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Sequences of Fuzzy Sets on $\mathbb{R}^n$ (Decision Theory in Mathematical Modelling)

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Sequences of Fuzzy Sets on $\mathbb{R}^n$

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Abstract

In this paper, we study the convergence of a sequence of fuzzy sets on $\mathbb{R}^n$ which in monotone w.r.t. a pseudo order $\preceq_K$ induced by a closed convex cone $K$. Our study is carried out by restricting the class of fuzzy sets into the subclass in which $\preceq_K$ becomes a partial order and a monotone convergence theorem is approved. This restricted subclass of fuzzy sets is created and characterized in the concept of a determining class.

Keywords: Pseudo-order, fuzzy max order, multidimensional fuzzy sets, monotone convergence theorem, determining class, rectangle-type fuzzy sets.

1. Introduction and notations

In our previous paper [3], we have introduced a pseudo order, $\preceq_K$, in the class of fuzzy sets, which is natural extension of fuzzy max order (cf. [2], [6]) in fuzzy numbers on $\mathbb{R}$ and induced by a closed convex cone $K$ in $\mathbb{R}^n$. For a lattice-structure of the fuzzy max order, see [1], [10]. Here, we study the convergence of a sequence of fuzzy sets on $\mathbb{R}^n$ which is monotone w.r.t. a pseudo order $\preceq_K$. Our study is done by restricting the class of fuzzy sets into the subclass in which $\preceq_K$ becomes a partial order and a monotone convergence theorem is approved. This restricted subclass of fuzzy sets is created and characterized in the concept of a determining class.

In the remainder of this section, we will give some notations and review a vector ordering of $\mathbb{R}^n$ by a convex cone. In Section 2, a pseudo order of fuzzy sets on $\mathbb{R}^n$ is reviewed referring to our previous paper [3]. In Section 3, we introduce a concept determining class and give a convergence theorem for convex compact subclass $\mathbb{R}^n$. These results are applied to obtain a monotone convergence theorem for fuzzy sets on $\mathbb{R}^n$ in Section 4.

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}^n$ an $n$-dimensional Euclidean space. We write fuzzy sets on $\mathbb{R}^n$ by their membership functions $\tilde{s}: \mathbb{R}^n \rightarrow [0,1]$ (see Novák [5] and Zadeh [9]). The $\alpha$-cut ($\alpha \in [0,1]$) of the fuzzy set $\tilde{s}$ on $\mathbb{R}^n$ is defined as

$$\tilde{s}_\alpha := \{x \in \mathbb{R}^n | \tilde{s}(x) \geq \alpha \} \ (\alpha > 0) \quad \text{and} \quad \tilde{s}_0 := \text{cl}\{x \in \mathbb{R}^n | \tilde{s}(x) > 0\},$$

where cl denotes the closure of the set. A fuzzy set $\tilde{s}$ is called convex if

$$\tilde{s}(\lambda x + (1 - \lambda)y) \geq \tilde{s}(x) \land \tilde{s}(y) \quad x, y \in \mathbb{R}^n, \ \lambda \in [0,1],$$

where $\land$ denotes the "and" operator in the sense of fuzzy logic.
where $a \wedge b = \min\{a, b\}$. Note that $\tilde{s}$ is convex iff the $\alpha$-cut $\tilde{s}_\alpha$ is a convex set for all $\alpha \in [0, 1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all convex fuzzy sets whose membership functions $\tilde{s} : \mathbb{R}^n \to [0, 1]$ are upper-semicontinuous and normal ($\sup_{x \in \mathbb{R}^n} \tilde{s}(x) = 1$) and have a compact support. When the one-dimensional case $n = 1$, the fuzzy sets are called fuzzy numbers and $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy numbers.

Let $C(\mathbb{R}^n)$ be the set of all compact convex subsets of $\mathbb{R}^n$, and $C_r(\mathbb{R}^n)$ be the set of all rectangles in $\mathbb{R}^n$. For $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$, we have $\tilde{s}_\alpha \in C(\mathbb{R}^n)$ ($\alpha \in [0, 1]$). We write a rectangle in $C_r(\mathbb{R}^n)$ by

$$[x, y] = [x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_n, y_n]$$

for $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ with $x_i \leq y_i$ ($i = 1, 2, \ldots, n$). For the case of $n = 1$, $C(\mathbb{R}) = C_r(\mathbb{R})$ and it denotes the set of all bounded closed intervals. When $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$ satisfies $\tilde{s}_\alpha \in C_r(\mathbb{R}^n)$ for all $\alpha \in [0, 1]$, $\tilde{s}$ is called a rectangle-type. We denote by $\mathcal{F}_r(\mathbb{R}^n)$ the set of all rectangle-type fuzzy sets on $\mathbb{R}^n$. Obviously $\mathcal{F}_r(\mathbb{R}) = \mathcal{F}(\mathbb{R})$.

The definitions of addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ are as follows: For $\bar{m}, \bar{n} \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda \geq 0$,

$$(1.1) \quad (\bar{m} + \bar{n})(x) := \sup_{x_1, x_2 \in \mathbb{R}^n; x_1 + x_2 = x} \{\bar{m}(x_1) \land \bar{n}(x_2)\},$$

$$(1.2) \quad (\lambda \bar{m})(x) := \begin{cases} \bar{m}(x/\lambda) & \text{if } \lambda > 0 \\ I_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbb{R}^n),$$

where $I_{\{1\}}(\cdot)$ is an indicator. By using set operations $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for any non-empty sets $A, B \subseteq \mathbb{R}^n$, the following holds immediately.

$$(1.3) \quad (\bar{m} + \bar{n})_\alpha := \bar{m}_\alpha + \bar{n}_\alpha \quad \text{and} \quad (\lambda \bar{m})_\alpha = \lambda \bar{m}_\alpha \quad (\alpha \in [0, 1]).$$

Let $K$ be a non-empty cone of $\mathbb{R}^n$. Using this $K$, we can define a pseudo-order relation $\preceq_K$ on $\mathbb{R}^n$ by $x \preceq_K y$ iff $y - x \in K$. Let $\mathbb{R}^n_+$ be the subset of entrywise non-negative elements in $\mathbb{R}^n$. When $K = \mathbb{R}^n_+$, the order $\preceq_K$ will be denoted by $\preceq_n$ and $x \preceq_n y$ means that $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$, where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

2. A pseudo-order on $\mathcal{F}(\mathbb{R}^n)$

In this section, we review a pseudo order introduced by [3]. Henceforth we assume that the convex cone $K \subseteq \mathbb{R}^n$ is given. A pseudo order $\preceq_K$ on $C(\mathbb{R}^n)$ is defined, whose idea is based on set-relation treated in [4], as follows.

For $A, B \subseteq C(\mathbb{R}^n)$, $A \preceq_K B$ means the following (C.a) and (C.b):

(C.a) For any $x \in A$, there exists $y \in B$ such that $x \preceq_K y$.

(C.b) For any $y \in B$, there exists $x \in A$ such that $x \preceq_K y$.

When $K = \mathbb{R}^n_+$, the relation $\preceq_K$ on $C(\mathbb{R}^n)$ will be written simply by $\preceq_n$ and for $[x, y], [x', y'] \in C_r(\mathbb{R}^n)$, $[x, y] \preceq_n [x', y']$ means $x \preceq_n x'$ and $y \preceq_n y'$.

Using a pseudo order $\preceq_K$ on $C(\mathbb{R}^n)$, a pseudo order $\preceq_K$ on $\mathcal{F}(\mathbb{R}^n)$ is defined as follows. For $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\tilde{s} \preceq_K \tilde{r}$ means the following (F.a) and (F.b):

(F.a) For any $x \in \mathbb{R}^n$, there exists $y \in \mathbb{R}^n$ such that $x \preceq_K y$.

(F.b) For any $y \in \mathbb{R}^n$, there exists $x \in \mathbb{R}^n$ such that $x \preceq_K y$.
(F.a) For any \( x \in \mathbb{R}^n \), there exists \( y \in \mathbb{R}^n \) such that \( x \preceq_K y \) and \( \bar{s}(x) \leq \bar{r}(y) \).

(F.b) For any \( y \in \mathbb{R}^n \), there exists \( x \in \mathbb{R}^n \) such that \( x \preceq_K y \) and \( \bar{s}(x) \geq \bar{r}(y) \).

In [3], for \( \bar{s}, \bar{r} \in \mathcal{F}(\mathbb{R}^n) \), it is shown that \( \bar{s} \preceq_K \bar{r} \) if and only if \( \bar{s}_\alpha \preceq_K \bar{r}_\alpha \) on \( C(\mathbb{R}^n) \) for all \( \alpha \in [0, 1] \). Define the dual cone of a cone \( K \) by

\[
K^+ := \{ a \in \mathbb{R}^n \mid a \cdot x \geq 0 \text{ for all } x \in K \},
\]

where \( a \cdot y \) denotes the inner product on \( \mathbb{R}^n \) for \( x, y \in \mathbb{R}^n \). For a subset \( A \subset \mathbb{R}^n \) and \( a \in \mathbb{R}^n \), we define

\[
a \cdot A := \{ a \cdot x \mid x \in A \} \subset \mathbb{R}.
\]

(2.1)

The equation (2.1) means the projection of \( A \) on the extended line of the vector \( a \) if \( a \cdot a = 1 \). It is trivial that \( a \cdot A \in C(\mathbb{R}) \) if \( A \in C(\mathbb{R}^n) \) and \( a \in \mathbb{R}^n \).

**Lemma 2.1** ([3]). Let \( A, B \in C(\mathbb{R}^n) \). \( A \preceq_K B \) on \( C(\mathbb{R}^n) \) if and only if \( a \cdot A \preceq_1 a \cdot B \) on \( C(\mathbb{R}) \) for all \( a \in K^+ \), where \( \preceq_1 \) is the natural order on \( C(\mathbb{R}) \).

For \( a \in \mathbb{R}^n \) and \( \bar{s} \in \mathcal{F}(\mathbb{R}^n) \), we define a fuzzy number \( a \cdot \bar{s} \in \mathcal{F}(\mathbb{R}) \) by

\[
a \cdot \bar{s}(x) := \sup_{\alpha \in [0, 1]} \min \{\alpha, 1_{a \cdot \bar{s}_\alpha}(x)\}, \quad x \in \mathbb{R}.
\]

where \( 1_D(\cdot) \) is the classical indicator function of a closed interval \( D \in C(\mathbb{R}) \).

We define a partial relation \( \preceq_M \) on \( \mathcal{F}(\mathbb{R}) \) as follows ([6]): For \( \bar{s}, \bar{r} \in \mathcal{F}(\mathbb{R}) \), \( \bar{s} \preceq_M \bar{r} \) means that \( \bar{s}_\alpha \preceq \bar{r}_\alpha \) for all \( \alpha \in [0, 1] \).

The following theorem gives the correspondence between the pseudo-order \( \preceq_K \) on \( \mathcal{F}(\mathbb{R}^n) \) and the fuzzy max order \( \preceq_M \) on \( \mathcal{F}(\mathbb{R}) \).

**Lemma 2.2** ([3]). For \( \bar{s}, \bar{r} \in \mathcal{F}(\mathbb{R}^n) \), \( \bar{s} \preceq_K \bar{r} \) if and only if \( a \cdot \bar{s} \preceq_M a \cdot \bar{r} \) for all \( a \in K^+ \).

Let \( \rho_n \) be the Hausdorff metric on \( C(\mathbb{R}^n) \), that is, for \( A, B \in C(\mathbb{R}^n) \), \( \rho_n(A, B) = \max_{a \in A} d(a, B) \vee \max_{b \in B} d(b, A) \), where \( d \) is a metric in \( \mathbb{R}^n \) and \( d(x, Y) = \min_{y \in Y} d(x, y) \) for \( x \in \mathbb{R}^n \) and \( Y \in \mathcal{F}(\mathbb{R}^n) \). It is well-known that \( (C(\mathbb{R}^n), \rho_n) \) is a complete separable metric space. A sequence \( \{D_\ell\}_{\ell=1}^\infty \subset C(\mathbb{R}^n) \) converges to \( D \in C(\mathbb{R}^n) \) w.r.t. \( \rho_n \) if \( \rho_n(D_\ell, D) \to 0 \) as \( \ell \to \infty \).

**Definition** (Convergence of fuzzy set, [8]).

For \( \{\bar{s}_\ell\}_{\ell=1}^\infty \subset \mathcal{F}(\mathbb{R}^n) \) and \( \bar{r} \in \mathcal{F}(\mathbb{R}^n) \), \( \bar{s}_\ell \) converges to \( \bar{r} \) w.r.t. \( \rho_n \) if \( \rho_n(\bar{s}_\ell, \bar{r}) \to 0 \) as \( \ell \to \infty \) except at most countable \( \alpha \in [0, 1] \).

In the sequel, the monotone convergence theorems for fuzzy sets are given under the concept of the above convergence.
3. Sequences in \( C(\mathbb{R}^n) \)

In this section, restricting \( C(\mathbb{R}^n) \) into the subclass by use of the concept of determining class, we prove the monotone convergence theorem for \( C(\mathbb{R}^n) \).

Let \( \mathcal{L} \subset C(\mathbb{R}^n) \) and \( A \subset \mathbb{R}^n \). Then we say that \( \mathcal{L} \) is determined by \( A \) if \( a \cdot D = a \cdot F \) for \( a \in A \) and \( D, F \in \mathcal{L} \) implies \( D = F \).

Note that \( \mathcal{L} \) determined by some \( A \subset \mathbb{R}^n \) is closed w.r.t. \( \rho_n \). Obviously, \( \mathcal{C}_r(\mathbb{R}^n) \) is determined by \( \{e_1, e_2, \ldots, e_n\} \). Also, by the separation theorem, \( C(\mathbb{R}^n) \) is determined \( \mathbb{R}^n \).

**Theorem 3.1.** Let \( K \) be a closed convex cone of \( \mathbb{R}^n \). Suppose that \( \mathcal{L} \subset C(\mathbb{R}^n) \) is determined by \( K^+ \). Then, the pseudo order \( \preceq_K \) is a partial one in the restricted class \( \mathcal{L} \).

**Proof.** It suffices to show that \( \preceq_K \) is antisymmetric in \( \mathcal{L} \). Let \( D, F \in \mathcal{L} \) satisfy that \( D \preceq_K F \) and \( F \preceq_K D \). By Lemma 2.1, \( aD \preceq_K aF \) and \( aF \preceq_K aD \) for all \( a \in K^+ \). Since \( \preceq_1 \) is a partial order, \( aF = aD \) for all \( a \in K^+ \), which implies \( F = D \) from the determining property of \( K^+ \). Q.E.D.

The sequence \( \{D_\ell\}_{\ell=1}^\infty \subset C(\mathbb{R}^n) \) is said to be bounded w.r.t. \( \preceq_K \) if there exists \( F, D \in C(\mathbb{R}^n) \) such that \( F \preceq_K D_\ell \preceq_K D \) for all \( \ell \geq 1 \) and said to be monotone w.r.t. \( \preceq_K \) if \( D_1 \preceq_K D_2 \preceq_K \cdots \). Then, as an application of Theorem 3.1, we have the following, whose proof is omitted.

**Theorem 3.2.** Let \( K \) be a closed convex cone of \( \mathbb{R}^n \) with \( K^+ \cap (\mathbb{R}^n)^o \neq \emptyset \). Suppose that \( \mathcal{L} \subset C(\mathbb{R}^n) \) is determined by \( K^+ \). Then, any sequence \( \{D_\ell\}_{\ell=1}^\infty \subset \mathcal{L} \) which is monotone and bounded w.r.t. \( \preceq_K \) converges w.r.t. \( \rho_n \), where \( A^o \) is a set of inner points in \( A \).

The following results are concerned with the scalarization method.

**Corollary 3.1.** Let \( K = \{\lambda a \mid \lambda \geq 0\} \) for some \( a \in \mathbb{R}_+^n \). Then, any sequence in \( C(\mathbb{R}^n) \) with monotonicity and boundedness w.r.t. \( \preceq_K \) converges w.r.t. \( \rho_n \).

**Corollary 3.2.** Any sequence in \( \mathcal{C}_r(\mathbb{R}^n) \) with monotonicity and boundedness w.r.t. \( \preceq_n \) converges w.r.t. \( \rho_n \).

4. Sequences in \( \mathcal{F}(\mathbb{R}^n) \)

In this section, applying the results in Section 3, we give the monotone convergence theorem in \( \mathcal{F}(\mathbb{R}^n) \). Let \( \tilde{\mathcal{L}} \subset \mathcal{F}(\mathbb{R}^n) \) and \( A \subset \mathbb{R}^n \). Then we say that \( \tilde{\mathcal{L}} \) is determined by \( A \) if \( a\tilde{s} = a\tilde{r} \) for all \( a \in A \) and \( \tilde{s}, \tilde{r} \in \tilde{\mathcal{L}} \) implies \( \tilde{s} = \tilde{r} \).

Note that \( \tilde{\mathcal{L}} \) determined by some \( A \subset \mathbb{R}^n \) is closed in the convergence given in Definition 1. Applying Lemma 2.2, the same proof as Theorem 3.1 is useful in proving the following.

**Theorem 4.1.** Let \( K \) be a closed convex cone of \( \mathbb{R}^n \). Suppose that \( \tilde{\mathcal{L}} \subset \mathcal{F}(\mathbb{R}^n) \) is
determined by $K^+$. Then, a pseudo order $\preceq_K$ is a partial order in the restricted class $\mathcal{L}$.

In order to get the convergence theorem, we need the concept of directionality given in [8]. Put the surface of the unit ball by $U := \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$. Let $V \subset U$. Then, for $D, D' \in C(\mathbb{R}^n)$ with $D \subseteq D'$, we call $D'$ $V$-directional to $D$ (written by $D' \supseteq_V D$) if there exists a real $\lambda > 0$, $y \in D$ and $z \in D'$ such that

(i) $d(z, y) = \rho_n(D', D)$ and

(ii) $z - y = \lambda v$ for some $v \in V$.

**Definition 2.** Let $V \subset U \subset \mathbb{R}^n$. For $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$, $\tilde{s}$ is called $V$-directional if $\tilde{s}_\alpha \supseteq_V \tilde{s}_{\alpha'}$ for $0 \leq \alpha \leq \alpha' \leq 1$.

**Theorem 4.2.** Let $K$ be a closed convex cone of $\mathbb{R}^n$ with $K^+ \cap (\mathbb{R}^n)^+ \neq \emptyset$. Suppose that $\mathcal{L} \subset \mathcal{F}(\mathbb{R}^n)$ is determined by $K^+$. Then, any monotone and bounded sequence $\{\tilde{s}_\ell\}_{\ell=1}^\infty \subset \mathcal{L}$ with $\tilde{s}_\ell$ ($\ell \geq 1$) is $V$-directional for a finite set $V$ converges w.r.t. $\rho_n$.

**Corollary 4.1.** Let $K = \{ \lambda a \mid \lambda \geq 0 \}$ for some $a \in \mathbb{R}^n_+$. Then, any sequence $\{\tilde{s}_\ell\} \subset \mathcal{F}(\mathbb{R}^n)$ satisfying that it is monotone and bounded w.r.t. $\preceq_K$ and $\tilde{s}_\ell$ ($\ell \geq 1$) is $V$-directional for a finite set $V$ converges w.r.t. $\rho_n$.

**Corollary 4.2.** Any sequence in $\mathcal{F}_r(\mathbb{R}^n)$ with monotonicity and boundedness w.r.t. $\preceq_n$ converges w.r.t. $\rho_n$.

**References**


