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京都大学
REMARKS ON NONSMOOTH DYNAMIC VECTOR OPTIMIZATION PROBLEMS

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1. Introduction. This paper deals with vector optimization problems. By convention, throughout this paper we will use the following notations. For $y = (y_1, \ldots, y_n)$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we say that

(i) $y \leq x$, if and only if $y_i \leq x_i$ for any $i \in \{1, \ldots, n\}$,
(ii) $y < x$ if and only if $y_i \leq x_i$ for any $i \in \{1, \ldots, n\}$ with $y \neq x$,
(iii) $y \ll x$ if and only if $y_i < x_i$ for any $i \in \{1, \ldots, n\}$.

Recently, many papers have been devoted to optimality conditions for the vector-valued programming and optimal control problems under some smooth or convex assumptions (see [2], [6], [7], [9], [10]). In [11], we derived the Kuhn-Tucker type proper-efficiency conditions for vector optimal control problems in general case. In this paper we use analogous method to discuss weak-efficiency and efficiency conditions for the following problem,

$$(P): \begin{array}{ll}
\text{minimize : } & \mathcal{F}(x, u) := (\mathcal{F}_1(x, u), \ldots, \mathcal{F}_k(x, u)) \\ 
\text{subject to : } & \dot{x}(t) = \Phi(t, x(t), u(t)) \text{ a.e.}, \\
& x(0) \in D, \ u(t) \in U(t) \text{ a.e.}, \\
& \mathcal{G}(x, u) := (\mathcal{G}_1(x, u), \ldots, \mathcal{G}_l(x, u)) \leq 0
\end{array}$$

where

$$\mathcal{F}_i(x, u) := \int_0^1 F_i(t, x(t), u(t)) dt + f_i(x(1)) \text{ for } i \in I := \{1, \ldots, k\};$$
$$\mathcal{G}_j(x, u) := \int_0^1 G_j(t, x(t), u(t)) dt + g_j(x(1)) \text{ for } i \in J := \{1, \ldots, l\};$$

$x(\cdot) \in AC([0,1], \mathbb{R}^m)$ and $u(\cdot) \in M([0,1], \mathbb{R}^n)$; $F_i, G_j : [0,1] \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$, $f_i, g_j : \mathbb{R}^m \rightarrow \mathbb{R}$ for $i \in I, j \in J$ and $\Phi : [0,1] \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ are given functions; $D$ is a subset of $\mathbb{R}^m$ and $U(\cdot) : [0,1] \rightarrow 2^{\mathbb{R}^n}$ is a set-valued function. Here, $AC([0,1], \mathbb{R}^m)$ is the space of absolutely continuous functions on $[0,1]$ with value in $\mathbb{R}^m$, $M([0,1], \mathbb{R}^n)$ is the space of Lebesgue measurable functions on $[0,1]$ with value in $\mathbb{R}^n$.

For this optimal control problem $(P)$, we say that $(x, u)$ is an admissible process iff $F_i(\cdot, x(\cdot), u(\cdot)) \text{ and } G_j(\cdot, x(\cdot), u(\cdot))$ are integrable for every $i \in I$ and $j \in J$, $(x, u)$ satisfies state equation $\dot{x}(t) = \Phi(t, x(t), u(t)) \text{ a.e.}$ with $x(0) \in D$, $u(t) \in U(t) \text{ a.e.}$ and $\mathcal{G}(x, u) \leq 0$. The first component of a process $(x, u)$ is called a state and the second is called a control. We denote by $\Omega$ the set of all admissible processes of $(P)$. The optimal solutions for $(P)$ are defined in the following meaning.

**Definition 1:** $(x_*, u_*) \in \Omega$ is said to be

(i) a weakly-efficient solution for $(P)$ if there exists no $(x, u) \in \Omega$ such that

$$\mathcal{F}(x, u) \ll \mathcal{F}(x_*, u_*);$$

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(ii) an efficient solution for \( (P) \) if there exists no \( (x, u) \in \Omega \) such that

\[
F(x, u) < F(x_*, u_*).
\]

**Definition 2:** \( (x_*, u_*) \in \Omega \) is called a local weakly-efficient solution of type (I) (resp. (II)) for \( (P) \) if and only if there is no \( (x, u) \in \Omega \) with \( \|x - x_*\|_\infty \leq \varepsilon \) for some \( \varepsilon > 0 \) (resp. with \( x(t) \in x_*(t) + \epsilon B_m \) and \( u(t) \in u_*(t) + \epsilon B_n \) for some \( \varepsilon > 0 \), where \( B_m \) and \( B_n \) are unit closed balls of \( \mathbb{R}_m \) and \( \mathbb{R}_n \), respectively) such that \( F(x, u) \leq F(x_*, u_*) \).

The main method to obtain optimality conditions for multiobjective optimization problems is based on a replacement of the multiobjective problems by single-objective (scalar) optimization problems. The following results give the relationship between \( (P) \) and scalar optimization problems.

**Lemma 1:** \( (x_*, u_*) \in \Omega \) is a weakly-efficient (local weakly-efficient) solution of \( (P) \) if and only if \( (x_*, u_*) \) is an optimal (local optimal) solution of the following scalar optimization problem,

\[
\min : \max_{i \in I} (F_i(x, u) - F_i(x_*, u_*)) \\
\text{s. t.} : (x, u) \in \Omega.
\]

**Proof.** By the definitions, it is easy to see that \( (x_*, u_*) \) is a weakly efficient of \( (P) \) if and only if there is no \( (x, u) \in \Omega \) satisfying

\[
\max_{i \in I} (F_i(x, u) - F_i(x_*, u_*)) < 0.
\]

Thus, this lemma hold. \( \square \)

**Lemma 2:** \( ([6, \text{Lemma 3.1}]) \) \( (x_*, u_*) \in \Omega \) is an efficient solution of \( (P) \) if and only if \( (x_*, u_*) \) is an optimal solution of the following scalar optimal control problem \( (P_i) \) for each \( i \in I \).

\[
(P_i) : \begin{array}{l}
\text{minimize} : F_i(x, u) \\
\text{subject to} : (x, u) \in \Omega \\
\quad F_j(x, u) - F(x_*, u_*) \leq 0 \quad j \in I/\{i\}.
\end{array}
\]

**Lemma 3:** Suppose that \( \Omega \) is convex set and \( F_i(x, u), i = 1, \ldots, k \) are convex functions. Then, \( (x_*, u_*) \in \Omega \) is a weakly-efficient solution of \( (P) \) if and only if \( (x_*, u_*) \) is an optimal solution of \( (P_i) \) stated in Lemma 2 for some \( i \in I \).

**Proof.** Assume that \( (x_*, u_*) \) is a weakly-efficient solution of \( (P) \). If for every \( (P_i), \) \( (x_*, u_*) \) is not an optimal solution, i.e. for any \( i \in I \) there exists \( (x_i, u_i) \in \Omega \) with

\[
F_i(x_i, u_i) < F_i(x_*, u_*) \\
F_j(x_i, u_i) - F_j(x_*, u_*) \leq 0 \quad \text{for} \ j \in I/\{i\}.
\]

Putting \( (x_0, u_0) := \frac{1}{k} \sum_{i \in I} (x_i, u_i) \), we see that \( (x_0, u_0) \in \Omega \). Notice that \( F_i(x, u) \) is convex, we have

\[
F_i(x_0, u_0) \leq \sum_{j \in I} \frac{1}{k} F_i(x_j, u_j) < F_i(x_*, u_*).
\]
Thus, $\mathcal{F}(x_0, u_0) \ll \mathcal{F}(x_*, u_*)$, which contradicts that $(x_*, u_*)$ is a weakly-efficient solution of $(P)$.

Conversely, let $(x_*, u_*)$ be an optimal solution of $(P_i)$ for some $i \in I$. If $(x_*, u_*)$ is not a weakly-efficient solution of $(P)$, then there is $(x, u) \in \Omega$ satisfying

$$\mathcal{F}_i(x, u) < \mathcal{F}_i(x_*, u_*)$$

and

$$\mathcal{F}_j(x, u) - \mathcal{F}_j(x_*, u_*) < 0$$

for $j \in I \setminus \{i\}$,

which contradicts that $(x_*, u_*)$ is an optimal solution of $(P_i)$.

2. Optimality conditions. For simplicity, throughout this section we omit the variable $t$ when it does not cause confusion, and abbreviate the arguments $(t, x_*(t), u_*(t))$ to $[t]$, for instance, we write $G_i[t] = G_i(t, x_*(t), u_*(t))$. In Theorem 1 and 2 below, the notations $\partial$ denote the Clarke generalized gradients and $N_D, N_{U(I)}$ indicate the Clarke normal cones, while in Theorem 3 and 4, these notations stand for the subdifferentials and the normal cones in the sense of convex analysis, respectively.

The following assumptions are required. The pair $(x_*, u_*)$ in (A2) and (A3) will be assumed to be a local weakly efficient solution of type (I) for $(P)$.

(A1): $D$ is closed, $U(\cdot)$ is a nonempty compact set-valued map and the graph $GrU$ is $\mathcal{L} \times B$ measurable.

(A2): $f_i(\cdot), g_j(\cdot)$ $(i \in I$, $j \in J)$ are Lipschitz continuous in a neighborhood of $x_*(1) \in \mathbb{R}^m$.

(A3): For every admissible control $u(\cdot)$, there are real-valued measurable functions $\epsilon(t) > 0$ and $h_i(t) \geq 0$, $i = 0, \cdots, k + l$, such that

$$|F_i(t, x, u(t)) - F_i(t, x', u(t))| \leq h_i(t)|x - x'|$$

for $i \in I$ 

$$|G_j(t, x, u(t)) - G_j(t, x', u(t))| \leq h_{k+j}(t)|x - x'|$$

for $j \in J$

$$|\Phi(t, x, u(t)) - \Phi(t, x', u(t))| \leq h_0(t)|x - x'|$$

whenever $|x - x_*(t)| \leq \epsilon(t), |x' - x_*(t)| \leq \epsilon(t)$, $t \in [0, 1]$; for $u(\cdot) = u_*(\cdot)$ these functions can be chosen in such a way that $\epsilon(t) = \epsilon > 0$ and $h_i(t)$ $(i = 0, \cdots, k + l)$ are integrable.

(A4): For any $u(\cdot) \in \mathcal{U} := \{u(\cdot) \in M([0, 1], \mathbb{R}^n) : u(t) \in U(t) \text{ a.e.}\}$, $F_i(t, x, u(t))$ for $i \in I$, $G_j(t, x, u(t))$ for $j \in J$ and $\Phi(t, x, u(t))$ are measurable.

Theorem 1. Let assumptions (A1)-(A4) be satisfied. Suppose that $(x_*, u_*)$ is a local weakly efficient solution of type (I) for $(P)$. Then, there exist $\lambda = (\lambda_1, \cdots, \lambda_{k+l}) > 0$ and an absolutely continuous function $p(\cdot) : [0, 1] \to \mathbb{R}^n$, such that

(1) $- p(t) \in \partial_x H(t, x_*(t), p(t), u_*(t), \lambda)$ \ a.e. \\
(2) $p(0) \in N_D(x_*(0))$, $-p(1) \in \sum_{i \in I} \lambda_i \partial f_i(x_*(1)) + \sum_{j \in J} \lambda_{k+j} \partial g_j(x_*(1))$ \\
(3) $H(t, x_*(t), p(t), u_*(t), \lambda) = \max_{v \in U(t)} H(t, x_*(t), p(t), v, \lambda)$ \ a.e. \\
(4) $\lambda_{k+j} \left( \int_0^1 G_j[t] dt + g_j(x_*(1)) \right) = 0$ \ for $j \in J$
where $H(t, x, p, u, \lambda) := \langle p, \Phi(t, x, u) \rangle - \sum_{i \in I} \lambda_i F_i(t, x, u) - \sum_{j \in J} \lambda_{k+j} G_j(t, x, u)$

**Proof.** We consider the following problem,

(P') \begin{align*}
&\min : \quad \Gamma_0(y) := \max_{i \in I} \{ y_i(1) + f_i(x(1)) - F_i(x_*, u_*) \} \\
&\text{s. t. : } \quad L_0(y, u) := x(t) - x(0) - \int_0^t \Theta(t, x(t), u(t)) dt = 0 \\
&\quad L_i(y, u) := y_i(t) - \int_0^t F_i(t, x(t), u(t)) dt = 0 \quad i \in I \\
&\quad L_{k+j}(y, u) := y_{k+j}(t) - \int_0^t G_j(t, x(t), u(t)) dt = 0 \quad j \in J \\
&\quad \Gamma_j(y) := y_{k+j}(1) + g_j(x(1)) \leq 0 \quad j \in J \\
&\quad y(\cdot) \in S, \ u(\cdot) \in U,
\end{align*}

where $y(\cdot) := (x(\cdot), y_1(\cdot), \cdots, y_{k+l}(\cdot)) \in C([0, 1], \mathbb{R}^{m+k+l})$ is the state and $u(\cdot) \in M([0, 1], \mathbb{R}^n)$ is the control, $S := \{ x \in C([0, 1], \mathbb{R}^m) : x(0) \in D \} \times C([0, 1], \mathbb{R}^{2k})$.

Let $y_i(t) := \int_0^t F_i[t] dt$ for $i \in I$ and $y_{k+j}(t) := \int_0^t G_j[t] dt$ for $j \in J$. Thus, by Lemma 1, we see that $y_* := (x_*, y_1, \cdots, y_{k+l})$ corresponding $u_*$ minimizes $\Gamma_0(y)$ over all admissible processes $(y, u)$ for (P') with $x$ being sufficiently close to $x_*$ in the norm of $L^\infty$.

By [4, Theorem 2], we see that there exist Lagrange multipliers $\delta := (\delta_0, \cdots, \delta_l) \geq 0$, $x^* \in C^*([0, 1], \mathbb{R}^m)$, and $y_i^* \in C^*([0, 1], \mathbb{R})$ $i = 1, \cdots, k + l$ not all zero such that

$$0 \in \partial_y \mathcal{L}(y^*, y^*, u, \kappa) + N_S(y_*)$$

$$\mathcal{L}(y^*, y^*, u, \kappa) = \min_{u \in U} \mathcal{L}(y^*, y^*, u, \kappa)$$

$$\delta_j \Gamma_j(y_*) = 0 \quad j \in J$$

where $\mathcal{L}(y^*, y^*, u, \kappa) := \sum_{i=0}^l \delta_i \Gamma_i(y) + \langle x^*, L_0(y, u) \rangle + \sum_{i=1}^{k+l} \langle y_i^*, L_i(y, u) \rangle$.

According to the formulas of the Clarke gradients (see [3]), we see that

(i) For any $\xi \in \partial \mathcal{L}(y_*)$, there are $\tilde{\lambda}_i \geq 0, \nu_i \in \partial f_i(x_*(1))$ for $i \in I$ with $\sum_{i \in I} \tilde{\lambda}_i = 1$ such that for any $y \in C([0, 1], \mathbb{R}^{n+2k})$

$$\langle \xi, y \rangle = \sum_{i \in I} \tilde{\lambda}_i y_i(1) + \sum_{i \in I} \nu_i \langle \nu_i, x(1) \rangle.$$

for every $\xi \in \sum_{i=1}^l \delta_i \Gamma_i(y_*)$, there exist $\nu_{k+j} \in \partial g_j(x_*(1))$ for $j \in J$ such that for any $y \in C([0, 1], \mathbb{R}^{n+2k})$

$$\langle \xi, y \rangle = \sum_{j \in J} \delta_j y_j(1) + \sum_{j \in J} \delta_j \langle \nu_{k+j}, x(1) \rangle.$$

Analyzing as in [4], we have the following.

(ii) The above multipliers $x^*, y_1^*, \cdots, y_{k+l}^*$ can be expressed by pairs of the nonnegative Radon measure and Radon-integrable functions $(\mu_i, \xi_i)$, $i = 0, \cdots, 2k$. For every $\xi \in \partial_{y} \left( \langle x^*, L_0(x_*, u_*) \rangle + \sum_{i=1}^{k+l} \langle y_i^*, L_i(x_*, u_*) \rangle \right)$, there is a Lebesgue measurable function $\eta(\cdot)$ with

$$\eta(t) \in \partial_x \left( \int_0^1 \xi_0 d\mu_0, \Phi[t] \right) + \sum_{i \in I} \left( \int_1^t \xi_i d\mu_i, F_i(t, x_*(t), u_*(t)) \right)$$

$$+ \sum_{j \in J} \left( \int_t^1 \xi_{k+j} d\mu_{k+j}, G_j(t, x_*(t), u_*(t)) \right) \text{ a.e.},$$
such that for any $y \in C([0, 1], R^{n+2k}),$

$$
\langle \xi, y \rangle = \int_0^1 \langle x(t) - x(0), \xi_0 \rangle d\mu_0 + \sum_{i=1}^{k+l} \int_0^1 \langle y_i, \xi_i \rangle d\mu_i - \int_0^1 \langle \eta, x \rangle dt.
$$

(iii) For each $\xi \in N_S(y^*),$ there is $\alpha \in N_D(x^*(0)),$ such that

$$
\langle \xi, y \rangle := \langle \alpha, x(0) \rangle \quad \text{for any } y \in C([0, 1], R^{n+k}).
$$

Combining (i), (ii) and (iii), from (5) we see that there are $\tilde{\lambda}_i, i = 1, \ldots, l; \nu_i, i = 1, \ldots, k + l; (\mu_i, \xi_i), i = 0, \ldots, k + l; \eta$ and $\alpha$ stated above such that

$$
0 = \sum_{i \in I} \delta^0 \tilde{\lambda}_i y_i(1) + \sum_{j \in J} \delta_j \nu_{k+j}(1) + \sum_{i \in I} \delta^0 \tilde{\lambda}_i \langle \nu_i, x(1) \rangle + \sum_{j \in J} \delta_j \langle \nu_{k+j}, x(1) \rangle + \sum_{i=1}^{k+l} \int_0^1 \langle y_i, \xi_i \rangle d\mu_i + \int_0^1 \langle x(t) - x(0), \xi_0 \rangle d\mu_0 - \int_0^1 \langle \eta, x \rangle dt + \langle \alpha, x(0) \rangle
$$

for any $x \in C([0, 1], R^n)$ and $y_i \in C([0, 1], R), i = 1, \ldots, k + l.$

Setting $\lambda_i = \delta^0 \tilde{\lambda}_i$ for $i \in I, \lambda_{k+j} := \delta_j$ for $j \in J$ and $p(t) := \int_0^t \xi_0 d\mu_0,$ from the above equation, we see that

$$
\lambda_i y_i(1) + \int_0^1 \left( \int_0^1 \xi_i d\mu_i, \dot{y}_i \right) dt = 0 \quad (\forall y_i \in AC \text{ with } y_i(0) = 0, i \in I \cup J),
$$

$$
\langle \alpha, x(0) \rangle + \sum_{i=1}^{k+l} \lambda_i \langle \nu_i, x(1) \rangle + \int_0^1 \langle p(t) - \int_t^1 \eta d\tau, \dot{x} \rangle dt = 0 \quad (\forall x \in AC).
$$

These yield that (refer to the proof of [4, Theorem 3])

$$
\int_0^1 \xi_i d\mu_i = -\lambda_i, \quad i = 1, \ldots, k + l
$$

(9)

$$
\dot{p}(t) = -\eta(t) \text{ a.e., } p(0) = \alpha, \quad p(1) = -\sum_{i=1}^{k+l} \lambda_i \nu_i.
$$

Therefore, (9), (8) and (7) imply (1), (2) and (4)

Here, if $\delta = 0,$ then $(\lambda_1, \ldots, \lambda_{k+l}) = (y_1^*, \ldots, y_{k+l}^*) = 0.$ From (1) and (2), we can get $p(\cdot) = 0.$ Thus, $y^* = 0$ which contradicts that $\delta$ and $y^*$ are not all zero. Hence, we have $(\lambda_1, \ldots, \lambda_{k+l}) > 0.$

On other hand, By (6) and (9), we see that

$$
\int_0^1 H(t, x^*, p, u^*, \lambda) dt = \max_{u \in U} \int_0^1 H(t, x^*, p, u, \lambda) dt.
$$

Discussing as in the proof of [4, Theorem 3], we can obtain (3). □

According to the results of [8], we see that the above necessary conditions (1)-(4) (Maximum Principle-type) may fail to be sufficient conditions for weak-efficient solutions of (P) even in the “convex” case given below. Next, we give another type necessary weak-efficiency conditions for (P), which is an extension of [8]. In the “convex” case, the latter necessary conditions are necessary-sufficient for weakly-efficiency under Slater constraint qualifications. Moreover, these conditions are also necessary-sufficient for efficient solutions of (P) under further assumptions.
We impose the following assumption, in which the process \((x_*, u_*)\) will be assumed to be a weakly-efficient solution of type (II) for \((P)\).

(A5): \(F_i(t, x, u), G_i(t, x, u), i = 1, \cdots, k, \Phi(t, x, u)\) are Lebesgue measurable, and there exist \(\epsilon > 0\) and \(h_i(t) \in L^1([0, 1], R), i = 0, \cdots, k + l\), such that

\[
|F_i(t, x, u) - F_i(t, x', u')| \leq h_i(t) (|x - x'| + |u - u'|) \quad \text{for } i \in I
\]

\[
|G_j(t, x, u) - G_j(t, x', u')| \leq h_{k+j}(t) (|x - x'| + |u - u'|) \quad \text{for } j \in J
\]

\[
|\Phi(t, x, u(t)) - \Phi(t, x', u')| \leq h_0(t) (|x - x'| + |u - u'|)
\]

whenever \(x, x' \in x_*(t) + \epsilon B_n, u, u' \in u_*(t) + \epsilon B_m\) a.e..

**Theorem 2**: Assume that (A1), (A2) and (A5) be satisfied. Let \((x_*, u_*)\) be a local weakly efficient solution of type (II) for \((P)\). Then there exist \(\lambda = (\lambda_1, \cdots, \lambda_{k+l}) > 0\), an absolutely continuous function \(p(\cdot): [0, 1] \rightarrow R^n\) and an integrable function \(\zeta(\cdot): [0, 1] \rightarrow R^n\) such that

\[
(-\hat{p}(t), \zeta(t)) \in \partial_{(x, u)} H(t, x_*(t), p(t), u_*(t), \lambda) \quad \text{a.e.}
\]

\[
p(0) \in N_D(x_*(0)), \quad p(1) = \sum_{i \in I} \lambda_i \partial f_i(x_*(1)) + \sum_{j \in J} \lambda_{k+j} \partial g_j(x_*(1))
\]

\[
\zeta(t) \in N_{U(t)}(u_*(t)) \quad \text{a.e.}
\]

\[
\lambda_{k+j} \left(\int_0^1 G_j[t]dt + g_j(x_*(t))\right) = 0 \quad \text{for } j \in J
\]

where \(H(t, x, p, u, \lambda)\) is defined in Theorem 1.

**Proof.** It is obvious that the scalar optimization problem in Lemma 1 can be rewritten as follows

\[
(P_1): \quad \text{minimize : } \Gamma(y(1)) := \max_{\overline{I}, \overline{J}} \{ y_i(1) + f_i(x(1)) - F_i(x_*, u_*), y_{k+j}(1) + g_j(x(1)) \}
\]

subject to:

\[
\dot{x}(t) = \Phi(t, x(t), u(t)) \quad \text{a.e.}
\]

\[
\dot{y}_i(t) = F_i(t, x(t), u(t)) \quad \text{a.e.} \quad i \in I
\]

\[
\dot{y}_{k+i}(t) = G_i(t, x(t), u(t)) \quad \text{a.e.} \quad i \in I
\]

\[
x(0) \in C, y_i(0) = 0 \quad i = 1, \cdots, 2k,
\]

\[
u(t) \in U(t) \quad \text{a.e.}
\]

where \(y := (x, y_1, \cdots, y_{2k}) \in AC([0, 1], R^{n+2k})\) is the state and \(u \in M([0, 1], R^n)\) is the control.

Define \(y_*\) as in proof of Theorem 1. By Lemma 1, we see that \((y_*, u_*)\) is a minimizer over all admissible process for \((P_1)\) with \(x(t) = x_*(t) + \epsilon B_n, u(t) = u_*(t) + \epsilon B_m\) a.e. for some \(\epsilon > 0\). Thus, by [8, Proposition 6.1], there exist an absolutely continuous function \(\tilde{p} = (p, p_1, \cdots, p_{k+l})\) and an integrable function \(\zeta\) such that (12) and the following hold

\[
(-\hat{p}(t), \tilde{\zeta}(t)) \in \partial_{(y, \tilde{p}, u)} \tilde{H}(t, y_*(t), \tilde{p}(t), u_*(t)) \quad \text{a.e.}
\]

\[
\tilde{p}(0) \in N_{\overline{C} \times \{0\} \times \cdots \times \{0\}}(y_*(0))
\]
where $\tilde{H}(t, y, \tilde{p}, u) := \langle p, \Phi(t, x, u) \rangle + \sum_{i \in I} \langle p_i, F_i(t, x, u) \rangle + \sum_{i \in I} \langle p_{k+i}, G_i(t, x, u) \rangle$.

First, let us discuss inclusion (16). Notice that for every $i \in I$ and $j \in J$,

$$
\begin{align*}
\Gamma_i(y(1)) &= y_i(1) + f_i(x(1)) - F_i(x_*, u_*), \\
\Gamma_j(y(1)) &= y_{k+j}(1) + g_j(x(1))
\end{align*}
$$

only contains the arguments $x$ and $y_i$, and $\Gamma_i(y_*(1)) = \Gamma(y_*(1)) = 0$. So by the formulas of the Clarke gradients, there are $\gamma_i \in \partial_x f_i(x_*(1))$ for $i \in I$, $\gamma_{k+j} \in \partial_x g_j(x_*(1))$ for $j \in J$ and $(\lambda_1, \ldots, \lambda_{k+l}) > 0$ such that

$$
- \tilde{p}(1) = \sum_{i \in I} \lambda_i \gamma_i, \quad - p_i(1) = \lambda_i, \quad i = 1, \ldots, k + l.
$$

where we can set $\lambda_j = 0$ for $j \in \{ j \in J : G_i(x_*, u_*) < 0 \}$.

Thus, (11) and (13) follow from (15) and (17).

On the other hand, since $\tilde{H}$ does not contain the arguments $y_i$, $i = 1, \ldots, k + l$, (14) implies that $\hat{p}_i(\cdot) = 0$, $i = 1, \ldots, k + l$. Thus, $p_i(\cdot) = -\lambda_i$, $i = 1, \ldots, k + l$ and

$$
(- \hat{p}(t), \dot{z}(t), \zeta(t)) \in \partial_{(x, p, u)} \left( \langle p(t), \Phi[t] \rangle - \sum_{i \in I} \lambda_i F_i[t] - \sum_{i \in I} \lambda_{k+i} G_i[t] \right)
$$

a.e.

From this inclusion, by the definition of the Clarke generalized gradients, we can easily deduce (10).

Next, we proceed to the optimality conditions for the following problem.

$$(P^*): \quad \min \quad F(x, u)$$

s.t. \quad $\dot{x}(t) = A(t)x(t) + B(t)u(t) + b(t)$ \quad a.e.

$\begin{align*}
\dot{x}(0) &\in D, \quad u(t) \in U(t) \quad a.e. \\
G(x, u) &\leq 0
\end{align*}$

where $x(\cdot) \in AC([0, 1], R^n)$ and $u(\cdot) \in L^1([0, 1], R^m)$, $F$ and $G$ are given above, $A(\cdot) : [0, 1] \rightarrow R^{n \times n}$, $B(\cdot) : [0, 1] \rightarrow R^{n \times m}$ are integrable, $b(\cdot) : [0, 1] \rightarrow R^m$ is measurable.

We impose the following hypotheses:

(H1): For every $i \in I$, $F_i(\cdot, x(\cdot), u(\cdot))$ and $G_i(\cdot, x(\cdot), u(\cdot))$ are integrable for any $(x, u) \in AC \times L^1$.

(H2): $F_i(t, \cdot, \cdot)$ for $i \in I$ and $G_i(t, \cdot, \cdot)$ for $j \in J$ are convex lower semicontinuous, and there are $v_i(t) \in L^\infty([0, 1], R^{m+n})$ and $w_i(t) \in L^1([0, 1], R)$, $i = 1, \ldots, k + l$ such that for any $x \in R^n$, $u \in R^m$, $F_i(t, x, u) \geq \langle v_i(t), (x, u) \rangle + w_i(t)$ for $i \in I$ and $G_j(t, x, u) \geq \langle v_j(t), (x, u) \rangle + w_j(t)$ for $j \in J$ a.e..

(H3): The functions $f_i(\cdot)$ for $i \in I$ and $g_i(\cdot)$ for $j \in J$ are proper convex and lower semicontinuous.

(H4): The set $C$ is convex, $U(t)$ is convex a.e., and there is $\rho(t) \in L^1$ such that $|u| \leq \rho(t)$ for any $u \in U(t)$ a.e..

(H5): There exists an admissible process $(x_i, u_i)$ for $(P^*)$, such that $G_j(x_i, u_i) - G_j(x_*, u_*) < 0$ for any $j \in \{ j \in J : G_j(x_*, u_*) = 0 \}$.

Here, $(x_*, u_*)$ will be assumed to be an admissible process for $(P^*)$. 
Theorem 3: Assume that (H1)-(H5) and (A1) be satisfied. An admissible process \((x_*, u_*)\) is a weakly-efficient solution for \((P^*)\) if and only if there exist \(\lambda = (\lambda_1, \ldots, \lambda_{k+l}) \geq 0\) with \((\lambda_1, \ldots, \lambda_k) > 0\), \(p(\cdot) \in AC([0,1], R^n)\), and \(\zeta(\cdot) \in L^{\infty}([0,1], R^n)\) such that

(18) \((\dot{p}(t) + p(t)A(t), p(t)B(t) - \zeta(t)) \in \partial_{(x,u)} \sum_{i \in I} \lambda_i F_i(t) + \sum_{j \in J} \lambda_{k+j} G_j(t)\) a.e.

(19) \(p(0) \in N_C(x_*(1)), -p(1) \in \sum_{i \in I} \lambda_i \partial f_i(x_*(1)) + \sum_{j \in J} \lambda_{k+j} \partial g_i(x_*(1))\)

(20) \(\zeta(t) \in N_{U\{f\}}(u_*(t)), \quad a.e\).

(21) \(\lambda_{k+j} (\int_0^1 G_j(t) dt + g_j(x_*(1))) = 0 \quad \text{for} \ j \in J\).

Proof. [Necessity] By Lemma 3, we know that there exists \(i \in I\) such that \((x_*, u_*)\) is an optimal solution for the following scalar optimal control problem,

\[
\text{minimize:} \quad F_i(x, u) \\
\text{subject to:} \quad \dot{x}(t) - A(t)x(t) - B(t)u(t) - b(t) = 0 \quad a.e. \\
\quad \mathcal{G}_j(x, u) \leq 0 \quad j \in J \\
\quad x \in \{x \in AC([0,1], R^m): x(0) \in D\} \\
\quad u \in C := \{u \in L^1([0,1], R^n): u(t) \in U(t) \ a.e.\}.
\]

This means that \((x_*, u_*, x_*(0), x_*(1))\) is a minimizer for the following scalar optimization problem.

\[
\text{minimize:} \quad \Lambda_i(x, u, \alpha, \beta) := \int_0^1 F_i(t, x, u) dt + f_i(\beta) \\
\text{subject to:} \quad \Gamma_1(x, u, \alpha, \beta) := x(t) - \alpha - \int_0^t (Az + Bu + b) d\tau = 0 \quad a.e. \\
\quad \Gamma_2(x, u, \alpha, \beta) := \beta - \alpha - \int_0^1 (Az + Bu + b) d\tau = 0 \\
\quad \Lambda_j(x, u, \alpha, \beta) := \int_0^1 F_j(t, x, u) dt + f_j(\beta) \quad \text{for} \ j \in I \setminus \{i\} \\
\quad \Lambda_j(x, u, \alpha, \beta) := \int_0^1 G_j(t, x, u) dt + g_j(\beta) \quad \text{for} \ j \in J \\
\quad (x, u, \alpha, \beta) \in \mathcal{M} := L^1([0,1], R^n) \times C \times D \times R^n, 
\]

where \((x, u, \alpha, \beta) \in L^1([0,1], R^n) \times L^1([0,1], R^n) \times R^n \times R^n\).

Put \(\theta := (x, u, \alpha, \beta)\) and \(\theta_* := (x_*, u_*, x_*(0), x_*(1))\). It is obvious that \(\Lambda_i(\theta)\) is convex, \(\Gamma_1(\theta)\) and \(\Gamma_2(\theta)\) are affine mappings. By [5, Theorem 5 p74], there exist \(\lambda := (\lambda_1, \ldots, \lambda_{k+l}) \geq 0\), \(q(\cdot) \in (L^1)^*\) and \(\sigma \in R^n\) not all zero, such that

\[
\sum_{j=1}^{k+l} \lambda_j \Lambda_j(\theta) + \int_0^1 \langle q, \Gamma_1(\theta) \rangle dt + \langle \sigma, \Gamma_2(\theta) \rangle = \min_{\theta \in \mathcal{M}} \left( \sum_{j=1}^{k+l} \lambda_j \Lambda_j(\theta) + \int_0^1 \langle q, \Gamma_1(\theta) \rangle dt + \langle \sigma, \Gamma_2(\theta) \rangle \right),
\]
\[ \lambda_{k+j} \Lambda_j(\theta_*) = \lambda_{k+j} \left( \int_0^1 G_j[t] dt + g_j(x_*(1)) \right) = 0 \quad \text{for } j \in J \]

Let \( I_M(\theta) \) denote the indicator function of \( M \). Notice that the functions \( I_M, \Lambda_j \) (\( j \in J \)), \( \int_0^1 (p, \Gamma_1) dt \), \( (\sigma, \Gamma_2) \) are proper convex and lower semicontinuous, from (22) we see that

\[ 0 \in \sum_{j=1}^{k+l} \lambda_j \partial \Lambda_j(\theta_*) + \partial \int_0^1 (q, \Gamma_1(\theta_*)) dt + \partial (\sigma, \Gamma_2(\theta_*)) + N_M(\theta_*) \]

(refer to Section 1 of Chapter 1 in [1]).

Now, we analyze (23). By the formulas of subdifferential (see [1], [5]), we have the following conclusions.

For every \( \xi \in \sum_{j=1}^{k+l} \lambda_j \partial \Lambda_j(\theta_*) \), there are \( \mu_j, \eta_j \in L^\infty \) with \( (\mu_j(t), \eta_j(t)) \in \partial_{(x,u)} F_i[t] \) and \( \nu_j \in \partial f_j(x_*(1)) \) for \( j \in I \), \( (\mu_{k+j}, \eta_{k+j}) \in L^\infty \) with \( (\mu_{k+j}(t), \eta_{k+j}(t)) \in \partial_{(x,u)} G_i[t] \) and \( \nu_{k+j} \in \partial g_j(x_*(1)) \) for \( j \in J \) such that for any \( \theta \in L^1 \times L^1 \times R^m \times R^m \)

\[ \langle \xi, \theta \rangle = \sum_{j=1}^{k+l} \lambda_j \left( \int_0^1 ((\mu_j, x) + (\eta_j, u)) dt + \nu_j, \beta) \right). \]

Corresponding to any \( \xi \in N_M(\theta_*) \), there are \( \gamma \in N_D(x_*(0)) \), and \( \zeta(\cdot) \in N_C(u_*(\cdot)) \) such that for any \( \theta \in L^1 \times L^1 \times R^m \times R^m \), one has

\[ \langle \xi, \theta \rangle = \langle \gamma, \alpha \rangle + \int_0^1 \langle \zeta, u \rangle dt. \]

Notice that \( \int_0^1 \langle q, \Gamma_1(\theta) \rangle dt \) is affine on \( \theta \), thus \( \partial \int_0^1 \langle q, \Gamma_1(\theta_*\rangle dt = \{ \xi \} \) with

\[ \langle \xi, \theta \rangle = \int_0^1 \langle q, z - \alpha - \int_0^1 (A z - B u) d\tau \rangle dt \]

for any \( \theta \in L^1 \times L^1 \times R^m \times R^m \).

Similarly, \( \partial (\sigma, \Gamma_2(\theta_*)) = \{ \xi \} \) with

\[ \langle \xi, \theta \rangle = \langle \sigma, \beta - \alpha - \int_0^1 (A z - B u) d\tau \rangle \]

for any \( \theta \in L^1 \times L^1 \times R^m \times R^m \).

Then, (23) implies that there are \( \mu_j, \eta_j, \nu_j, j = 1, \cdots, k+l, \gamma \) and \( \zeta \) stated above such that

\[ \sum_{j=1}^{k+l} \lambda_j \int_0^1 \langle (\mu_j, x) + (\eta_j, u) \rangle dt \]

\[ + \sum_{j=1}^{k+l} \lambda_j \langle \nu_j, \beta \rangle + \int_0^1 \langle q, z - \int_0^1 (A z + B u) d\tau \rangle dt \]

\[ - \int_0^1 q dt, \alpha \rangle + \langle \sigma, \beta - \alpha - \int_0^1 (A z + B u) d\tau \rangle + \langle \gamma, \alpha \rangle + \int_0^1 \langle \zeta, u \rangle dt = 0 \]

for any \( (z, u, \alpha, \beta) \in L^1 \times L^1 \times R^m \times R^m \).

Put \( p(t) := \int_0^1 q(\tau) d\tau + \alpha. \) From (24) we see that

\[ \int_0^1 \left( \sum_{i=1}^{k+l} \lambda_i \mu_i, z \right) dt - \int_0^1 \langle p + p A, z \rangle dt + \int_0^1 \left( \sum_{i=1}^{k+l} \lambda_i \eta_i, u \right) dt - \int_0^1 \langle p B - \zeta, u \rangle dt \]

\[ + \langle \sum_{i=1}^{k+l} \lambda_i \nu_i, \beta \rangle + \langle \sigma, \beta \rangle - \int_0^1 q dt, \alpha \rangle - \langle \sigma, \alpha \rangle + \langle \gamma, \alpha \rangle = 0 \]
for any \((z, u, \alpha, \beta) \in L^1 \times L^1 \times R^m \times R^n\), which implies that
\[
\dot{p} + pA = \sum_{i=1}^{k+l} \lambda_i \mu_i, \quad pB - \zeta = \sum_{i=1}^{k+l} \lambda_i \eta_i,
\]
\[
p(1) = \sigma = -\sum_{i=1}^{k+l} \lambda_i \nu_i.
\]
(25)
\[
p(1) = \sigma = -\sum_{i=1}^{k+l} \lambda_i \nu_i, \quad p(0) = \int_0^1 q(\tau) d\tau + \sigma = \gamma.
\]

From (25), we obtain (18) and (19).

By \(\zeta(\cdot) \in NC(u_*(\cdot))\), we have \(\zeta(t)(u(t) - u_*(t)) \leq 0\) for any \(u(\cdot) \in \mathcal{U}\).

Finally, if \(\lambda = 0\), then (28) and (29) imply that \(\sigma = 0\) and \(p(\cdot) = 0\), thus \(\lambda, q\) and \(\sigma\) all are zero. Hence, \(\lambda > 0\). By the Slater constraint qualifications (H5) and the conditions (18)-(21), we have that
\[
0 > \sum_{j \in J} \lambda_{k+j}(G_j(x_i, u_i) - G_j(x_*, u_*))
\]
\[
= \sum_{j \in I/I(i)} \lambda_j \left( \int_0^1 (G_j(t, x_i, u_i) - G_j(t)) dt + g_j(x_i(1)) - g_j(x_*(1)) \right)
\]
\[
\geq \int_0^1 ((\dot{p} + pA, x_i - x_*) + (pB - \zeta, u_i - u_*)) dt - p(1)(x_i(1) - x_*(1))
\]
\[
= -p(0)(x_i(0) - x_*(0)) - \int_0^1 \langle \zeta, u_i - u_* \rangle dt
\]
\[
\geq 0,
\]
a contradiction. Hence, \((\lambda_1, \cdots, \lambda_k) > 0\).

[Sufficiency] Assume that there exist \((\lambda_1, \cdots, \lambda_k) > 0\), \(p(\cdot) \in AC\), and \(\zeta(\cdot) \in L^\infty\) satisfying (18)-(21). Notice that \(\sum_{i \in J} \lambda_i > 0\), so we can set \(\sum_{i \in I} \lambda_i = 1\). Let \((x, u)\) be an arbitrary admissible process for \((P^*)\). Using (18)-(21) again, we see that
\[
\max \{ F_i(x, u) - F_i(x_*, u_*) : i \in I \}
\]
\[
\geq \sum_{i \in I} \lambda_i \left( \int_0^1 F_i(t, x, u) dt + f_i(x(1)) - \int_0^1 F_i[t] dt - f_i(x_*(1)) \right)
\]
\[
+ \sum_{j \in J} \lambda_{k+j} \left( \int_0^1 G_i(t, x, u) dt + g_i(x(1)) - \int_0^1 G_i[t] dt - g_i(x_*(1)) \right)
\]
\[
+ \int_0^1 \langle p, \dot{x} - Ax - Bu - b \rangle dt - \int_0^1 \langle \zeta, \dot{x}_* - Ax_* - Bu_* - b \rangle dt
\]
\[
= \int_0^1 \left( \sum_{i \in I} \lambda_i F_i(t, x, u) + \sum_{j \in J} \lambda_{k+j} G_i(t, x, u) dt - \sum_{i \in I} \lambda_i F_i[t] - \sum_{j \in J} \lambda_{k+j} G_j[t] \right) dt
\]
\[
+ \sum_{i \in I} \lambda_i f_i(x(1)) + \sum_{j \in J} \lambda_{k+j} g_j(x(1)) - \sum_{i \in I} \lambda_i f_i(x_*(1)) - \sum_{j \in J} \lambda_{k+j} g_j(x_*(1))
\]
\[
- \int_0^1 ((\dot{p} + pA, x - x_*) + (pB - \zeta, u - u_*)) dt - \int_0^1 \langle \zeta, u - u_* \rangle dt
\]
\[
+ \langle p(1), (x(1) - x_*(1)) - p(0), (x(0) - x_*(0)) \rangle
\]
\[
\geq 0.
\]

By Lemma 1, \((x_*, u_*)\) is a weakly-efficient solution for \((P^*)\). \(\square\)

Using Theorem 3 and Lemma 3, we can easily show that the conditions (18)-(21) in Theorem 3 are also necessary-sufficient for efficient solutions of \((P^*)\) under the following Slater constraint qualifications (H6).
(H6): For every \( i \in I \), there is an admissible process \((x_i, u_i)\) for \((P^*)\), such that 
\[ F_j(x_i, u_i) - F_i(x_*, u_*) < 0 \]
for any \( j \in I / \{ i \} \) and 
\[ G_j(x_i, u_i) - G_j(x_*, u_*) < 0 \]
for any \( j \in \{ j \in J : G_j(x_*, u_*) = 0 \} \).

**Theorem 4:** Assume that (H1)-(H6) and (A1) are satisfied. An admissible process \((x_*, u_*)\) is an efficient solution for \((P^*)\) if and only if there exist \((\lambda_1, \ldots, \lambda_k, \mu) \geq 0\) with \((\lambda_1, \cdots, \lambda_k, \mu) \geq 0\), \( p(\cdot) \in AC([0, 1], R^n) \), and \( \zeta(\cdot) \in L^\infty([0, 1], R^n) \) such that (18)-(21) hold.

**Remark:** It is easy to see that the sufficiency in Theorem 3 and Theorem 4 also hold under the following simpler assumptions: \( F_i \) for \( i \in I \) and \( G_j \) for \( j \in I \) are convex in \((x, u)\) and measurable in \( t, f \); for \( i \in I \) and \( g_j \) for \( j \in I \) are convex functions, \( C \) is convex set and \( U(t) \) is convex a.e.

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