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Sequences of Fuzzy Sets on $\mathbb{R}^n$

千葉大学教育学部 蔵野正美 (Masami KURANO)
千葉大学理学部 安田正實 (Masami YASUDA)
千葉大学理学部 中神潤一 (Jun-ichi NAKAGAMI)
北九州大学経済学部 吉田祐治 (Yuji YOSHIDA)

Abstract

In this paper, we study the convergence of a sequence of fuzzy sets on $\mathbb{R}^n$ which in monotone w.r.t. a pseudo order $\preccurlyeq_K$ induced by a closed convex cone K. Our study is carried out by restricting the class of fuzzy sets into the subclass in which $\preccurlyeq_K$ becomes a partial order and a monotone convergence theorem is approved. This restricted subclass of fuzzy sets is created and characterized in the concept of a determining class.

Keywords: Pseudo-order, fuzzy max order, multidimensional fuzzy sets, monotone convergence theorem, determining class, rectangle-type fuzzy sets.

1. Introduction and notations

In our previous paper [3], we have introduced a pseudo order, $\preccurlyeq_K$, in the class of fuzzy sets, which is natural extension of fuzzy max order (cf. [2], [6]) in fuzzy numbers on $\mathbb{R}$ and induced by a closed convex cone K in $\mathbb{R}^n$. For a lattice-structure of the fuzzy max order, see [1], [10]. Here, we study the convergence of a sequence of fuzzy sets on $\mathbb{R}^n$ which is monotone w.r.t. a pseudo order $\preccurlyeq_K$. Our study is done by restricting the class of fuzzy sets into the subclass in which $\preccurlyeq_K$ becomes a partial order and a monotone convergence theorem is approved. This restricted subclass of fuzzy sets is created and characterized in the concept of a determining class.

In the remainder of this section, we will give some notations and review a vector ordering of $\mathbb{R}^n$ by a convex cone. In Section 2, a pseudo order of fuzzy sets on $\mathbb{R}^n$ is reviewed referring to our previous paper [3]. In Section 3, we introduce a concept determining class and give a convergence theorem for convex compact subclass $\mathbb{R}^n$. These results are applied to obtain a monotone convergence theorem for fuzzy sets on $\mathbb{R}^n$ in Section 4.

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}^n$ an n-dimensional Euclidean space. We write fuzzy sets on $\mathbb{R}^n$ by their membership functions $\tilde{s}:\mathbb{R}^n\to[0,1]$ (see Novák [5] and Zadeh [9]). The α-cut ($\alpha\in[0,1]$) of the fuzzy set $\tilde{s}$ on $\mathbb{R}^n$ is defined as

$$\tilde{s}_\alpha := \{x\in\mathbb{R}^n \mid \tilde{s}(x)\geq\alpha\}\ (\alpha>0) \quad\text{and}\quad \tilde{s}_0 := \mathrm{cl}\{x\in\mathbb{R}^n \mid \tilde{s}(x)>0\},$$

where cl denotes the closure of the set. A fuzzy set $\tilde{s}$ is called convex if

$$\tilde{s}(\lambda x+(1-\lambda)y)\geq\tilde{s}(x)\wedge\tilde{s}(y) \quad x,y\in\mathbb{R}^n,\ \lambda\in[0,1],$$

where $a \wedge b = \min\{a, b\}$. Note that $\tilde{s}$ is convex iff the α-cut $\tilde{s}_\alpha$ is a convex set for all $\alpha \in [0,1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all convex fuzzy sets whose membership functions $\tilde{s} : \mathbb{R}^n \to [0,1]$ are upper-semicontinuous and normal ($\sup_{x\in\mathbb{R}^n} \tilde{s}(x) = 1$) and have a compact support. When the one-dimensional case $n = 1$, the fuzzy sets are called fuzzy numbers and $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy numbers.

Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all compact convex subsets of $\mathbb{R}^n$, and $\mathcal{C}_r(\mathbb{R}^n)$ be the set of all rectangles in $\mathbb{R}^n$. For $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$, we have $\tilde{s}_\alpha \in \mathcal{C}(\mathbb{R}^n)$ ($\alpha \in [0,1]$). We write a rectangle in $\mathcal{C}_r(\mathbb{R}^n)$ by

$$[x, y] = [x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_n, y_n]$$

for $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ with $x_i \leq y_i$ ($i = 1, 2, \cdots, n$). For the case of $n = 1$, $\mathcal{C}(\mathbb{R}) = \mathcal{C}_r(\mathbb{R})$ and it denotes the set of all bounded closed intervals. When $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$ satisfies $\tilde{s}_\alpha \in \mathcal{C}_r(\mathbb{R}^n)$ for all $\alpha \in [0,1]$, $\tilde{s}$ is called a rectangle-type. We denote by $\mathcal{F}_r(\mathbb{R}^n)$ the set of all rectangle-type fuzzy sets on $\mathbb{R}^n$. Obviously $\mathcal{F}_r(\mathbb{R}) = \mathcal{F}(\mathbb{R})$.

The definitions of addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ are as follows: For $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda \geq 0$,

$$(\tilde{m} + \tilde{n})(x) := \sup_{x_1, x_2 \in \mathbb{R}^n;\ x_1 + x_2 = x} \{\tilde{m}(x_1) \wedge \tilde{n}(x_2)\}, \tag{1.1}$$

$$(\lambda \tilde{m})(x) := \begin{cases} \tilde{m}(x/\lambda) & \text{if } \lambda > 0 \\ I_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbb{R}^n), \tag{1.2}$$

where $I_{\{\cdot\}}(\cdot)$ is an indicator. By using set operations $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for any non-empty sets $A, B \subset \mathbb{R}^n$, the following holds immediately.

$$(\tilde{m} + \tilde{n})_\alpha := \tilde{m}_\alpha + \tilde{n}_\alpha \quad \text{and} \quad (\lambda \tilde{m})_\alpha = \lambda \tilde{m}_\alpha \quad (\alpha \in [0,1]). \tag{1.3}$$

Let K be a non-empty cone of $\mathbb{R}^n$. Using this K, we can define a pseudo-order relation $\preccurlyeq_K$ on $\mathbb{R}$ by $x \preccurlyeq_K y$ iff $y - x \in K$. Let $\mathbb{R}^n_+$ be the subset of entrywise non-negative elements in $\mathbb{R}^n$. When $K = \mathbb{R}^n_+$, the order $\preccurlyeq_K$ will be denoted by $\preccurlyeq_n$ and $x \preccurlyeq_n y$ means that $x_i \leq y_i$ for all $i = 1, 2, \cdots, n$, where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$.

2. A pseudo-order on $\mathcal{F}(\mathbb{R}^n)$

In this section, we review a pseudo order introduced by [3]. Henceforth we assume that the convex cone $K \subset \mathbb{R}^n$ is given. A pseudo order $\preccurlyeq_K$ on $\mathcal{C}(\mathbb{R}^n)$ is defined, whose idea is based a set-relation treated in [4], as follows.

For $A, B \in \mathcal{C}(\mathbb{R}^n)$, $A \preccurlyeq_K B$ means the following (C.a) and (C.b) :

(C.a) For any $x \in A$, there exists $y \in B$ such that $x \preccurlyeq_K y$.

(C.b) For any $y \in B$, there exists $x \in A$ such that $x \preccurlyeq_K y$.

When $K = \mathbb{R}^n_+$, the relation $\preccurlyeq_K$ on $\mathcal{C}(\mathbb{R}^n)$ will be written simply by $\preccurlyeq_n$ and for $[x, y]$, $[x', y'] \in \mathcal{C}_r(\mathbb{R}^n)$, $[x, y] \preccurlyeq_n [x', y']$ means $x \preccurlyeq_n x'$ and $y \preccurlyeq_n y'$.

Using a pseudo order $\preccurlyeq_K$ on $\mathcal{C}(\mathbb{R}^n)$, a pseudo order $\preccurlyeq_K$ on $\mathcal{F}(\mathbb{R}^n)$ is defined as follows. For $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\tilde{s} \preccurlyeq_K \tilde{r}$ means the following (F.a) and (F.b):

(F.a) For any $x \in \mathbb{R}^n$, there exists $y \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\tilde{s}(x) \le \tilde{r}(y)$.

(F.b) For any $y \in \mathbb{R}^n$, there exists $x \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\tilde{s}(x) \ge \tilde{r}(y)$.

In [3], for $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n)$, it is shown that $\tilde{s} \preccurlyeq_K \tilde{r}$ if and only if $\tilde{s}_\alpha \preccurlyeq_K \tilde{r}_\alpha$ on $\mathcal{C}(\mathbb{R}^n)$ for all $\alpha \in [0,1]$. Define the dual cone of a cone K by

$$K^+ := \{a \in \mathbb{R}^n \mid a \cdot x \ge 0 \text{ for all } x \in K\},$$

where $x \cdot y$ denotes the inner product on $\mathbb{R}^n$ for $x, y \in \mathbb{R}^n$. For a subset $A \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, we define

$$a \cdot A := \{a \cdot x \mid x \in A\} \ (\subset \mathbb{R}). \tag{2.1}$$

The equation (2.1) means the projection of A on the extended line of the vector a if $a \cdot a = 1$. It is trivial that $a \cdot A \in \mathcal{C}(\mathbb{R})$ if $A \in \mathcal{C}(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$.

Lemma 2.1([3]). *Let $A, B \in \mathcal{C}(\mathbb{R}^n)$. $A \preccurlyeq_K B$ on $\mathcal{C}(\mathbb{R}^n)$ if and only if $a \cdot A \preccurlyeq_1 a \cdot B$ on $\mathcal{C}(\mathbb{R})$ for all $a \in K^+$, where $\preccurlyeq_1$ is the natural order on $\mathcal{C}(\mathbb{R})$.*

For $a \in \mathbb{R}^n$ and $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$, we define a fuzzy number $a \cdot \tilde{s} \in \mathcal{F}(\mathbb{R})$ by

$$a \cdot \tilde{s}(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{a \cdot \tilde{s}_\alpha}(x)\}, \quad x \in \mathbb{R}. \tag{2.2}$$

where $1_D(\cdot)$ is the classical indicator function of a closed interval $D \in \mathcal{C}(\mathbb{R})$.

We define a partial relation $\preccurlyeq_M$ on $\mathcal{F}(\mathbb{R})$ as follows ([6]): For $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R})$, $\tilde{s} \preccurlyeq_M \tilde{r}$ means that $\tilde{s}_\alpha \preccurlyeq_1 \tilde{r}_\alpha$ for all $\alpha \in [0,1]$.

The following theorem gives the correspondence between the pseudo-order $\preccurlyeq_K$ on $\mathcal{F}(\mathbb{R}^n)$ and the fuzzy max order $\preccurlyeq_M$ on $\mathcal{F}(\mathbb{R})$.

Lemma 2.2([3]). *For $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\tilde{s} \preccurlyeq_K \tilde{r}$ if and only if $a \cdot \tilde{s} \preccurlyeq_M a \cdot \tilde{r}$ for all $a \in K^+$.*

Let ρ_n be the Hausdorff metric on $\mathcal{C}(\mathbb{R}^n)$, that is, for $A, B \in \mathcal{C}(\mathbb{R}^n)$, $\rho_n(A,B) = \max_{a \in A} d(a,B) \vee \max_{b \in B} d(b,A)$, where d is a metric in $\mathbb{R}^n$ and $d(x,Y) = \min_{y \in Y} d(x,y)$ for $x \in \mathbb{R}^n$ and $Y \in \mathcal{C}(\mathbb{R}^n)$. It is well-known that $(\mathcal{C}(\mathbb{R}^n), \rho_n)$ is a complete separable metric space. A sequence $\{D_\ell\}_{\ell=1}^\infty \subset \mathcal{C}(\mathbb{R}^n)$ converges to $D \in \mathcal{C}(\mathbb{R}^n)$ w.r.t. ρ_n if $\rho_n(D_\ell, D) \to 0$ as $\ell \to \infty$.

Definition(Convergence of fuzzy set, [8]).
For $\{\tilde{s}_\ell\}_{\ell=1}^\infty \subset \mathcal{F}(\mathbb{R}^n)$ and $\tilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\tilde{s}_\ell$ converges to $\tilde{r}$ w.r.t. ρ_n if $\rho_n(\tilde{s}_{\ell,\alpha}, \tilde{r}_\alpha) \to 0$ as $\ell \to \infty$ except at most countable $\alpha \in [0,1]$.

In the sequel, the monotone convergence theorems for fuzzy sets are given under the concept of the above convergence.

3. Sequences in $\mathcal{C}(\mathbb{R}^n)$

In this section, restricting $\mathcal{C}(\mathbb{R}^n)$ into the subclass by use of the concept of determining class, we prove the monotone convergence theorem for $\mathcal{C}(\mathbb{R}^n)$.

Let $\mathfrak{L} \subset \mathcal{C}(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$. Then we say that $\mathfrak{L}$ is determined by A if $a \cdot D = a \cdot F$ for $a \in A$ and $D, F \in \mathfrak{L}$ implies $D = F$.

Note that $\mathfrak{L}$ determined by some $A \subset \mathbb{R}^n$ is closed w.r.t. ρ_n. Obviously, $\mathcal{C}_r(\mathbb{R}^n)$ is determined by $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$. Also, by the separation theorem, $\mathcal{C}(\mathbb{R}^n)$ is determined $\mathbb{R}^n$.

Theorem 3.1. *Let K be a closed convex cone of $\mathbb{R}^n$. Suppose that $\mathfrak{L} \subset \mathcal{C}(\mathbb{R}^n)$ is determined by K^+. Then, the pseudo order $\preccurlyeq_K$ is a partial one in the restricted class $\mathfrak{L}$.*

Proof. It suffices to show that $\preccurlyeq_K$ is antisymmetric in $\mathfrak{L}$. Let $D, F \in \mathfrak{L}$ satisfy that $D \preccurlyeq_K F$ and $F \preccurlyeq_k D$. By Lemma 2.1, $aD \preccurlyeq_1 aF$ and $aF \preccurlyeq_1 aD$ for all $a \in K^+$. Since $\preccurlyeq_1$ ia a partial order, $aF = aD$ for all $a \in K^+$, which implies $F = D$ from the determining property of K^+. Q.E.D.

The sequence $\{D_\ell\}_{\ell=1}^{\infty} \subset \mathcal{C}(\mathbb{R}^n)$ is said to be bounded w.r.t. $\preccurlyeq_K$ if there exists $F, D \in \mathcal{C}(\mathbb{R}^n)$ such that $F \preccurlyeq_K D_\ell \preccurlyeq_K D$ for all $\ell \geq 1$ and said to be monotone w.r.t. $\preccurlyeq_K$ if $D_1 \preccurlyeq_K D_2 \preccurlyeq_K \cdots$. Then, as an application of Theorem 3.1, we have the following, whose proof is omitted.

Theorem 3.2. *Let K be a closed convex cone of $\mathbb{R}^n$ with $K^+ \cap (\mathbb{R}^n_+)^o \neq \emptyset$. Suppose that $\mathfrak{L} \subset \mathcal{C}(\mathbb{R}^n)$ is determined by K^+. Then, any sequence $\{D_\ell\}_{\ell=1}^{\infty} \subset \mathfrak{L}$ which is monotone and bounded w.r.t. $\preccurlyeq_K$ converges w.r.t. ρ_n, where A^o is a set of inner points in A.*

The following results are concerned with the scalarization method.

Corollary 3.1. *Let $K = \{\lambda a \mid \lambda \geq 0\}$ for some $a \in \overline{\mathbb{R}^n_+}$. Then, any sequence in $\mathcal{C}(\mathbb{R}^n)$ with monotonicity and boundedness w.r.t. $\preccurlyeq_K$ converges w.r.t. ρ_n.*

Corollary 3.2. *Any sequence in $\mathcal{C}_r(\mathbb{R}^n)$ with monotonicity and boundedness w.r.t. $\preccurlyeq_n$ converges w.r.t. ρ_n.*

4. Sequences in $\mathcal{F}(\mathbb{R}^n)$

In this section, applying the results in Section 3, we give the monotone convergence theorem in $\mathcal{F}(\mathbb{R}^n)$. Let $\tilde{\mathfrak{L}} \subset \mathcal{F}(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$. Then we say that $\tilde{\mathfrak{L}}$ is determined by A if $a\tilde{s} = a\tilde{r}$ for all $a \in A$ and $\tilde{s}, \tilde{r} \in \tilde{\mathfrak{L}}$ implies $\tilde{s} = \tilde{r}$.

Note that $\tilde{\mathfrak{L}}$ determined by some $A \subset \mathbb{R}^n$ is closed in the convergence given in Definition 1. Applying Lemma 2.2, the same proof as Theorem 3.1 is useful in proving the following.

Theorem 4.1. *Let K be a closed convex cone of $\mathbb{R}^n$. Suppose that $\tilde{\mathfrak{L}} \subset \mathcal{F}(\mathbb{R}^n)$ is* *determined by K^+. Then, a pseudo order $\preccurlyeq_K$ is a partial order in the restricted class $\tilde{\mathfrak{L}}$.*

In order to get the convergence theorem, we need the concept of directionality given in [8]. Put the surface of the unit ball by $U := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$. Let $V \subset U$. Then, for $D, D' \in \mathcal{C}(\mathbb{R}^n)$ with $D \subset D'$, we call D' V-directional to D (written by $D' \supset_V D$) if there exists a real $\lambda > 0, y \in D$ and $z \in D'$ such that
(i) $d(z, y) = \rho_n(D', D)$ and (ii) $z - y = \lambda v$ for some $v \in V$.

Definition 2. *Let $V \subset U \subset \mathbb{R}^n$. For $\tilde{s} \in \mathcal{F}(\mathbb{R}^n)$, $\tilde{s}$ is called V-directional if $\tilde{s}_\alpha \supset_V \tilde{s}_{\alpha'}$ for $0 \leq \alpha \leq \alpha' \leq 1$.*

Theorem 4.2. *Let K be a closed convex cone of $\mathbb{R}^n$ with $K^+ \cap (\mathbb{R}^n_+)^o \neq \emptyset$. Suppose that $\tilde{\mathfrak{L}} \subset \mathcal{F}(\mathbb{R}^n)$ is determined by K^+. Then, any monotone and bounded sequence $\{\tilde{s}_\ell\}_{\ell=1}^\infty \subset \tilde{\mathfrak{L}}$ with $\tilde{s}_\ell$ $(\ell \geq 1)$ is V-directional for a finite set V converges w.r.t. ρ_n.*

Corollary 4.1. *Let $K = \{\lambda a \mid \lambda \geq 0\}$ for some $a \in \overline{\mathbb{R}^n_+}$. Then, any sequence$\{\tilde{s}_\ell\} \subset \mathcal{F}(\mathbb{R}^n)$ satisfying that it is monotone and bounded w.r.t. $\preccurlyeq_K$ and $\tilde{s}_\ell$ $(\ell \geq 1)$ is V-directional for a finite set V converges w.r.t. ρ_n.*

Corollary 4.2. *Any sequence in $\mathcal{F}_r(\mathbb{R}^n)$ with monotonicity and boundedness w.r.t. $\preccurlyeq_n$ converges w.r.t. ρ_n.*

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