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Some Duality Theorems of Set-Valued Optimization*

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Abstract

A set optimization problem with a set-valued objective function is investigated, and duality result is considered.

1 Introduction

Set-valued optimization has been investigated for about twenty years by many authors and various results concerned with the problem were obtained, see [1, 2, 3, 4, 6, 8, 9] and so on. Usually, this optimization is interpreted as a vector optimization problem with a set-valued objective function as follows:

$$\text{(VP)} \qquad \begin{array}{ll} \text{Minimize} & F(x) \\ \text{subject to} & x \in S \end{array}$$

where S is a nonempty set, $(Z, \leq)$ is an ordered space, F is a set-valued map from S to Z, that is, $F : S \to 2^Z$. The aim of vector optimization problem (VP) is to find $x_0 \in S$, called solution, satisfying $F(x_0)$ includes a Pareto extremal point of $\bigcup_{x\in S} F(x)$, that is, there exists $z_0 \in F(x_0)$ such that if $z \in \bigcup_{x\in S} F(x)$ and $z \leq z_0$ then $z_0 = z$.

However, the aim of (VP) is not suitable for 'set-valued optimization' because such solutions are decided by one of the extremal elements of solution's value. Recently, a set optimization problem with a set-valued objective function was introduced against vector optimization problem (VP), see [5]. Criteria of solutions of the optimization problem are obtained by comparisons of set-values of the objective function, these are called natural criteria. Our aim of this paper is to establish duality theory of such a set optimization problem with a set-valued objective function.

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The construction of this paper is the following: In Section 2, we mention some notations and definitions concerned with such a set optimization problem. In Section 3, we show an embedding theorem, and also we prove a strong duality theorem. In Section 4, we prove a saddle point theorem for our set optimization problem.

2 Set Optimization of Set-Valued Maps

Let X be a nonempty set, Y and Z topological vector spaces, K and L solid, pointed, convex cones of Y, Z, respectively, F and G set-valued maps from X to Y and Z, respectively, that is $F : X \to 2^Z$, $G : X \to 2^Y$, and assume that $F(x) \neq \emptyset$ and $G(x) \neq \emptyset$ for each $x \in S$, and $S := \{x \in X \mid G(x) \cap (-K) \neq \emptyset\}$. Now we define problem (SP) as follows:

$$\text{(SP)} \qquad \begin{array}{ll} \text{Minimize} & F(x) \\ \text{subject to} & x \in S. \end{array}$$

Before to define notions of solutions of problem (SP), we mention about some set relations in ordered vector space $(Z, \leq_L)$. For $\emptyset \neq A,\ B \subset Z$,

$$A \leq_L^l B \overset{\text{def}}{\iff} A + L \supset B,$$

$$A \leq_L^u B \overset{\text{def}}{\iff} A \subset B - L.$$

In these notations, l means lower and u means upper: $A \leq_L^l B$ iff each element b of B has a lower bound in A, and $A \leq_L^u B$ iff each element a of A has an upper bound in B. We treat only relation $\leq_L^l$ in this paper.

The aim of problem (SP) is to find the following solutions:

Definition 2.1 A vector $x_0 \in X$ is said to be

(i) a feasible solution of (SP) if $x \in S$

(ii) a minimal solution of (SP) if $x_0 \in S$, and if $x \in S$ and $F(x) \leq_L^l F(x_0)$ are satisfied, then $F(x_0) \leq_L^l F(x)$ is fulfilled.

If $F(x)$ is a singleton, that is $F(x)$ is written by $F(x) = \{f(x)\}$ for some map f from X to Z, these notions are equivalent to usual ones of 'set-valued optimization.'

3 Embedding Theorem and Duality Theorem

In the rest of paper, we assume that all values of set-valued map F are nonempty compact convex. We denote $\mathcal{C}(Z)$ as the family of all nonempty compact convex sets in Z.

First, we construct an ordered normed linear space $\mathcal{V}$ in which $\mathcal{C}(Z)$ is embedded. On $\mathcal{C}(Z)^2$ we define an equivalent relation $\sim$: for (A, B), $(C, D) \in \mathcal{C}(Z)^2$,

$$(A, B) \sim (C, D) \overset{\text{def}}{\iff} A + D + L = B + C + L.$$

Let $[(A, B)]$ be the equivalence class includes (A, B), and let $\mathcal{V}$ be $\mathcal{C}(Z)^2/\sim$, the sets of all equivalence classes $[(A, B)]$. We define a vector structure on $\mathcal{V}$ as follows: for $[(A, B)]$, $[(C, D)] \in \mathcal{V}$, sum and scalar product are defined by:

$$[(A, B)] + [(C, D)] := [(A + C, B + D)]$$

$$\lambda \cdot [(A, B)] := \begin{cases} [(\lambda A, \lambda B)], & \lambda \geq 0 \\ [(-\lambda B, -\lambda A)], & \lambda < 0 \end{cases}$$

Then we can show that $\mathcal{V}$ is a vector space over the realfield. Moreover, we define a norm $\|\cdot\|$. For $[(A, B)] \in \mathcal{V}$,

$$\|[(A, B)]\| := \inf\{\lambda \geq 0 \mid A + \lambda U \leq_L^l B,\ B + \lambda U \leq_L^l A\}$$

then, $(\mathcal{V}, \|\cdot\|)$ is a normed space. Let $\Pi := \{[(A, B)] \in \mathcal{V} \mid B \leq_L^l A\}$, then Π is a solid, pointed, convex cone in $\mathcal{V}$, and we can derive a partial order $\leq_\Pi$ in $\mathcal{V}$:

$$[(A, B)] \leq_\Pi [(C, D)] \overset{\text{def}}{\iff} [(C, D)] - [(A, B)] \in \Pi$$

Finally, $\mathcal{V}$ is an ordered normed space over the realfield.

Now we show the following embedding theorem:

Theorem 3.1 Let $\varphi : \mathcal{C}(Z) \to \mathcal{V}$ by

$$\varphi(A) := [(A, \{\theta\})],\ A \in \mathcal{C}(Z)$$

then, the following are satisfied:

(i) For each $A, B \in \mathcal{C}(Z)$,

$$A \leq_L^l B \iff \varphi(A) \leq_\Pi \varphi(B);$$

(ii) conditions a) and b) are equivalent:

a) $x_0 \in S$ is a solution of set optimization (SP),

b) $x_0 \in S$ is a solution of the following vector optimization (EP):

$$\text{(EP)} \qquad \begin{array}{ll} \text{Minimize} & \varphi(F(x)) \\ \text{subject to} & x \in S. \end{array}$$

From this result, we can use results of vector optimization with set-valued maps to solve set optimization with set-valued maps.

Theorem 3.2 Let the following assumptions are satisfied:

(A1) F is nonempty compact convex values

(A2) $\forall x_1, x_2 \in X$, $\forall y_1 \in G(x_1)$, $y_2 \in G(x_2)$, $\forall \lambda \in (0,1)$, $\exists (x,y) \in \mathrm{Gr}(G)$ such that

$$\begin{cases} F(x) \leq_L^l (1-\lambda)F(x_1) + \lambda F(x_2) \\ y \leq_K (1-\lambda)y_1 + \lambda y_2 \end{cases}$$

(A3) $\exists x' \in X$ such that $G(x') \cap (-\mathrm{int}K) \neq \emptyset$

(A4) x_0 is a proper solution of set optimization of (SP)

then there exist $y_0^* \in K^+ \setminus \{\theta\}$ and $\mu : \mathrm{int}L \to (0,\infty)$ such that

(i) $1/\mu$ is affine on $\mathrm{int}L$

(ii) for each $a \in \mathrm{int}L$, $(T_a, \varphi(F(x_0)))$ is a weak maximizer of the weak dual problem of (EP),

where $T_a(y) = \langle y_0^*, y\rangle \mu(a)a$, $y \in Y$.

Corollary 3.1 Under same assumption of the last theorem, there exist $y_0^* \in K^+ \setminus \{\theta\}$ and $\mu : \mathrm{int}L \to (0,\infty)$ with $1/\mu$ is affine on $\mathrm{int}L$ such that

for any $a \in \mathrm{int}L$, there does not exist $(x,y) \in \mathrm{Gr}(G)$ such that

$$F(x) + T_a(y) \leq_{\mathrm{int}L}^l F(x_0)$$

where $T_a(y) = \langle y_0^*, y\rangle \mu(a)a$, $y \in Y$.

4 Saddle Point Theorem

In this section, we consider a saddle point theorem of (SP). First, for primal problem (SP), we define dual problem (SD):

$$\text{(SD)} \qquad \begin{array}{ll} \text{Maximize} & \Phi(T) \\ \text{subject to} & T \in \mathcal{M} \end{array}$$

where

- $\Phi(T) = \mathrm{Min}(\varphi(L(X,T))|\Pi)$
- $L(x,T) = F(x) + T(G(x))$
- $\mathcal{M} = \{T \in \mathcal{L}(Y,Z)_+ \mid T = \langle y^*, \cdot\rangle a, y^* \in K^+ \setminus \{\theta\}, a \in \mathrm{int}L\}$

Definition 4.1 (Saddle Point) $(x_0, T_0) \in X \times \mathcal{M}$ is said to be a saddle point of L if

$$\varphi(L(x_0,T_0)) \cap \mathrm{Max}(\varphi(L(x_0,\mathcal{M}))|\Pi) \cap \mathrm{Min}(\varphi(L(X,T_0))|\Pi) \neq \emptyset.$$

Proposition 4.1 $(x_0, T_0) \in X \times \mathcal{M}$ is a saddle point of L iff there exists $y_0 \in G(x_0)$ such that

(i) $F(x) + T_0(y) \leq^l F(x_0) + T_0(y_0), (x, y) \in \mathrm{Gr}(G)$
$\Rightarrow F(x_0) + T_0(y_0) \leq^l F(x) + T_0(y)$

(ii) $F(x_0) + T_0(y_0) \leq^l F(x_0) + T(y_0), T \in \mathcal{L}_+(Y, Z)$
$\Rightarrow F(x_0) + T(y_0) \leq^l F(x_0) + T_0(y_0)$

Theorem 4.1 (Saddle Point Theorem) If (x_0, T_0) is a saddle point of L, then

1) x_0 is an optimal of (SP);

2) T_0 is an optimal of (SD);

3) $\varphi(F(x_0)) \cap \Phi(T_0) \cap \mathrm{Max}(\Phi(\mathcal{M})|\Pi) \neq \emptyset$;

4) $G(x_0) \subset -K$;

5) $T_0(y) = \theta$ for all $y \in G(x_0)$.

Conversely, if 1) through 5) above and $F(x) = \mathrm{Min}(F(x)|K)$ for each $x \in X$ hold, then (x_0, T_0) is a saddle point of L.

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