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AN OPTIMAL STOPPING PROBLEM FOR A GEOMETRIC BROWNIAN MOTION WITH POISSONIAN JUMPS

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SUMMARY

This paper examines an optimal stopping problem for a geometric Brownian motion with random jumps. It is assumed that jumps occur according to a time-homogeneous Poisson process and the proportions of these sizes are independent and identically distributed. The objective is to find an optimal stopping time of maximizing the expected discounted terminal reward which is defined as a power function of the stopped state. By applying what is called the smooth pasting technique (Dixit [2], and Dixit and Pindyck [3]) and taking a martingale approach, we derive almost explicitly an optimal stopping rule of a threshold type and the optimal value function of the initial state. That is, we express the critical state of the optimal stopping region and the optimal value function by formulae which include only given problem parameters except an unknown to be uniquely determined by a nonlinear equation.

KEY WORDS: geometric Brownian motion, Poisson jump process, expected discounted terminal reward, optimal stopping, smooth pasting.

1. INTRODUCTION

Dixit [2], Dixit and Pindyck [3], and their papers cited therein formulate various investment problems under uncertainty as optimal stopping problems and apply what is called the smooth pasting technique to derive optimal value functions and optimal stopping rules. Although they emphasize its power and easiness, it seems that its mathematical validity and scope are not sufficiently discussed.

This paper examines an optimal stopping problem for a geometric Brownian motion with random jumps. It is assumed that jumps occur according to a time-homogeneous Poisson process and the relative amplitudes of these sizes are independent and identically distributed. The objective is to find an optimal stopping time of maximizing the expected discounted terminal reward which is defined as a power function of the stopped state. By applying the smooth pasting technique, we derive almost explicitly an optimal stopping rule of a threshold type and the optimal value function of the initial state. That is, we express the critical state of the optimal stopping region and the optimal value function by formulae which include only given problem parameters except an unknown to be uniquely determined by a solution of a nonlinear equation.

Although Dixit [2], Dixit and Pindyck [3], and their papers cited therein formulate various investment problems under uncertainty as optimal stopping problems similar to this paper and derive optimal value functions

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and optimal investment policies by applying the smooth pasting technique, it seems that its mathematical validity is not sufficiently discussed. In this paper, by taking a martingale approach, we shows that it is indeed mathematically valid under a set of some mild conditions on the parameters of the problem.

2. DESCRIPTION OF PROBLEM

Let $(\Omega, \mathcal{F}, P)$ denote the underlying probability space, and consider the following random elements which are defined on this space:

$\mathcal{W} = (W_t;\ t \in \mathcal{R}_+)$: a standard Brownian motion.

$\mathcal{N} = (N_t;\ t \in \mathcal{R}_+)$: a time–homogeneous (right–continuous) Poisson counting process with intensity $\lambda \geq 0$.

$\mathcal{U} = (U_i;\ i \in \mathcal{Z}_{++})$: a sequence of independent and identically distributed $(-1, +\infty)$–valued random variables. Their generic random variable is denoted by U and their common cumulative distribution function is denoted by F_U. It is assumed that it has a finite mean m_U. That is, we assume that

$$F_U(-1) = 0; \quad m_U = E[U] = \int_{-1}^{+\infty} u dF_U(u) < +\infty. \tag{2.1}$$

Furthermore, we assume these random elements are mutually independent.

Now, we let

$\mathcal{T} = (T_i;\ i \in \mathcal{Z}_+)$: the sequence of the event times of the Poisson counting process $\mathcal{N}$ $(0 = T_0 \leq T_1 \leq \cdots)$

and consider a right–continuous $\mathcal{R}_{++}$–valued stochastic process $\mathcal{X} = (X_t;\ t \in \mathcal{R}_+)$ described as follows.

(D1) On the time interval $[T_i, T_{i+1})$ $(i \in \mathcal{Z}_+)$, for some constants μ and $\sigma \geq 0$, it follows the following stochastic differential equation:

$$dX_t = X_t\left(\mu dt + \sigma dW_t\right). \tag{2.2}$$

(D2) At every event time T_i $(i \in \mathcal{Z}_{++})$ of the Poisson counting process $\mathcal{N}$, $\mathcal{X}$ jumps in a random size whose proportion, i.e., relative amplitude to the state just before the jump is given by U_i, that is,

$$X_{T_i} = X_{T_i-}(1 + U_i). \tag{2.3}$$

Then, since the state X_t at time instant $t \in [T_i, T_{i+1})$ $(i \in \mathcal{Z}_+)$ is represented by

$$X_t = X_{T_i} \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}, \tag{2.4}$$

we can show, by induction in $i \in \mathcal{Z}_+$, for any time instant $t \in \mathcal{R}_+$,

$$X_t = X_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}\left[\prod_{i=1}^{N_t}(1 + U_i)\right] \tag{2.5}$$

(see, e.g., Lamberton and Lapeyre [5]). In the sequel, we denote the X_t of eq. (2.5) when the initial state is $X_0 = x$ $(\in \mathcal{R}_{++})$ by X_t^x for notational convenience.

For this state process $\mathcal{X}$, let $p > 0$, $q > 0$, and $\beta \geq 0$ be constants, and define the terminal reward function by

$$R(x) := px^\beta - q, \quad x \in \mathcal{R}_{++}. \tag{2.6}$$

Now, let us consider the optimal stopping problem whose objective is to find an optimal stopping time τ of attaining the following supremum of the expected discounted terminal reward:

$$v^*(x) := \sup_{\tau} E\left[e^{-\alpha\tau} R(X_\tau^x)\, 1_{\{\tau<+\infty\}}\right], \quad x \in \mathcal{R}_{++}, \tag{2.7}$$

where $\alpha > 0$ is a discount rate, and the sup of the right hand side of eq. (2.7) is taken over the set of all stopping times with respect to the state process $\mathcal{X}$.

Remark 2.1

(1) When $\beta = 0$, the above optimal stopping problem has the following trivial optimal stopping times.

(+) If $R(x) \equiv p - q \geq 0$ then $\tau^* = 0$, a.s. is an optimal stopping time, and the optimal value function is given by $v^*(x) \equiv p - q$;

(−) If $R(x) \equiv p - q \leq 0$ then $\tau^* = +\infty$, a.s. is an optimal stopping time, and the optimal value function is given by $v^*(x) \equiv 0$.

(2) Assuming that, at each time instant until the stopping time τ, a cost (rate per unit of time) is incurred dependently on the state process $\mathcal{X}$, we consider a seemingly more general criterion:

$$E\left[-\int_0^\tau e^{-\alpha s}(X_s^x)^\beta ds + e^{-\alpha\tau}\left\{p'(X_\tau^x)^\beta - q'\right\} 1_{\{\tau<+\infty\}}\right], \quad x \in \mathcal{R}_{++}. \tag{2.8}$$

Under a set of some mild integrability conditions, however, this could be reduced to a equivalent criterion of the form of eq. (2.7).

(3) As it will be seen later, only the positive part of the terminal reward function is relevant, so that we could take it as

$$R(x) := \left[px^\beta - q\right]_+, \quad x \in \mathcal{R}_{++}, \tag{2.9}$$

where, for a real number a, we define its positive part by $[a]_+ := \max\{a, 0\}$. □

3. ANALYSIS

We first introduce the infinitesimal generater L of the Markovian state process $\mathcal{X}$ as follows: for twice continuously differentiable function $w : \mathcal{R}_{++} \to \mathcal{R}$,

$$[Lw](x) := \lim_{h\downarrow 0+} \frac{e^{-\alpha h} E\left[w(X_h^x)\right] - w(x)}{h}, \quad x \in \mathcal{R}_{++}. \tag{3.1}$$

Then, by Itô formula and properties of Poisson process, we have

$$[Lw](x) = \frac{1}{2}\sigma^2 x^2 w''(x) + \mu x w'(x) - \alpha w(x) + \lambda\left(\int_{-1}^{+\infty} w((1+u)x)\, dF_U(u) - w(x)\right) \tag{3.2}$$

provided that

$$\int_{-1}^{+\infty} w((1+u)x)\, dF_U(u) \;\; (= E\left[w((1+U)x)\right]) \tag{3.3}$$

is well defined.

Now, let us consider a functional equation

$$[Lw](x) = 0, \quad x \in \mathcal{R}_{++}, \tag{3.4}$$

where $w : \mathcal{R}_{++} \to \mathcal{R}$ is an unknown function to be determined. In order to solve this functional equation, for two real numbers a and b, we apply a trial solution function of the form

$$w(x) = ax^b, \quad x \in \mathcal{R}_{++}. \tag{3.5}$$

Substituting it into the functional equation (3.4), we have

$$\begin{aligned}
[Lw](x) &= \frac{1}{2}\sigma^2 x^2 \left(ab(b-1)x^{b-2}\right) + \mu x \left(abx^{b-1}\right) - \alpha \left(ax^b\right) + \lambda \left(\int_{-1}^{+\infty} \left(a\{(1+u)x\}^b\right) dF_U(u) - \left(ax^b\right) \right) \\
&= ax^b g(b) \\
&= w(x) g(b) \\
&= 0, \quad x \in \mathcal{R}_{++},
\end{aligned} \tag{3.6}$$

where the function $g : \mathcal{R} \to \mathcal{R}$ is defined by

$$g(b) := \frac{1}{2}\sigma^2 b^2 + \left(\mu - \frac{1}{2}\sigma^2\right) b - \alpha + \lambda \left(\int_{-1}^{+\infty} (1+u)^b dF_U(u) - 1 \right), \quad b \in \mathcal{R} \tag{3.7}$$

provided that

$$\int_{-1}^{+\infty} (1+u)^b dF_U(u) \quad \left(= E\left[(1+U)^b\right]\right) \tag{3.8}$$

is well defined.

By using this notation, we have

$$\begin{aligned}
E\left[(X_t^x)^b\right] &= x^b E\left[\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) bt + \sigma b W_t\right\}\right] E\left[\prod_{i=1}^{N_t} (1+U_i)^b\right] \\
&= x^b \exp\{(g(b) + \alpha)t\},
\end{aligned} \tag{3.9}$$

where we use the formulae:

$$E\left[\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) bt + \sigma b W_t\right\}\right] = \exp\left\{\left(\frac{1}{2}\sigma^2 b^2 + \left(\mu - \frac{1}{2}\sigma^2\right) b\right) t\right\}, \tag{3.10}$$

and

$$\begin{aligned}
E\left[\prod_{i=1}^{N_t} (1+U_i)^b\right] &= \sum_{n=0}^{+\infty} E\left[\prod_{i=1}^{n} (1+U_i)^b\right] P(N_t = n) \\
&= \sum_{n=0}^{+\infty} E\left[(1+U)^b\right]^n \frac{(\lambda t)^n \exp\{-\lambda t\}}{n!} \\
&= \exp\left\{\lambda \left(E\left[(1+U)^b\right] - 1\right) t\right\}.
\end{aligned} \tag{3.11}$$

We assume the followings.

Assumption 3.1

(A1)

$$g(1) = \mu - \alpha + \lambda m_U \leq 0. \tag{3.12}$$

□

Under this assumption, we see from eq. (3.9) that the discounted state process

$$\widetilde{\mathcal{X}} := \left(e^{-\alpha t} X_t^x;\ t \in \mathcal{R}_+\right) \tag{3.13}$$

becomes a super-martingale. In particular, if $g(1) = \mu - \alpha + \lambda m_U = 0$ then the process $\widetilde{\mathcal{X}}$ becomes a martingale.

Lemma 3.1 Let us assume (A1). Then, the nonlinear equation $g(b) = 0$ has two distinct real roots, the larger one, b_+ of which satisfies

$$b_+ \geq 1. \tag{3.14}$$

Proof. The function $g(b)$ is decomposed into the sum of two functions:

$$g_{\mathrm{D}}(b) := \frac{1}{2}\sigma^2 b^2 + \left(\mu - \frac{1}{2}\sigma^2\right) b - \alpha, \tag{3.15}$$

$$g_{\mathrm{J}}(b) := \lambda \left(\int_{-1}^{+\infty} (1+u)^b dF_U(u) - 1 \right). \tag{3.16}$$

Since the former g_{D} is a (strictly) convex quadratic function and the latter g_{P} consists of a mixture of (strictly) convex exponential functions $(1+u)^b$, $u \in (-1, +\infty)$, we assure that $g(b)$ is a strictly convex function. Furthermore, we have

$$g(0) = -\alpha < 0; \quad g(1) = \mu - \alpha + \lambda m_U \leq 0. \tag{3.17}$$

Therefore, the nonlinear equation $g(b) = 0$ has two distinct real roots b_- and b_+ such that $b_- < 0$ and $1 \leq b_+$ respectively. □

We also assume the followings.

Assumption 3.2

(A2)

$$0 < \beta < b_+. \tag{3.18}$$

□

Now, let us define a function $w^* : \mathcal{R}_{++} \to \mathcal{R}$ by

$$w^*(x) := \begin{cases} w(x) = a^* x^{b_+}, & 0 < x < x^*, \\ R(x) = px^\beta - q, & x^* \leq x, \end{cases} \tag{3.19}$$

where $a^* > 0$ and $x^* > 0$ are constants which are uniquely determined by the following simultaneous equations (see Dixit [2], Dixit and Pindyck [3]):

Value Matching Condition:

$$w(x^*) = R(x^*); \tag{3.20}$$

Smooth Pasting Condition:

$$w'(x^*) = R'(x^*). \tag{3.21}$$

That is,

$$a^* = q \left(\frac{q}{p}\right)^{-\frac{b_+}{\beta}} \frac{\beta}{b_+ - \beta} \left(\frac{b_+}{b_+ - \beta}\right)^{-\frac{b_+}{\beta}}; \quad x^* = \left(\frac{q}{p}\right)^{\frac{1}{\beta}} \left(\frac{b_+}{b_+ - \beta}\right)^{\frac{1}{\beta}}. \tag{3.22}$$

The next assumption assumes that the sizes of Poissonian jumps are always nonpositive.

Assumption 3.3

(A3)

$$F_U(0) = 0. \tag{3.23}$$

□

Further we assume the followings:

Assumption 3.4

(A4)

$$\frac{1}{2}\sigma^2\beta + \left(\mu - \frac{1}{2}\sigma^2 - \frac{\alpha}{b_+}\right) \leq 0. \tag{3.24}$$

□

Remark 3.2 Although the physical and/or economical meaning of the condition (A4) is not clear, when $\beta = 1$, that is, the terminal reward function $R(x)$ is an affine function, (A4) becomes a more simple condition:

$$\mu - \frac{\alpha}{b_+} \leq 0. \tag{3.25}$$

Under the condition (A2), we could say that the above condition is slightly stronger than the condition:

$$\mu - \alpha \leq 0. \tag{3.26}$$

Furthermore, since the assumption (A3) implies

$$m_U \leq 0, \tag{3.27}$$

ineq. (3.26) implies the condition (A1):

$$\mu - \alpha + \lambda m_U \leq 0. \tag{3.28}$$

That is, when $\beta = 1$, (A2), (A3), and (A4) imply (A1). □

Lemma 3.2 Let us assume (A1), (A2), (A3), and (A4). Then, the function $w^* : \mathcal{R}_{++} \to \mathcal{R}$ satisfies the following properties (P1), (P2), (P4), (P5), and (P3):

(P1) For any $x \in \mathcal{R}_{++}$ and $t \in \mathcal{R}_+$,

$$E\left[|w^*(X_t^x)|\right] < +\infty; \quad E\left[\int_0^t e^{-\alpha s}\,|[Lw^*](X_s^x)|\,ds\right] < +\infty. \tag{3.29}$$

(P2) For any $x \in \mathcal{R}_{++}$,

$$w^*(x) \geq R(x). \tag{3.30}$$

(P3) $w^*(x)$ is strictly increasing in x.

(P4) For any $x \in \mathcal{R}_{++}$ $(x \neq x^*)$,

$$[Lw^*](x) \leq 0. \tag{3.31}$$

(P5) For any $x \in \mathcal{R}_{++}$, either of ineqs. (3.30) or (3.31) holds with equality.

Proof.

(P1) Straightforward from eq. (3.9).

(P2) Define a function $h : \mathcal{R}_{++} \to \mathcal{R}$ by the difference of the two functions w and R, that is,

$$h(x) := w(x) - R(x) = a^* x^{b_+} - \left(px^\beta - q\right). \tag{3.32}$$

Then, differentiating $h(x)$ with respect to x, we have

$$\begin{aligned} h'(x) &:= w'(x) - R'(x) \\ &= a^* b_+ x^{b_+ - 1} - p\beta x^{\beta - 1} \\ &= x^{\beta-1}\left(a^* b_+ x^{b_+ - \beta} - p\beta\right). \end{aligned} \tag{3.33}$$

Since, by the assumption (A2), $a^* b_+ x^{b_+ - \beta} - p\beta$ strictly increases from $-p\beta$ to $+\infty$ as x moves from 0 to $+\infty$, and its unique zero point is x^*, the sign of the derivative $h'(x)$ changes from $-$ to $+$ as x moves from 0 to $+\infty$, and its unique zero point is x^*. Therefore, $h(x)$ first strictly decreases from q to 0 as x moves from 0 to x^*, and it then strictly increases from 0 to $+\infty$ as x moves from x^* to $+\infty$. That is,

$$w(x) \left\{ \begin{matrix} = \\ > \end{matrix} \right\} R(x), \quad x \left\{ \begin{matrix} = \\ \neq \end{matrix} \right\} x^*. \tag{3.34}$$

Accordingly,

$$w^*(x) = \begin{cases} w(x) > R(x), & 0 < x < x^*, \\ R(x) = w(x), & x = x^*, \\ R(x) < w(x), & x^* < x. \end{cases} \tag{3.35}$$

(P3) Obvious from the definition (3.19) of the function w^*.

(P4) (a) For $0 < x < x^*$, since $w^*(x) = w(x)$ from the proof of (P2), we have

$$\begin{aligned} [Lw^*](x) &= \frac{1}{2}\sigma^2 x^2 w^{*\prime\prime}(x) + \mu x w^{*\prime}(x) - \alpha w^*(x) + \lambda\left(\int_{-1}^{+\infty} w^*((1+u)x)dF_U(u) - w^*(x)\right) \\ &= \frac{1}{2}\sigma^2 x^2 w''(x) + \mu x w'(x) - \alpha w(x) + \lambda\left(\int_{-1}^{0} w^*((1+u)x)dF_U(u) - w(x)\right) \\ &= \frac{1}{2}\sigma^2 x^2 w''(x) + \mu x w'(x) - \alpha w(x) + \lambda\left(\int_{-1}^{0} w((1+u)x)dF_U(u) - w(x)\right) \\ &= [Lw](x) \\ &= 0, \end{aligned} \tag{3.36}$$

where the second equality follows from (A3), and the third equality holds because $0 < x < x^*$ and $-1 < u \le 0$ imply $0 < (1+u)x \le x < x^*$ which in turn implies

$$w^*((1+u)x) = w((1+u)x), \quad 0 < x < x^*. \tag{3.37}$$

(b) For $x^* \le x$, since $w^*(x) = R(x)$ from the proof of (P2), we have

$$\begin{aligned} [Lw^*](x) &= \frac{1}{2}\sigma^2 x^2 w^{*\prime\prime}(x) + \mu x w^{*\prime}(x) - \alpha w^*(x) + \lambda\left(\int_{-1}^{+\infty} w^*((1+u)x)dF_U(u) - w^*(x)\right) \\ &= \frac{1}{2}\sigma^2 x^2 R''(x) + \mu x R'(x) - \alpha R(x) + \lambda\left(\int_{-1}^{0} w^*((1+u)x)dF_U(u) - R(x)\right) \\ &\le \frac{1}{2}\sigma^2 x^2 R''(x) + \mu x R'(x) - \alpha R(x) + \lambda\left(\int_{-1}^{0} R(x)dF_U(u) - R(x)\right) \\ &= \frac{1}{2}\sigma^2 x^2 R''(x) + \mu x R'(x) - \alpha R(x), \end{aligned} \tag{3.38}$$

where the second equality follows from (A3), and the third equality holds because $0 < x$ and $-1 < u \le 0$ imply $0 < (1+u)x \le x$ which, together with (P3), implies

$$w^*((1+u)x) \le w^*(x) = R(x), \quad x^* \le x. \tag{3.39}$$

Therefore, it suffices to show that

$$r(x) := \frac{1}{2}\sigma^2 x^2 R''(x) + \mu x R'(x) - \alpha R(x) \leq 0, \quad x^* \leq x. \tag{3.40}$$

By substituting $R(x) = px^\beta - q$ into the right hand side of eq. (3.40), we have

$$\begin{aligned} r(x) &= \frac{1}{2}\sigma^2 x^2 \left(p\beta(\beta-1)x^{\beta-2}\right) + \mu x \left(p\beta x^{\beta-1}\right) - \alpha\left(px^\beta - q\right) \\ &= px^\beta \left\{\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - \alpha\right\} + \alpha q \\ &= px^\beta g_{\mathrm{D}}(\beta) + \alpha q. \end{aligned} \tag{3.41}$$

Since (A4) implies

$$\begin{aligned} 0 &\geq \beta\left\{\frac{1}{2}\sigma^2\beta + \left(\mu - \frac{1}{2}\sigma^2 - \frac{\alpha}{b_+}\right)\right\} \\ &= \frac{1}{2}\sigma^2\beta^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\beta - \alpha\frac{\beta}{b_+} \\ &= g_{\mathrm{D}}(\beta) + \alpha\frac{b_+ - \beta}{b_+}, \end{aligned} \tag{3.42}$$

it holds that

$$g_{\mathrm{D}}(\beta) \leq 0. \tag{3.43}$$

Therefore, for $x^* \leq x$, we have

$$\begin{aligned} r(x) &= px^\beta g_{\mathrm{D}}(\beta) + \alpha q \\ &\leq px^{*\beta} g_{\mathrm{D}}(\beta) + \alpha q \\ &= q\frac{b_+}{b_+ - \beta}\left(g_{\mathrm{D}}(\beta) + \alpha\frac{b_+ - \beta}{b_+}\right) \\ &\leq 0, \quad x^* \leq x \end{aligned} \tag{3.44}$$

where the last inequality holds by ineq. (3.42).

(P5) Obvious from the proofs of (P2) and (P4). □

Theorem 3.1 Let us assume (A1), (A2), (A3), and (A4). The function $w^* : \mathcal{R}_{++} \to \mathcal{R}$ is the optimal value function, that is,

$$v^*(x) = w^*(x), \quad x \in \mathcal{R}_{++}. \tag{3.45}$$

Moreover, the optimal stopping region S^* ($\subset \mathcal{R}_{++}$) and the optimal stopping time τ^* are given by the followings:

$$S^* := \{x \in \mathcal{R}_{++} : w^*(x) = R(x)\} = [x^*, +\infty); \quad \tau^* := \inf\{t \in \mathcal{R}_+ : X_t^x \in S^*\}. \tag{3.46}$$

Proof. Using the function $w^* : \mathcal{R}_{++} \to \mathcal{R}$, we define a new stochastic process $\mathcal{M} = (M_t;\ t \in \mathcal{R}_+)$ by

$$M_t := e^{-\alpha t}w^*(X_t^x) - w^*(X_0^x) - \int_0^t e^{-\alpha s}[Lw^*](X_s^x)ds, \quad t \in \mathcal{R}_+ \tag{3.47}$$

Then, the process $\mathcal{M}$ becomes a 0–mean martingale (see, e.g., Davis [1]). Therefore, applying the optional sampling theorem for martingales, we have, for any stopping time τ for the process $\mathcal{X}$ and any $t \in \mathcal{R}_+$, the following so called Dynkin formula:

$$E\left[e^{-\alpha(\tau\wedge t)}w^*(X_{\tau\wedge t}^x)\right] = w^*(x) + E\left[\int_0^{\tau\wedge t} e^{-\alpha s}[Lw^*](X_s^x)ds\right]. \tag{3.48}$$

Thus, the property (P4) of function w^* implies

$$E\left[e^{-\alpha(\tau\wedge t)}w^*(X^x_{\tau\wedge t})\right] \le w^*(x). \tag{3.49}$$

Taking $\liminf_{t\to+\infty}$ of the both hand sides of eq. (3.49), we have, by Fatou lemma,

$$E\left[e^{-\alpha\tau}w^*(X^x_\tau)1_{\{\tau<+\infty\}}\right] \le w^*(x). \tag{3.50}$$

Moreover, since the function w^* has the property (P2), it holds that

$$E\left[e^{-\alpha\tau}R(X^x_\tau)1_{\{\tau<+\infty\}}\right] \le E\left[e^{-\alpha\tau}w^*(X^x_\tau)1_{\{\tau<+\infty\}}\right] \le w^*(x). \tag{3.51}$$

On the other hand, for the stopping time τ^* defined by eqs. (3.46), we have

$$E\left[e^{-\alpha(\tau^*\wedge t)}w^*(X^x_{\tau^*\wedge t})\right] = w^*(x). \tag{3.52}$$

By the properties (P2), (P4), and (P5) of the function w^* we assure that the stopping region S^* coincides with the interval $[x^*, +\infty)$. Furthermore, by the assumption (A3) and the property (P3) of the function w^*, it holds that

$$0 \le w^*(X^x_{\tau^*\wedge t}) \le w^*(x^*), \quad \text{a.s.} \tag{3.53}$$

Taking $\lim_{t\to+\infty}$ of the both hand sides of eq. (3.52), we have, by the bounded convergence theorem of Lebesgue,

$$\begin{aligned} w^*(x) &= E\left[e^{-\alpha\tau^*}w^*(X^x_{\tau^*})1_{\{\tau^*<+\infty\}}\right] \\ &= E\left[e^{-\alpha\tau^*}R(X^x_{\tau^*})1_{\{\tau^*<+\infty\}}\right], \end{aligned} \tag{3.54}$$

where the second quality follows from the fact that, on the event $\{\tau^* < +\infty\}$,

$$w^*(X^x_{\tau^*}) = R(X^x_{\tau^*}). \tag{3.55}$$

By ineq. (3.51) and eq. (3.54), we conclude that

$$v^*(x) = w^*(x) = E\left[e^{-\alpha\tau^*}R(X^x_{\tau^*})1_{\{\tau^*<+\infty\}}\right]. \tag{3.56}$$

□

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