mathbb{R}^n \), and \( \mathcal{C}_r(\mathbb{R}^n) \) be the set of all rectangles in \( \mathbb{R}^n \). For \( \tilde{s} \in \mathcal{F}(\mathbb{R}^n) \), we have \( \tilde{s}_\alpha \in \mathcal{C}(\mathbb{R}^n) \) (\( \alpha \in [0, 1] \)). We write a rectangle in \( \mathcal{C}_r(\mathbb{R}^n) \) by

\[
[x, y] = [x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_n, y_n]
\]

for \( x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n \) with \( x_i \leq y_i \) (\( i = 1, 2, \cdots, n \)). For the case of \( n = 1 \), \( \mathcal{C}(\mathbb{R}) = \mathcal{C}_r(\mathbb{R}) \) and it denotes the set of all bounded closed intervals. When \( \tilde{s} \in \mathcal{F}(\mathbb{R}^n) \) satisfies \( \tilde{s}_\alpha \in \mathcal{C}_r(\mathbb{R}^n) \) for all \( \alpha \in [0, 1] \), \( \tilde{s} \) is called a rectangle-type. We denote by \( \mathcal{F}_r(\mathbb{R}^n) \) the set of all rectangle-type fuzzy sets on \( \mathbb{R}^n \). Obviously \( \mathcal{F}_r(\mathbb{R}) = \mathcal{F}(\mathbb{R}) \).

The definitions of addition and scalar multiplication on \( \mathcal{F}(\mathbb{R}^n) \) are as follows: For \( \tilde{m}, \tilde{n} \in \mathcal{F}(\mathbb{R}^n) \) and \( \lambda \geq 0 \),

\[
(1.1) \quad (\tilde{m} + \tilde{n})(x) := \sup_{x_1, x_2 \in \mathbb{R}^n; x_1 + x_2 = x} \{\tilde{m}(x_1) \land \tilde{n}(x_2)\},
\]

\[
(1.2) \quad (\lambda \tilde{m})(x) := \left\{ \begin{array}{ll} \tilde{m}(x/\lambda) & \text{if } \lambda > 0 \\ I_{\{0\}}(x) & \text{if } \lambda = 0 \end{array} \right. \quad (x \in \mathbb{R}^n),
\]

where \( I_{\{1\}}(\cdot) \) is an indicator. By using set operations \( A + B := \{x + y \mid x \in A, y \in B\} \) and \( \lambda A := \{\lambda x \mid x \in A\} \) for any non-empty sets \( A, B \subset \mathbb{R}^n \), the following holds immediately.

\[
(1.3) \quad (\tilde{m} + \tilde{n})_\alpha := \tilde{m}_\alpha + \tilde{n}_\alpha \quad \text{and} \quad (\lambda \tilde{m})_\alpha = \lambda \tilde{m}_\alpha \quad (\alpha \in [0, 1]).
\]

Let \( K \) be a non-empty cone of \( \mathbb{R}^n \). Using this \( K \), we can define a pseudo-order relation \( \preceq_K \) on \( \mathbb{R}^n \) by \( x \preceq_K y \) iff \( y - x \in K \). Let \( \mathbb{R}^n_+ \) be the subset of entrywise non-negative elements in \( \mathbb{R}^n \). When \( K = \mathbb{R}^n_+ \), the order \( \preceq_K \) will be denoted by \( \preceq_n \) and \( x \preceq_n y \) means that \( x_i \leq y_i \) for all \( i = 1, 2, \cdots, n \), where \( x = (x_1, x_2, \cdots, x_n) \) and \( y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n \).

2. A pseudo-order on \( \mathcal{F}(\mathbb{R}^n) \)

In this section, we review a pseudo order introduced by [3]. Henceforth we assume that the convex cone \( K \subset \mathbb{R}^n \) is given. A pseudo order \( \preceq_K \) on \( \mathcal{C}(\mathbb{R}^n) \) is defined, whose idea is based on set-relation treated in [4], as follows.

For \( A, B \in \mathcal{C}(\mathbb{R}^n) \), \( A \preceq_K B \) means the following (C.a) and (C.b):

(C.a) For any \( x \in A \), there exists \( y \in B \) such that \( x \preceq_K y \).

(C.b) For any \( y \in B \), there exists \( x \in A \) such that \( x \preceq_K y \).

When \( K = \mathbb{R}^n_+ \), the relation \( \preceq_K \) on \( \mathcal{C}(\mathbb{R}^n) \) will be written simply by \( \preceq_n \) and for \( [x, y], [x', y'] \in \mathcal{C}_r(\mathbb{R}^n), [x, y] \preceq_n [x', y'] \) means \( x \preceq_n x' \) and \( y \preceq_n y' \).

Using a pseudo order \( \preceq_K \) on \( \mathcal{C}(\mathbb{R}^n) \), a pseudo order \( \preceq_K \) on \( \mathcal{F}(\mathbb{R}^n) \) is defined as follows. For \( \tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n), \tilde{s} \preceq_K \tilde{r} \) means the following (F.a) and (F.b):

\[
(\text{F.a}) \quad (\tilde{s} + \tilde{r})_\alpha := \tilde{s}_\alpha + \tilde{r}_\alpha \quad \text{and} \quad (\lambda \tilde{s})_\alpha = \lambda \tilde{s}_\alpha \quad (\alpha \in [0, 1]).
\]

\[
(\text{F.b}) \quad \tilde{s} \preceq_K \tilde{r} \iff \tilde{s}_\alpha \preceq_K \tilde{r}_\alpha \quad (\alpha \in [0, 1]).
\]
(F.a) For any \( x \in \mathbb{R}^n \), there exists \( y \in \mathbb{R}^n \) such that \( x \preceq_K y \) and \( \tilde{s}(x) \leq \tilde{r}(y) \).

(F.b) For any \( y \in \mathbb{R}^n \), there exists \( x \in \mathbb{R}^n \) such that \( x \preceq_K y \) and \( \tilde{s}(x) \geq \tilde{r}(y) \).

In [3], for \( \tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n) \), it is shown that \( \tilde{s} \preceq_{K} \tilde{r} \) if and only if \( \tilde{s}_\alpha \preceq_{K} \tilde{r}_\alpha \) on \( C(\mathbb{R}^n) \) for all \( \alpha \in [0,1] \). Define the dual cone of a cone \( K \) by

\[ K^+ := \{ a \in \mathbb{R}^n \mid a \cdot x \geq 0 \text{ for all } x \in K \}, \]

where \( a \cdot y \) denotes the inner product on \( \mathbb{R}^n \) for \( x, y \in \mathbb{R}^n \). For a subset \( A \subset \mathbb{R}^n \) and \( a \in \mathbb{R}^n \), we define

\[ a \cdot A := \{ a \cdot x \mid x \in A \} (\subset \mathbb{R}). \]

The equation (2.1) means the projection of \( A \) on the extended line of the vector \( a \) if \( a \cdot a = 1 \). It is trivial that \( a \cdot A \in C(\mathbb{R}) \) if \( A \in C(\mathbb{R}^n) \) and \( a \in \mathbb{R}^n \).

**Lemma 2.1**([3]). Let \( A, B \in C(\mathbb{R}^n) \). \( A \preceq_K B \) on \( C(\mathbb{R}^n) \) if and only if \( a \cdot A \preceq_1 a \cdot B \) on \( C(\mathbb{R}) \) for all \( a \in K^+ \), where \( \preceq_1 \) is the natural order on \( C(\mathbb{R}) \).

For \( a \in \mathbb{R}^n \) and \( \tilde{s} \in \mathcal{F}(\mathbb{R}^n) \), we define a fuzzy number \( a \cdot \tilde{s} \in \mathcal{F}(\mathbb{R}) \) by

\[ a \cdot \tilde{s}(x) := \sup_{\alpha \in [0,1]} \min \{ \alpha, 1_{a \cdot \tilde{s}_\alpha}(x) \}, \quad x \in \mathbb{R}. \]

where \( 1_{D}(\cdot) \) is the classical indicator function of a closed interval \( D \in C(\mathbb{R}) \).

We define a partial relation \( \preceq_M \) on \( \mathcal{F}(\mathbb{R}) \) as follows ([6]): For \( \tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}) \), \( \tilde{s} \preceq_M \tilde{r} \) means that \( \tilde{s}_\alpha \preceq \tilde{r}_\alpha \) for all \( \alpha \in [0,1] \).

The following theorem gives the correspondence between the pseudo-order \( \preceq_K \) on \( \mathcal{F}(\mathbb{R}^n) \) and the fuzzy max order \( \preceq_M \) on \( \mathcal{F}(\mathbb{R}) \).

**Lemma 2.2**([3]). For \( \tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n) \), \( \tilde{s} \preceq_K \tilde{r} \) if and only if \( a \cdot \tilde{s} \preceq_M a \cdot \tilde{r} \) for all \( a \in K^+ \).

Let \( \rho_n \) be the Hausdorff metric on \( C(\mathbb{R}^n) \), that is, for \( A, B \in C(\mathbb{R}^n) \), \( \rho_n(A, B) = \max_{a \in A} d(a, B) \vee \max_{b \in B} d(b, A) \), where \( d \) is a metric in \( \mathbb{R}^n \) and \( d(x, y) = \min_{y \in Y} d(x, y) \) for \( x \in \mathbb{R}^n \) and \( Y \in C(\mathbb{R}^n) \). It is well-known that \( (C(\mathbb{R}^n), \rho_n) \) is a complete separable metric space. A sequence \( \{ D_\ell \}_{\ell=1}^\infty \subset C(\mathbb{R}^n) \) converges to \( D \in C(\mathbb{R}^n) \) w.r.t. \( \rho_n \) if \( \rho_n(D_\ell, D) \to 0 \) as \( \ell \to \infty \).

**Definition** (Convergence of fuzzy set, [8]). For \( \{ \tilde{s}_\ell \}_{\ell=1}^\infty \subset \mathcal{F}(\mathbb{R}^n) \) and \( \tilde{r} \in \mathcal{F}(\mathbb{R}^n) \), \( \tilde{s}_\ell \) converges to \( \tilde{r} \) w.r.t. \( \rho_n \) if \( \rho_n(\tilde{s}_\ell, \tilde{r}) \to 0 \) as \( \ell \to \infty \) except at most countable \( \alpha \in [0,1] \).

In the sequel, the monotone convergence theorems for fuzzy sets are given under the concept of the above convergence.
3. Sequences in $C(\mathbb{R}^n)$

In this section, restricting $C(\mathbb{R}^n)$ into the subclass by use of the concept of determining class, we prove the monotone convergence theorem for $C(\mathbb{R}^n)$.

Let $\mathcal{L} \subset C(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$. Then we say that $\mathcal{L}$ is determined by $A$ if $a \cdot D = a \cdot F$ for $a \in A$ and $D, F \in \mathcal{L}$ implies $D = F$.

Note that $\mathcal{L}$ determined by some $A \subset \mathbb{R}^n$ is closed w.r.t. $\rho_n$. Obviously, $C_r(\mathbb{R}^n)$ is determined by \{e_1, e_2, \ldots, e_n\}. Also, by the separation theorem, $C(\mathbb{R}^n)$ is determined $\mathbb{R}^n$.

**Theorem 3.1.** Let $K$ be a closed convex cone of $\mathbb{R}^n$. Suppose that $\mathcal{L} \subset C(\mathbb{R}^n)$ is determined by $K^+$. Then, the pseudo order $\preceq_K$ is a partial one in the restricted class $\mathcal{L}$.

**Proof.** It suffices to show that $\preceq_K$ is antisymmetric in $\mathcal{L}$. Let $D, F \in \mathcal{L}$ satisfy that $D \preceq_K F$ and $F \preceq_K D$. By Lemma 2.1, $aD \preceq_1 aF$ and $aF \preceq_1 aD$ for all $a \in K^+$. Since $\preceq_1$ is a partial order, $aF = aD$ for all $a \in K^+$, which implies $F = D$ from the determining property of $K^+$. Q.E.D.

The sequence $\{D\ell\}_{\ell=1}^\infty \subset C(\mathbb{R}^n)$ is said to be bounded w.r.t. $\preceq_K$ if there exists $F, D \in C(\mathbb{R}^n)$ such that $F \preceq_K D_t \preceq_K D$ for all $\ell \geq 1$ and said to be monotone w.r.t. $\preceq_K$ if $D_1 \preceq_K D_2 \preceq_K \ldots$. Then, as an application of Theorem 3.1, we have the following, whose proof is omitted.

**Theorem 3.2.** Let $K$ be a closed convex cone of $\mathbb{R}^n$ with $K^+ \cap (\mathbb{R}^n)^\circ \neq \emptyset$. Suppose that $\mathcal{L} \subset C(\mathbb{R}^n)$ is determined by $K^+$. Then, any sequence $\{D\ell\}_{\ell=1}^\infty \subset \mathcal{L}$ which is monotone and bounded w.r.t. $\preceq_K$ converges w.r.t. $\rho_n$, where $A^\circ$ is a set of inner points in $A$.

The following results are concerned with the scalarization method.

**Corollary 3.1.** Let $K = \{\lambda a \mid \lambda \geq 0\}$ for some $a \in \mathbb{R}^n_+$. Then, any sequence in $C(\mathbb{R}^n)$ with monotonicity and boundedness w.r.t. $\preceq_K$ converges w.r.t. $\rho_n$.

**Corollary 3.2.** Any sequence in $C_r(\mathbb{R}^n)$ with monotonicity and boundedness w.r.t. $\preceq_n$ converges w.r.t. $\rho_n$.

4. Sequences in $F(\mathbb{R}^n)$

In this section, applying the results in Section 3, we give the monotone convergence theorem in $F(\mathbb{R}^n)$. Let $\hat{\mathcal{L}} \subset F(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$. Then we say that $\hat{\mathcal{L}}$ is determined by $A$ if $a\hat{s} = a\hat{r}$ for all $a \in A$ and $\hat{s}, \hat{r} \in \hat{\mathcal{L}}$ implies $\hat{s} = \hat{r}$.

Note that $\hat{\mathcal{L}}$ determined by some $A \subset \mathbb{R}^n$ is closed in the convergence given in Definition 1. Applying Lemma 2.2, the same proof as Theorem 3.1 is useful in proving the following.

**Theorem 4.1.** Let $K$ be a closed convex cone of $\mathbb{R}^n$. Suppose that $\hat{\mathcal{L}} \subset F(\mathbb{R}^n)$ is
determined by $K^+$. Then, a pseudo order $\preceq_K$ is a partial order in the restricted class $\mathcal{L}^\circ$.

In order to get the convergence theorem, we need the concept of directionality given in [8]. Put the surface of the unit ball by $U := \{x \in \mathbb{R}^n \mid ||x|| = 1\}$. Let $V \subset U$. Then, for $D, D' \in \mathcal{C}(\mathbb{R}^n)$ with $D \subset D'$, we call $D'$ $V$-directional to $D$ (written by $D' \supset_V D$) if there exists a real $\lambda > 0, y \in D$ and $z \in D'$ such that

(i) $d(z, y) = \rho_n(D', D)$ and (ii) $z - y = \lambda v$ for some $v \in V$.

**Definition 2.** Let $V \subset U \subset \mathbb{R}^n$. For $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$, $\tilde{s}$ is called $V$-directional if $\tilde{s}_\alpha \supset_V \tilde{s}_{\alpha'}$ for $0 \leq \alpha \leq \alpha' \leq 1$.

**Theorem 4.2.** Let $K$ be a closed convex cone of $\mathbb{R}^n$ with $K^+ \cap (\mathbb{R}_{+}^n)^o \neq \emptyset$. Suppose that $\mathcal{L} \subset \mathcal{F}(\mathbb{R}^n)$ is determined by $K^+$. Then, any monotone and bounded sequence $\{\tilde{s}_\ell\}_{\ell=1}^\infty \subset \mathcal{L}$ with $\tilde{s}_\ell$ ($\ell \geq 1$) is $V$-directional for a finite set $V$ converges w.r.t. $\rho_n$.

**Corollary 4.1.** Let $K = \{\lambda a \mid \lambda \geq 0\}$ for some $a \in \mathbb{R}_{+}^n$. Then, any sequence $\{\tilde{s}_\ell\} \subset \mathcal{F}(\mathbb{R}^n)$ satisfying that it is monotone and bounded w.r.t. $\preceq_K$ and $\tilde{s}_\ell$ ($\ell \geq 1$) is $V$-directional for a finite set $V$ converges w.r.t. $\rho_n$.

**Corollary 4.2.** Any sequence in $\mathcal{F}_-(\mathbb{R}^n)$ with monotonicity and boundedness w.r.t. $\preceq_n$ converges w.r.t. $\rho_n$.

**References**


