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AN APPLICATION OF ALUTHGE TRANSFORM TO PUTNAM INEQUALITY FOR LOG-HYPONORMAL OPERATORS

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ABSTRACT. In this note, we give a short proof to the Putnam inequality for log-hyponormal operators due to Tanahashi: If $T$ is invertible and log-hyponormal, i.e., $\log T^*T \geq \log TT^*$, then

$$\|\log T^*T - \log TT^*\| \leq \frac{1}{\pi} \int_{\sigma(T)} r^{-1} dr d\theta,$$

where $\sigma(T)$ is the spectrum of $T$. It is based on his original idea that the log-hyponormality is regarded as 0-hyponormality.

1. Introduction. After the Furuta inequality was originated by Furuta [12], see also [5,13,16,18], we initiated to study it under the chaotic order $A \gg B$, i.e., $\log A \geq \log B$, for positive invertible operators $A$ and $B$ [11]. We finally characterized $A \gg B$ by a Furuta-type inequality [6] and [7,8]: For $A,B > 0$, $A \gg B$ if and only if

$$(1) \quad (A^r B^p A^*)^{\frac{2r}{p+2r}} \leq A^{2r}$$

holds for all $p,r \geq 0$.

Furthermore we interpolated between the Furuta inequality and Theorem A as follows [9,10]:

Theorem A. For a fixed $\delta > 0$, $A^\delta \geq B^\delta$ for $A,B \geq 0$ if and only if for each $r \geq 0$

$$(2) \quad A^{\frac{r+2r}{\delta}} \geq (A^r B^p A^*)^{\frac{1}{\delta}}$$

holds for $p \geq 0$ and $q \geq 1$ with

$$(3) \quad (\delta + 2r)q \geq p + 2r.$$

We note that (1) is equivalent to the case $\delta = 0$ in Theorem A and the Furuta inequality is just the case $\delta = 1$. The domain given by (3) is explained by the figure below:
From the viewpoint of this, Tanahashi [19] defined the log-hyponormality for invertible operators by $|T| \gg |T^*|$, where $|X|$ is the square root of $X^*X$, and constructed the Putnam inequality for log-hyponormal operators:

**Theorem T.** If $T$ is an invertible log-hyponormal operator, i.e., $\log T^* - \log TT^* \geq 0$, then

$$\| \log T^* - \log TT^* \| \leq \frac{1}{\pi} \int_{\sigma(T)} r^{-1} \, dr \, d\theta,$$

where $\sigma(T)$ is the spectrum of $T$.

It was conjectured from the Putnam inequality for $p$-hyponormal operators by Cho and Itoh [3]:

If $T$ is a $p$-hyponormal operator, i.e., $(T^*T)^p - (TT^*)^p \geq 0$, then

$$\left\| (T^*T)^p - (TT^*)^p \right\| \leq \frac{p}{\pi} \int_{\sigma(T)} r^{2p-1} \, dr \, d\theta.$$

As a matter of fact, he understood (5) as follows:

$$\left\| \frac{(T^*T)^p - (TT^*)^p}{p} \right\| \leq \frac{1}{\pi} \int_{\sigma(T)} r^{2p-1} \, dr \, d\theta.$$

By taking $p \to \infty$, he constructed Theorem T and proved it by the idea developed in (5).

The purpose of this note is to continue his consideration directly. That is, we here propose a straight and simple proof of Theorem T which might be along with his intention; Tanahashi might regards the log-hyponormality as the 0-hyponormality. Our tool in this note is the Aluthge transform which is grown up the $p$-hyponormality, see [1,2,4,14,15,20].

2. Preliminary.

For the sake of convenience, we cite the following characterization of chaotic order which implies (1) by the help of the Furuta inequality [7], see also [8] and [9].

**Theorem B.** For $A, B > 0, A \gg B$, i.e., $\log A \geq \log B$, if and only if for any $\delta \in (0, 1]$ there exists an $\alpha = \alpha_\delta > 0$ such that

$$(e^\delta A)^\alpha > B^\alpha.$$  

The essential part of Theorem B is as follows: If $A$ and $B$ are selfadjoint and $A > B$, then there exists an $\alpha \in (0, 1]$ such that

$$e^{\alpha A} > e^{\alpha B}. (*)$$

It has the following simple proof: The assumption $A > B$ means that $A - B \geq \epsilon > 0$ for some $\epsilon$. We here take $0 < \alpha < \epsilon/(\|A\| + \|B\|)$ and $\alpha \leq 1$. Then we have

$$e^{\alpha A} - e^{\alpha B} = \alpha (A - B) + \sum_{n=2}^{\infty} \frac{\alpha^n}{n!} (A^n - B^n)$$

$$\geq \alpha \epsilon + \alpha^2 \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!} (A^n - B^n)$$

$$\geq \alpha \epsilon - \alpha^2 || \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!} (A^n - B^n)||$$

$$\geq \alpha \epsilon - \alpha^2 \sum_{n=2}^{\infty} \frac{1}{n!} (\|A\|^n + \|B\|^n)$$

$$\geq \alpha (\epsilon - \alpha (\|A\| + \|B\|)) > 0.$$
Here we should note an interesting characterization of chaotic order recently obtained by Yamazaki and Yanagida [22], which is associated with Kantorovich inequality and consequently Specht's ratio, see [23].

3. Proof.

We begin with the Putnam inequality for p-hyponormal operators; we turn it into the following lemma via the Löwner-Heinz inequality:

**Lemma 1.** If $T$ is a q-hyponormal operator, then $T$ satisfies (6) for all $0 < p \leq q$. Consequently (4) holds for any q-hyponormal operators $T$.

The second half is ensured by the fact that $\lim_{t \to 0} \frac{A^t - 1}{t} = \log A$ for a positive invertible operator $A$.

Thus Lemma 1 suggests us to find a family $\{T_q; q > 0\}$ of $q$-hyponormal operators such that $\|T_q - T\| \to 0$ as $q \to 0$ for a given log-hyponormal operator $T$. In this situation, the Aluthge transform completely responds to our demand. As a matter of fact, Tanahashi prepared the following result in [20]:

**Lemma 2.** If $T$ is an invertible log-hyponormal operator with the polar decomposition $T = U|T|$, then the Aluthge transform $\tilde{T} = |T|^q U |T|^{1-q}$ is $q$-hyponormal for $0 < q \leq \frac{1}{2}$.

To prove Theorem T, we take $T_q = |T|^q U |T|^{1-q}$ for $0 < q < \frac{1}{2}$. Since $\sigma(T_q) = \sigma(T)$ for all $q$, it is complete.

We finally give a short proof to Lemma 2 via Theorem A.

Putting $p = 2q$ and $r = 1 - q$ in (1), we have

$$
(T_q T_q^*)^{1-q} = (|T|^{1-q} U^* |T|^{2q} U |T|^{1-q})^{1-q}
$$

$$
= U^* (U |T|^{1-q} U^* |T|^{2q} U |T|^{1-q} U^*)^{1-q} U
$$

$$
= U^* |T^*|^{2(1-q)} U
$$

$$
= |T|^{2(1-q)},
$$

so that $(T_q T_q^*)^{2q} \geq |T|^{2q}$ via the Löwner-Heinz inequality by $1 - q \geq q$. On the other hand, we have also

$$
(T_q T_q^*)^q = (|T|^q U |T|^{2(1-q)} U^* |T|^q)^q
$$

$$
= (|T|^q |T^*|^{2(1-q)} |T|^q)^q
$$

$$
\leq |T|^{2q},
$$

Therefore it follows that

$$
(T_q T_q^*)^q \geq |T|^{2q} \geq (T_q T_q^*)^q,
$$

as desired.

**Remark.** (1) (5) is obtained by Putnam [17] for $p = 1$, Xia [21] for $\frac{1}{2} \leq p < 1$ and Cho-Itoh [3] for $0 < p < \frac{1}{2}$.

(2) The proof of Lemma 2 is available to show Tanahashi's result [20; Theorem 4].

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