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Repeated Game of Criminal vs Police -----Incomplete-Information Case

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Abstract. In this paper a conflict between a potential criminal offender and a law-enforcement authorities is investigated. Continueing the previous work [10] the model we study is a non-zero-sum two-period game under incomplete information, where each player doesn't know whether the opponent is unable to act, or can act at most two times during the two periods. We study the game in the Bayesian approach and derive Bayesian equilibria of three one-period games and one two-period game under various information structures, each in an explicit form depending on the parameter values of the game. It is shown that, just as our common sense suggests, the equilibrium goes to "act-act" choice-pair, (i.e. criminal commits crime, when police places an alert against him) as offender's illeagal income, comming from an unpunished crime, increases. Also we give a numerical example which corroborates the theoretical analysis.

1. The Game of Criminal vs Police under Incomplete Information.

The game is played as a repeated game over n periods between a potenti al criminal offender (hereafter called a criminal, or player I) and a law-enforcement authorities (hereafter called police, or player II) Being a repeated game implies that the fundamentals of the game are the same in each period. There are two pure strategies available in each period to player I: to commit a crime (C) and to act honestly (H). Similarly, player II has two pure strategies: to enforce the law (E) or to do nothing (N). If player I chooses H he earns his leagal income $r>0$ (dollars). If he chooses C, illegal income in amount of $\pi>0$, in addition to his legal income r, may be earned. However if I's crime is detected and arrested by II, I is punished by having to pay a fine in amount of $f>0$, and inprisoned until the end of the game. When caught in prison, I earns no income at all, of course.

If player II chooses E, with a cost of $c>0$ (dollars), he can (cannot), catch I's crime with probability $p(\bar{p}=1-p)$. In case that I commits crime that goes unpunished, a loss of $l>0$ is inflicted upon society.

So a single stage of this game has the game tree as shown by Figure 1, and is represented by a bimatrix game with payoff bimatrix(1).

We assume that $c<pl$ i.e. the strategy E for player II has a positive merit of choosing. This condition is very important as is seen in the proofs of the subsequent theorems.

We shall disacuss the n-stage game, where player I wants to commit crime at most k of n periods, and player II attempts to prevent I's illegal act by taking enforcement action at most m times during n periods. After each period is over, the outcome in that period becomes known to both players. The total payoff during n periods is the sum of the payoffs on each period. We assume that all of the above information is known to both players.

(1) bimatrix game (I) vs (II):

(I) \ (II)	E	N
C	$-pf+\bar{p}(r+\pi),\quad -(c+\bar{p}\ell)$	$r+\pi,\quad -\ell$
H	$r,\quad -c$	$r,\quad 0$

Let $\Gamma_{k,m}(n)$ denote the game described above. (n,k,m) denotes the state of the system in which players I and II possess k and m times to take actions, respectively, and they have n periods to go as their "mission time." Let $(u_{k,m}(n), v_{k,m}(n))$ represent the equilibrium values of this non-zero-sum n-stage game $\Gamma_{k,m}(n)$. Then theOptimality Equation of dynamic programming gives a system of equations

$(u_{k,m}(n), v_{k,m}(n)) =$ Eq.Val.

(2)

	E	N
C	$-pf+\bar{p}\{r+\pi+u_{k-1,m-1}(n-1)\},\quad -(c+\bar{p}\ell)+\bar{p}v_{k-1,m-1}(n-1)$	$r+\pi+u_{k-1,m}(n-1),\quad -\ell+v_{k-1,m}(n-1)$
H	$r+u_{k,m-1}(n-1),\quad -c+v_{k,m-1}(n-1)$	$r+u_{k,m}(n-1),\quad v_{k,m}(n-1)$

(if the equilibrium values exist uniquely), with the boundary conditions :

(2a) $\quad (u_{0,m}(n), v_{0,m}(n)) = (nr, 0), \quad \text{for } 1 \leqq m \leqq n,$

(2b) $\quad (u_{k,0}(n), v_{k,0}(n)) = (nr+k\pi, -k\ell), \quad \text{for } 1 \leqq k \leqq n,$

(2c) $\quad (u_{0,0}(n), v_{0,0}(n)) = (nr, 0), \quad \text{for } n \geqq 1,$

(2d) $\quad u_{k,m}(0) = v_{k,m}(0) = 0, \quad \forall k, m \geqq 0,$

and

(2e) $\quad (u_{k,m}(n), v_{k,m}(n)) = (u_{k',m'}(n), v_{k',m'}(n)), \text{ with } k'=k\wedge n,\ m'=m\wedge n.$

The four conditions (2a)~(2d) imply that: (a) If II has m times of law-enforcement and his opponent has none of the opportunity of violation, then the decision-pair H-N is repeated throughout the whole period, (b) If I has k times of violating law and his opponent cannot do anything becaause of lack of budget, then I chooses C and H k and $n-k$ times, respectively, during the n periods, (c) If both players do not have any law-violation and law-enforcement intentions, the decision -pair H-N is repeated throughout the whole period, and (d) The problem with n=1 reduces to the bimatrix game with payoff matrix (1).

If release from prison and a second offense are not taken into account, we need not consider large n, and the optimality equation (2), with (2a)~(2e), can be, in principle solved by backward induction. The two-period games $\Gamma_{n,n}(n)$, $\Gamma_{1,1}(n)$, $\Gamma_{1,n}(n)$ and $\Gamma_{n,1}(n)$, all for n=2, are explicitly solved in the previous works [1, 10].

In the present paper we shall investigate the incomplete-information version of the above game. Each player may not know his opponent's and/or his own allowed number of actions, and is able to estimate only by some probability dis tribution Suppose that (k, m) is a bivariate random variable with independent Bernoulli marginal distributions with parameters α and β (See Table 1). This distribution is assumed to be a common knowledge for each player.

Table 1 Bivariate type distribution

(3)

		(II) $m=m'$	(II) $m=m''$	
(I)	$k=k'$	$\bar{\alpha}\bar{\beta}$	$\bar{\alpha}\beta$	$\bar{\alpha}$
(I)	$k=k''$	$\alpha\bar{\beta}$	$\alpha\beta$	α
		$\bar{\beta}$	β	

We consider the information structure (IS) of our game model that is described by a statement as to "who knows what ?"

Let $I^{ij:de}$ be the IS such that

$$\left\{\begin{matrix} i \\ e \end{matrix}\right\} = 1\,(0), \text{ if player} \left\{\begin{matrix} \text{I} \\ \text{II} \end{matrix}\right\} \text{does (doesn't) know his "type"} \left\{\begin{matrix} k \\ m \end{matrix}\right\}.$$

$$\left\{\begin{matrix} j \\ d \end{matrix}\right\} = 1\,(0), \text{ if player} \left\{\begin{matrix} \text{I} \\ \text{II} \end{matrix}\right\} \text{does (doesn't) know his opponents' "type"} \left\{\begin{matrix} m \\ k \end{matrix}\right\}.$$

Among the possible $2^4=16$ ISs we shall focus our attention to the following four ISs.

(1°) $I^{11:11}$ i.e. complete information : Both players know both of k and m.

(2°) $I^{10:01}$ i.e. symmetric (or private) information : Each player knows his owntype, but not the opponents.

(3°) $I^{11:01}$ and $I^{10:11}$ i.e. asymmetric information ; One player knows both players' types whereas the other can know his own type only.

In each case of (1°)~(3°) the information structure is known to both players. The complete information case (1°) was solved in [1] and [10] and cases (2°) and (3°) will be solved in subsequent sections. In Section 2, one-period games with incomplete information where $k'=m'=0$ and $k''=m''=1$ are solved. In Section 3, two-period game with symmetric information, where $k'=m'=0$ and $k''=m''=2$ is solved.

Aplications of two-person games under incomplete information to real economic or social world have not-a-small library of references, among which are, for example, Karlin [4; Chapter 9] and Sakaguchi [8,9] in poker, Sakaguchi [7] in noisy duels, Engelbrecht-Wiggans [3] in auction and bidding, Chatterjee and Samuelson [2] in bargaining, Lipnowski and Shilony [5] in traffic control by city-police, and Milgrom and Roberts [6] in limit pricing and entry to monopolistic market. The common feature shared by these examples is that each player, while certain of his own situation in the game, has only probabilistic information concerning the true situation of his opponent.

2 One-Period Games under Inconplete Information.

The first model we shall investigate is a one-period game in the case where $k'=m'=0$ and $k''=m''=1$ in (3) That is, players are uncertain whether they can perform their action or not.

2a. Symmetric information $I^{10:01}$

Player I's strategy is denoted by $X=(\langle 0,1\rangle,\langle x,\bar{x}\rangle)$ with the meaning that I chooses H (adopts the mixed strategy $\langle x,\bar{x}\rangle$) when he knows that $k=0\,(1)$.

Similarly Player II's strategy is denoted by $Y=(\langle 0,1\rangle,\langle y,\bar{y}\rangle)$ meaning that II chooses N (adopts the mixed strategy $\langle y,\bar{y}\rangle$) when he knows that $m=0\,(1)$.

Let $K_1(X,Y\mid k)$ and $K_2(X,Y\mid m)$ be the expected payoffs to I and II. respect ively, under each of the two possible types of information they have, and when strategies X for I and Y for II are adopted. If there exist strategies $X^*=(\langle 0,1\rangle\langle x^*,\bar{x}^*\rangle)$ and $Y^*=(\langle 0,1\rangle\langle y^*,\bar{y}^*\rangle)$, such that

$$(4)\qquad K_1(X^*,Y^*|k)\geq K_1(X,Y^*|k),\quad k=0,1,\ \forall X;$$
$$K_2(X^*,Y^*|m)\geq K_2(X^*,Y|m),\quad m=0,1,\ \forall Y,$$

then (X^*,Y^*) is a Bayesian equilibriam, and Bayesian equilibrium values are

$$(5)\qquad \bar{\alpha}K_1(X^*,Y^*|k=0)+\alpha K_1(X^*,Y^*|k=1),\quad \text{and}\quad \bar{\beta}K_2(X^*,Y^*|m=0)+\beta K_2(X^*,Y^*|m=1)$$

for I and II, respectively.

We first prove

Theorem 1. Assume that $c<\alpha p\ell$ and let $\pi_1'=\frac{\beta p}{1-\beta p}(f+r)$. Then the solution to the game $G_{\alpha,\beta}(1)$ under the IS $I^{10:01}$ is:

Case	$0<\pi<\pi_1'$	$\pi>\pi_1'$
Bayesian eq. play	X^*-Y^* with $x^*=\frac{c}{\alpha p\ell}$, $y^*=\frac{\pi}{\beta p(f+r+\pi)}$	X^*-Y^* with $x^*=y^*=1$
Bayesian eq. values	$U=r$ $V=-c/p$	$U=r+\alpha(1-\beta p)(\pi-\pi_1')$ $V=-\{\beta c+\alpha(1-\beta p)\ell\}$

Note. Hereafter we shall omit considerations about the bordering cases (e.g. $\pi=\pi_1'$ in Theorem 1) where a continuum of equilibria in the first period and correspondingly in the whole game exist. Direct calculations show that the game value for the offender is continuous, non-decreasing in π, and the game value for the defender involves the mixing parameter $z\in[0,1]$ chosen arbitrarily by the offender.

Proof is omitted.

2b. Asymmetric Information $I^{11:01}$

2c. Asymmetric Information $I^{10:11}$

Theorems 2 and 3 with their proofs are omitted.

Summarizing the results obtained by Theorems 1~3, we observe the following facts:

1° For any fixed α and β Bayesian eq. value U, as a function of $\pi>0$ is continuous and non-decreasing, whereas V is piece-wise constant and has a jump at a particular one point of π.

2° Bayesian eq. value U, as a function of $\alpha\in[0,1]$ is increasing in α, and V, as a function of $\beta\in[0,1]$, is also increasing in β. That is, "type 1" gives more benefit than "type 0" for both players.

3° In the special case $\alpha=\beta=1$, all of Theorems 1~3 reduce to Theorem 1 of the previous paper [10], i.e. the one-period incomplete-information games $G_{1,1}(1)$ reduce to the one-period perfect-information game $\Gamma_{1,1}(1)$.

4° Let $U^{ij:de}$ ($V^{ij:de}$) be the Bayesian eq. value for player I (II) of the game $G_{\alpha,\beta}$ under the IS $I^{ij:de}$. Assume that $c<\alpha p\ell$. Then we have the inequalities

$$U^{10:11}\leq U^{10:01}\leq U^{11:01},$$

and

$$V^{11:01}\leq V^{10:01}\leq V^{10:11}.$$

A player who obtains his rival's information privately makes a profit, and a player who leaks his information to his rival makes a loss.

3. Two-Period Game under Symmetric Information.

The second model we shall discuss is a two-period game in the case where $k'=m'=0$ and $k''=m''=2$ in (3). That is, players are uncertain whether they can perform their action at most two times or none in a given two periods. We consider the IS $J^{10:01}$ only, ie. symmetric information that is the same as $I^{10:01}$, with $k''=m''=1$ replaced by $k''=m''=2$.

Player I's strayegy is denoted by $X(2)=(\langle 0,1\rangle,\langle x,\bar{x}\rangle;\text{Opt. Cont.})$ meaning that: In the first period I chooses H (adopts the mixed strategy $\langle x,\bar{x}\rangle$) when he knows that $k=0$ (2). And in the second period, if either he chose H or chose C but remained unpunished, he uses his one-period equilibrium strategy starting from the outcome resulted by the strategy-pair used in the first period.

Similarly player II's strategy is denoted by $Y(2)=(\langle 0,1\rangle,\langle y,\bar{y}\rangle;\text{Opt. Cont.})$ meaning that: In the first period, II chooses N (adopts the mixed strategy $\langle y,\bar{y}\rangle$) when he knows that $m=0$ (2). And in the second period, he uses his one-period eq. strategy starting from the outcome resulted in the first period.

First we note that in the second period of the game, posterior knowledge of the true "type-pair" is

$$\begin{matrix} & m=0 & m=2 \\ k=0 & 0 & 0 \\ k=2 & 0 & 1 \end{matrix}, \quad \begin{bmatrix} 0 & 0 \\ \bar{\beta} & \beta \end{bmatrix}, \quad \begin{bmatrix} 0 & \bar{\alpha} \\ 0 & \alpha \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar{\alpha}\bar{\beta} & \bar{\alpha}\beta \\ \alpha\bar{\beta} & \alpha\beta \end{bmatrix},$$

after the choice-pairs C'-E, C-N, H-E, and H-N, respectively, were played in the first period. Here C' means commiting crime without being punished. Hence, by the boundary condition (2e) corresponding to these four choice-pairs the second-period games are $G_{1,1}(1)$, $G_{1,\beta}(1)$, $G_{\alpha,1}(1)$ and $G_{\alpha,\beta}(1)$, under the IS $I^{10:01}$, respectively. Let $K_1(X(2)Y(2)\mid k)$ and $K_2(X(2),Y(2)\mid m)$ be the same as in Section 2a, with X and Y replaced by X(2) and Y(2).

Let the Bayesian equilibrium values for the game $G_{\alpha,\beta}(1)$ be $U_{\alpha,\beta}$ for I, and $V_{\alpha,\beta}$ for II. Also let

$$\tilde{U}=\begin{bmatrix} \bar{p}U_{1,1} & U_{1,\beta} \\ U_{\alpha,1} & U_{\alpha,\beta} \end{bmatrix} \quad \text{and} \quad \tilde{V}=\begin{bmatrix} \bar{p}V_{1,1} & V_{1,\beta} \\ V_{\alpha,1} & V_{\alpha,\beta} \end{bmatrix}$$

then we have

(13) $$K_1(X(2),Y^*(2)\mid k=0)=(0,1)(M_1+\tilde{U})\left\{\bar{\beta}\begin{bmatrix}0\\1\end{bmatrix}+\beta\begin{bmatrix}y^*\\\bar{y}^*\end{bmatrix}\right\}$$

(14) $$K_1(X(2),Y^*(2)\mid k=2)=(x,\bar{x})(M_1+\tilde{U})\left\{\bar{\beta}\begin{bmatrix}0\\1\end{bmatrix}+\beta\begin{bmatrix}y^*\\\bar{y}^*\end{bmatrix}\right\}$$

(15) $$K_2(X^*(2),Y(2)\mid m=0)=\{\bar{\beta}(0,1)+\beta(x^*,\bar{x}^*)\}(M_2+\tilde{V})\begin{bmatrix}0\\1\end{bmatrix}$$

and

(16) $$K_2(X^*(2),Y(2)\mid m=2)=\{\bar{\beta}(0,1)+\beta(x^*,\bar{x}^*)\}(M_2+\tilde{V})\begin{bmatrix}y^*\\\bar{y}^*\end{bmatrix}$$

where

$$M_1=\begin{bmatrix} -pf+\bar{p}(r+\pi), & r+\pi \\ r, & r \end{bmatrix} \quad \text{and} \quad M_2=\begin{bmatrix} -(c+\bar{p}l) & -l \\ -c & 0 \end{bmatrix}$$

are the same as in Section 2. If there exist $X^*(2)$ and $Y^*(2)$, such that

(17) $$\begin{aligned} &K_1(X^*(2),Y^*(2)\mid k)\geqq K_1(X(2),Y^*(2)\mid k), \quad k=0,2, \ \forall X(2), \\ &K_2(X^*(2),Y^*(2)\mid m)\geqq K_2(X^*(2),Y(2)\mid m), \quad m=0,2, \ \forall Y(2), \end{aligned}$$

then $(X^*(2),Y^*(2))$ is a Bayesian equilibrium-strategy-pair.

Theorem 4. Assume that parameters satiisfy

$$(18)\quad \begin{cases} \alpha\bar\beta < c/\ell < p\min(\alpha, \beta/\bar\beta), \\ \bar\alpha\bar\beta^2 + 2\bar\beta(c/\ell) - 1 < 0, \\ \text{and } f(\bar p) \text{ given by (21), is negative (i.e. } p \text{ is not so small.)} \end{cases}$$

Let $\pi_1' \equiv \frac{\beta p}{1-\beta p}(f+r)$ and

$$\pi_2' \equiv \frac{\beta p(f+2r) + \{\beta\bar p^2 + (\bar\alpha-\beta)(1-\beta p) + \alpha\beta\bar\beta p\}\pi_1'}{\beta\bar p^2 + (\bar\alpha+\beta)(1-\beta p) + \alpha\beta\bar\beta p}$$

Then we have $0 < \pi_1' < \pi_2'$ and the solution to the game $G_{\alpha,\beta}(2)$ under the IS $J^{10;c1}$ is :

Case	$0<\pi<\pi_1'$	$\pi_1'<\pi<\pi_2'$	$\pi>\pi_2'$
Bayesian eq-play in the 1st period	X^*-Y^* with $x^* = \frac{c(1-\bar\beta/p)}{c(1-\bar\beta/p)+\beta\bar p\ell-\bar\beta c}$, $y^* = \pi/\beta p(f+2r+\pi)$	X^*-Y^* with $x^* = \frac{c/\ell-\alpha\bar\beta}{c/\ell-\alpha\bar\beta-f(\bar p)}$; $f(\bar p)$, y^* are given by (21), (22) resp.	X^*-Y^* with $x^*=y^*=1$
Bayesian eq-values $\begin{bmatrix} U(2) \\ V(2) \end{bmatrix}$	$\begin{bmatrix} 2r \\ -(c/p+\beta\ell x^*) \end{bmatrix}$	$\begin{bmatrix} 2r+\alpha(1-\beta p-\beta\bar\beta p y^*) \\ -[\beta c+\alpha(1-\beta p)\ell+\beta\ell(1+\bar\alpha(1-\beta p))x^*] \end{bmatrix}$	$\begin{bmatrix} (24) \\ (25) \end{bmatrix}$

Proof. We have from Theorem 1

Case	$0<\pi<\pi_1'$	$\pi>\pi_1'$
$U_{1,1}$ $V_{1,1}$	r, $-c/p$	$r+\bar p(\pi-\pi_1)$, $-(c+\bar p\ell)$
$U_{1,\beta}$ $V_{1,\beta}$	r, $-c/p$	$r+(1-\beta p)(\pi-\pi_1')$, $-\{\beta c+(1-\beta p)\ell\}$
$U_{\alpha,1}$ $V_{\alpha,1}$	r, $-c/p$	$r+\alpha\bar p(\pi-\pi_1)$, $-(c+\alpha\bar p\ell)$
$U_{\alpha,\beta}$ $V_{\alpha,\beta}$	r, $-c/p$	$r+\alpha(1-\beta p)(\pi-\pi_1')$, $-\{\beta c+\alpha(1-\beta p)\ell\}$

(We previously noted in Section 2b, that $\pi_1-\pi_1' \left\{\begin{matrix} \geq \\ = \end{matrix}\right\} 0$, if $\left\{\begin{matrix} 0<\beta<1 \\ \beta=1 \end{matrix}\right\}$.)

Computing (13)~(16) and substitutjng the above values into them, we get

$$(13')\quad K_1(X(2)Y(2)\mid k=0) = r + U_{\alpha,\beta} + \beta(U_{\alpha,1}-U_{\alpha,\beta})y^* = \begin{cases} 2r, & \text{if } 0<\pi<\pi_1' \\ 2r+\alpha(\pi-\pi_1')(1-\beta p-\beta\bar\beta p y^*) & \text{if } \pi>\pi_1'. \end{cases}$$

$$(14')\quad K_1(X(2)Y^*(2)\mid k=2) = r+U_{\alpha,\beta} + (x,\bar x)\begin{bmatrix} \beta\{-p(f+r)+\bar p(\pi+U_{1,1})\} + \bar\beta(\pi+U_{1,\beta}) - U_{\alpha,\beta} & \pi+U_{1,\beta}-U_{\alpha,\beta} \\ \beta(U_{\alpha,1}-U_{\alpha,\beta}) & 0 \end{bmatrix}\begin{bmatrix} y^* \\ \bar y^* \end{bmatrix}$$

$$= \begin{cases} 2r+(x,\bar x)\begin{bmatrix} (1-\beta p)\pi-\beta p(f+2r) & \pi \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y^* \\ \bar y^* \end{bmatrix}, & \text{if } 0<\pi<\pi_1' \\ 2r+\alpha(1-\beta p)(\pi-\pi_1')+(x,\bar x)\begin{bmatrix} \pi-\beta p(f+\pi+2r) + \{\beta\bar p^2+(\bar\alpha-\beta)(1-\beta p)\}(\pi-\pi_1') - \alpha\beta\bar\beta p(\pi-\pi_1') & \pi+\bar\alpha(1-\beta p)(\pi-\pi_1') \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y^* \\ \bar y^* \end{bmatrix}, & \text{if } \pi>\pi_1, \end{cases}$$

$$(15')\quad K_2(X^*(2), Y(2)\mid m=0) = V_{\alpha\beta} + \beta(-\ell + V_{1\beta} - V_{\alpha\beta})x^* = \begin{cases} -c/p-\beta\ell x^*, & \text{if } 0<\pi<\pi_1' \\ -\{\beta c+\alpha(1-\beta p)\ell\} - \beta\{1+\bar\alpha(1-\beta p)\}\ell x^*, & \text{if } \pi>\pi_1' \end{cases}$$

and

(16') $$K_2(X^*(2),Y(2)\mid m=2)=\bar{V}_{\alpha\beta}+(x^*,\bar{x}^*)\begin{bmatrix}-c-\bar{V}_{\alpha\beta}-\beta\bar{p}(\ell-\bar{V}_{11})+\bar{\beta}\bar{V}_{\alpha 1} & \beta(-\ell+\bar{V}_{1\beta}-\bar{V}_{\alpha\beta}) \\ -c-\bar{V}_{\alpha\beta}+\beta\bar{V}_{\alpha 1} & 0\end{bmatrix}\begin{bmatrix}y\\ \bar{y}\end{bmatrix}$$

$$=\begin{cases}-c/p+(x^*,\bar{x}^*)\begin{bmatrix}-(\bar{\beta}c+\beta\bar{p}\ell) & -\beta\ell \\ -c(1-\bar{\beta}/p) & 0\end{bmatrix}\begin{bmatrix}y\\ \bar{y}\end{bmatrix}, & \text{if } 0<\pi<\pi_1' \\ -\{\beta c+\alpha(1-\beta p)\ell\}+(x^*,\bar{x}^*)\cdot \\ \begin{bmatrix}-(\bar{\beta}+1-\beta p)c+[\alpha(\beta\bar{p}+\bar{\beta}p)-\beta(\bar{p}+\bar{p}^2)]\ell & -\beta\ell(1+\bar{\alpha}(1-\beta p)) \\ -c+\alpha\bar{\beta}\ell & 0\end{bmatrix}\begin{bmatrix}y\\ \bar{y}\end{bmatrix}, & \text{if } \pi>\pi_1'.\end{cases}$$

Therefore x^*-y^* satisfying (17) will be derived as the eq. strategy-pair of the bimatrix games

(19)

$(1-\beta p)\pi-\beta p(f+2r),\ -(\bar{\beta}c+\beta\bar{p}\ell)$	$\pi,\ -\beta\ell$
$0,\ -c(1-\bar{\beta}/p)$	$0,\ 0$

if $0<\pi<\pi_1'$; and

(20)

$\pi-\beta p(f+\pi+2r)+\{\beta\bar{p}^2+(\bar{\alpha}-\beta)(1-\beta p)\}(\pi-\pi_1'),$ $-(\bar{\beta}+1-\beta p)c+\{\alpha(\beta\bar{p}+\bar{\beta}p)-\beta(\bar{p}+\bar{p}^2)\}\ell$	$\pi+\bar{\alpha}(1-\beta p)(\pi-\pi_1'),$ $-\beta\ell(1+\bar{\alpha}(1-\beta p))$
$-\alpha\beta\bar{\beta}p(\pi-\pi_1'),\ -c+\alpha\bar{\beta}\ell$	$0,\ 0$

if $\pi>\pi_1'$.

We consider the followjng threecases. Assume that $\alpha\bar{\beta}<c/\ell<p\min(\alpha,\beta/\bar{\beta})$.

Case 1. $0<\pi<\pi_1'$.

Since we have, from (18), $c/p\ell<\alpha\wedge(\beta/\bar{\beta})$ and $\beta>\bar{p}$ the equilibrium for the bimatrix (19) is obtained in the same way as followed in [10], with the result

$$x^*=\frac{c(1-\bar{\beta}/p)}{c(1-\bar{\beta}/p)+\beta p\ell-\bar{\beta}c},\qquad y^*=\frac{\pi}{\beta p(f+2r+\pi)},$$

and eq. values 0, for I, and $-\beta\ell x^*$, for II. It is evident that x^* and $y^*\in[0,1]$.

The eq. values for the game $G_{\alpha,\beta}(2)$ are

$$\bar{\alpha}K_1(X^*(2),Y^*(2)\mid k=0)+\alpha K_1(X^*(2),Y^*(2)\mid k=2)=\bar{\alpha}\cdot 2r+\alpha(2r+0)=2r$$

for I, and

$$\bar{\beta}K_2(X^*(2),Y^*(2)\mid m=0)+\beta K_2(X^*(2),Y^*(2)\mid m=2)$$
$$=-\bar{\beta}(c/p+\beta\ell x^*)+\beta(-c/p-\beta\ell x^*)=-(c/p+\beta\ell x^*).$$

Case 2. $\pi_1'<\pi<\pi_2'$.

Consider the bimatrix game (20). We have $-c+\alpha\bar{\beta}\ell<0\Leftrightarrow\alpha\bar{\beta}<c/\ell$, and

$$-(\bar{\beta}+1-\beta p)c+\{\alpha(\beta\bar{p}+\bar{\beta}p)-\beta(\bar{p}+\bar{p}^2)\}\ell>-\beta\ell\{1+\bar{\alpha}(1-\beta p)\}\Leftrightarrow$$

(21) $$f(\bar{p})\equiv\beta\bar{p}^2-(\bar{\alpha}\beta^2-\bar{\alpha}\beta-\alpha\bar{\beta}-\beta(c/\ell))\bar{p}+\bar{\alpha}\bar{\beta}^2+2\bar{\beta}(c/\ell)-1<0,$$

with

$$f(0)=\bar{\alpha}\bar{\beta}^2+2\bar{\beta}(c/\ell)-1<0 \text{ (by the assumption (18))},$$
$$f(1)=(1+\bar{\beta})(c/\ell)>0,$$

implying that $f(\bar{p})<0$ except for some small $p>0$.

Moreover

$$(22)\quad \pi-\beta p(f+\pi+2r)+\{\beta\bar{p}^2+(\bar{\alpha}-\beta)(1-\beta p)\}(\pi-\pi_1')<-\alpha\beta\bar{\beta}p(\pi-\pi_1')$$
$$\Leftrightarrow\quad \pi<\frac{\beta p(f+2r)+\{\beta\bar{p}^2+(\bar{\alpha}-\beta)(1-\beta p)+\alpha\beta\bar{\beta}p\}\pi_1'}{\beta\bar{p}^2+(\bar{\alpha}+\bar{\beta})(1-\beta p)+\alpha\beta\bar{\beta}p}\equiv\pi_2'.$$

Here we note that

$$\pi_2'-\pi_1'=(\text{positive facter})\times\{\beta p(f+2r)-(1-\beta p)\pi_1'\}$$
$$=(\text{positive facter})\times\beta pr>0.$$

Combinimg these inequalities, we assertain that the equilibrium for the bimatrix (19) for $\pi_1'<\pi<\pi_2'$ is obtained in the same way as in [10] ,with the result

$$(23)\quad x^*=\frac{c/\ell-\alpha\bar{\beta}}{c/\ell-\alpha\bar{\beta}-f(\bar{p})},\quad y^*=\frac{\pi+\bar{\alpha}(1-\beta p)(\pi-\pi_1')}{\pi+\bar{\alpha}(1-\beta p)(\pi-\pi_1')+\{\beta\bar{p}^2+(\bar{\alpha}+\bar{\beta})(1-\beta p)+\alpha\beta\bar{\beta}p\}\cdot(\pi_2'-\pi)}$$

and eq. values $-\alpha\beta\bar{\beta}p(\pi-\pi_1')\bar{y}$ for I, and $-\beta\ell(1+\bar{\alpha}(1-\beta p))x^*$ for II. It is evident that x^* and $y^*\in[0,1]$.

Thus we find, from (13)~(16′), that for $\pi_1'<\pi<\pi_2'$,

$$K_1(X^*(2),Y^*(2)\mid k)=2r+\alpha(\pi-\pi_1')(1-\beta p-\beta\bar{\beta}py^*),\quad k=0,2$$
$$K_2(X^*(2),Y^*(2)\mid m)=-[\beta c+\alpha(1-\beta p)\ell+\beta\ell(1+\bar{\alpha}(1-\beta p))x^*],\quad m=0,2,$$

and these common values are equal to the Bayesian eq. values for the game $G_{\alpha,\beta}(2)$.

Case 3. $\pi>\pi_2'$.

Since the bimatrix game (20) has the eq. point at C-E element, we have from(13′), (14′), and (22),

$$K_1(X^*(2),Y^*(2)\mid k=0)=2r+\alpha(1-\beta p-\beta\bar{\beta}p)(\pi-\pi_1'),$$
$$K_1(X^*(2),Y^*(2)\mid k=2)=2r+\alpha(1-\beta p-\beta\bar{\beta}p)(\pi-\pi_1')$$
$$+\{\beta\bar{p}^2+(\bar{\alpha}+\bar{\beta})(1-\beta p)+\alpha\beta\bar{\beta}p\}(\pi-\pi_2').$$

Also from (15), (16′) and (21), we get

$$K_2(X^*(2),Y^*(2)\mid m=0)=-[(1+c/\ell)\beta+(1-\bar{\alpha}\bar{\beta})(1-\beta p)]\ell.$$
$$K_2(X^*(2),Y^*(2)\mid m=2)=-[(1+c/\ell)\beta+(1-\bar{\alpha}\bar{\beta})(1-\beta p)]\ell-f(\bar{p}).$$

Thus the Bayesian eq. values are

$$(24)\quad \bar{\alpha}K_1(X^*(2),Y^*(2)\mid k=0)+\alpha K_1(X^*(2),Y^*(2)\mid k=2)$$
$$=2r+\alpha(1-\beta p-\beta\bar{\beta}p)(\pi-\pi_1')+\{\beta\bar{p}^2+(\bar{\alpha}+\bar{\beta})(1-\beta p)+\alpha\beta\bar{\beta}p\}(\pi-\pi_2'),$$

and

$$(25)\quad \bar{\beta}K_2(X^*(2),Y^*(2)\mid m=0)+\beta K_2(X^*(2),Y^*(2)\mid m=2)$$
$$=[(1+c/\ell)\beta+(1-\bar{\alpha}\bar{\beta})(1-\beta p)]\ell-\beta f(\bar{p}),$$

for I and II, respectively.

This completes the proof of Theorem 4. □

From Theorem 4, we remark the following facts :

1° When α and β tend to 1, the game reduces to $\Gamma_{2,2}(2)$ in the previous paper [10]. In fact we can find that $\pi_1' \to (p/\bar{p})(f+r) \equiv \pi_1$, $\pi_2' \to \pi_1 + (p/\bar{p}^2) r \equiv \pi_2$, and $f(\bar{p}) \to \bar{p}^2 + (c/\ell)\bar{p} - 1$, so that $f(\bar{p}) < 0$, for all $p \in (p_0, 1)$, where $p_0 \equiv (1/2)\{2 + c/\ell - \sqrt{4 + (c/\ell)^2}\}$. The solution to the game $G_{\alpha,\beta}(2)$, stated in Theorem4, becomes Theorem 2 in [10]

2° For any fixed α and β, Bayesian eq. value $U(2)$ as a function of $\pi > 0$ is continuous and is conjectured to be non-decreasing (note that y^* involes π in a complicate manner), whereas $V(2)$ is piecewise-constant (since $f(\bar{p})$ doesn't involve π), and has jumps at two particular points π_1' and π_2'.

3° It is not sure that $U(2)$ ($V(2)$) is non-decreasing in α (β) since x^* and y^* involve α and β in a complicate manner. This fact is largely different from the corresponding statement 2° at the end of Section 2, for one-period games.

4° Combining Theorems 4 and 1 we find the two-period eq. play under the IS $J^{10:01}$ as follows

If $0 < \pi < \pi_1'$ the eq. play is $X^*(2)$-$Y^*(2)$, with

$$(\underline{x}^*(2))_1 = \frac{c(1-\bar{\beta}/p)}{c(1-\bar{\beta}/p) + \beta p \ell - \bar{\beta} c} \quad \text{and} \quad (\underline{y}^*(2))_1 = \frac{\pi}{\beta p (f + 2r + \pi)}$$

in the firtst period, and $X^*(1)$-$Y^*(1)$ with

$(\underline{x}^*(1))_1$	$(\underline{y}^*(1))_1$	Choice-pair resulted by $\underline{x}^*(2)$-$\underline{y}^*(2)$
$c/p\ell$	$\pi/p(f+r+\pi)$	C′-E
$c/p\ell$	$\pi/\beta p(f+r+\pi)$	C-N
$c/(\alpha p \ell)$	$\pi/p(f+r+\pi)$	H-E
$c/(\alpha p \ell)$	$\pi/\beta p(f+r+\pi)$	H-N

in the second period.

If $\pi_1' < \pi < \pi_2'$ the eq. play is $X^*(2)$-$Y^*(2)$ with

$$(\underline{x}^*(2))_1 = \frac{c/\ell - \alpha\bar{\beta}}{c/\ell - \alpha\bar{\beta} - f(\bar{p})} \quad \text{and} \quad (\underline{y}^*(2))_1 \text{ given by (22)}$$

in the first period, and C-E is chosen in the second period.

If $\pi > \pi_2'$ the eq. play is to choose C-E in both periods.

Table 2 gives a numerical example of the solutions to the games $G_{\alpha,\beta}(1)$ and $G_{\alpha,\beta}(2)$ under symmetric information for the parameters $r = \frac{1}{2}$, $p = \frac{2}{3}$, $c = 1$, $f = \ell = 2$ and $\alpha = \beta = 4/5$. All conditions (18), stated in the begining of Theorem 4, are satisfied, since $p_0 = \frac{1}{4}(5 - \sqrt{17}) \fallingdotseq 0.219 < p = 2/3$ (in remark 1°) and $f(\bar{p}) = f(1/3) = -\frac{569}{1125}$. We obtain $\pi_1' = 20/7$ and $\pi_2' = 730/203$. For comparison we reproduced here the solutions to the complete-information games $\Gamma_{1,1}(1)$ and $\Gamma_{2,2}(2)$ from [10].

Table 2. Solutions to the games under symmetric information

Game	Case	Eq.play $(\underline{x}^*(2))_1 - (\underline{y}^*(2))_1$ in the first period	Bayesian eq. payoffs	Based on
$\Gamma_{1,1}(1)$	1	$3/4 - (3\pi/2\pi+5)$	$1/2 - (-3/2)$	Th. 1 in [10]
	2	C-E	$(1/6)(2\pi-7) - (-5/3)$	
$\Gamma_{2,2}(2)$	1	$3/7 - 3\pi/2(\pi+3)$	$1 - (-33/14)$	Th. 2 in [10]
	2′	$9/22 - 9\pi/8(\pi+1)$	$(1/3)(\pi-2) - (-82/33)$	
	2″	C-E	$(2/9)(2\pi-7) - (-20/9)$	
$G_{\alpha,\beta}(1)$	a	$15/16 - 15\pi/4(2\pi+5)$	$1/2 - (-3/2)$	Th. 1
	b	C-E in (7)	$\left(\frac{28}{75}\pi - \frac{17}{30}\right) - \left(-\frac{116}{75}\right)$	
$G_{\alpha,\beta}(2)$	a	$21/47 - 15\pi/8(\pi+3)$	$1 - \left(-\frac{1041}{470}\right)$	Th. 4
	b′	$\frac{765}{1903} - \left(\frac{15}{4}\right)\left(\frac{41\pi-10}{103\pi+145}\right)$	$\left[1+\left(\frac{4}{21}\right)\cdot\frac{(7\pi-20)(19\pi+3)}{103\pi+145}\right] - \left[\begin{matrix}-1.85\\ \text{appr.}\end{matrix}\right]$	
	b″	C-E in (20)	$\left(\frac{146}{225}\pi - \frac{353}{315}\right) - \left(-\frac{16264}{5625}\right)$	

(Cases 1, 2, 2′, 2″ mean $0<\pi<5$, $\pi>5$, $5<\pi<8$, $\pi>8$, respectively.
Cases a, b, b′, b″ mean $0<\pi<\frac{20}{7}$, $\pi>\frac{20}{7}$, $\frac{20}{7}<\pi<\frac{730}{203}$, $\pi>\frac{730}{203}$, respectively.)

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