作用素不等式二題

Exponential operator inequalities

福岡教育大学 内山 充 (Mitsuru Uchiyama)
Fukuoka University of Education
Munakata, Fukuoka, 811-41 Japan,
e-mail uchiyama@fukuoka-edu.ac.jp

Section 1.

Let $X$ be a unital Banach algebra over $\mathbb{R}$ or $\mathbb{C}$, that is, a complete normed algebra with a unit 1 such that $||1|| = 1$.

The aim of this note is, roughly speaking, to show that if $f : [0, \infty) \to X$ satisfies $f(0) = 1$, $f'(0) = a$, then $f(\frac{t}{n})^n$ converges to $e^{ta}$ as $n \to \infty$, where

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

by definition.

If $X = \mathbb{R}$, this assertion clearly follows from the L’Hospital theorem. Since a set of all bounded operators on a Banach space is a unital Banach algebra, for a bounded operator $A, e^A$ is defined as above. In this case for bounded operators $A, B$ the Lie product formula:

$$\exp(A + B) = (n) \lim_{n \to \infty} \{\exp(\frac{A}{n}) \exp(\frac{B}{n})\}^n$$

is well-known, where $(n)$ means that the limit is in the sense of the (operator) norm topology. This implies that the above assertion holds for $f(t) = \exp(tA) \exp(tB)$ as well.

The above definition $e^x$ is not useful for unbounded operator. However it is well-known that if $A$ is a generator of $(C_0)$ contractive semigroup, then

$$e^{tA} = (s) \lim_{n \to \infty} (1 - \frac{t}{n} A)^{-n} \text{ for } t > 0,$$

where $(s)$ means that the limit is in the sense of strong topology. The Lie product formula was extended to the case of unbounded operators on a Banach space in [2][4].
Chernoff [1] showed a product formula in a more general form as follows:

Let $f(t)$ be a strongly continuous function from $[0, \infty)$ to the linear contractions on a Banach space. Suppose that $f(0) = 1$ and the strong derivative $f'(0)$ has a closure $A$ which is a generator of a $(C_0)$ contractive semigroup. Then $f(t/n)^n$ strongly converges to $e^{tA}$.

In the proof of this theorem the condition $||f(t)|| \leq 1$ plays an important role, so it is not easy to relax it. However we encounter many cases where $f(t)$ is not a contraction and the derivative $A$ is bounded: in this case

$$\frac{f(t)}{||f(t)||}$$

is a contraction, but may not be differentiable at $t = 0$; so we can not use the Chernoff's theorem. Therefore we need to make a new product formula for bounded operators. See [3] for product formulas.

**Theorem 1.** Let $X$ be a unital Banach algebra, and let $f(t)$ be a function from an interval $0 \leq t < \zeta$ to $X$. If $f(0) = 1$ and $f(t)$ has a norm right derivative $a$ at $t = 0$, then

$$||f(t)^{n} - \exp(ta)|| \to 0 \ (n \to \infty) \ for \ 0 \leq t < \infty.$$  

**Proof.** For every $t : 0 \leq t < \infty$, $f(t^n)$ is defined for sufficiently large $n$, so we may assume $f$ is defined on $[0, \infty)$. We claim that

there is $r > 0$ such that $||f(t)||^{t}$ is bounded on $(0, r)$.

To see this we may show that $\frac{1}{t} \log ||f(t)||$ is bounded above on $0 < t < r$.

Since

$$||\frac{f(t) - 1}{t} - a|| \to 0 \ (t \to +0),$$

\(\frac{1}{t} (||f(t)|| - 1)\) is bounded, and $||f(t)|| \to 1 \ (t \to +0)$. Thus

$$\frac{\log ||f(t)||}{t} = \left\{ \begin{array}{ll} \frac{\log ||f(t)|| - \log 1}{||f(t)|| - 1} & (||f(t)|| \neq 1) \\ 0 & (||f(t)|| = 1) \end{array} \right.$$  

is bounded on some interval $(0, r)$.

Now take an arbitrary $t : 0 < t < \infty$, and fix it. By the claim above, we can see that $\{||f(t^n)||^n\}_n$ is bounded. Thus there is $M > 0$ such that

$$e^{||a||} \leq M, \ ||f(t^n)||^n \leq M \ for \ every \ n.$$
From
\[ f \left( \frac{t}{n} \right)^n - e^{ta} = \sum_{m=0}^{n-1} f \left( \frac{t}{n} \right)^m \left\{ f \left( \frac{t}{n} \right) - e^{\frac{t}{n}a} \right\} (e^{\frac{t}{n}a})^{n-1-m}, \]

it follows that
\[
\| f \left( \frac{t}{n} \right)^n - e^{ta} \| \leq \| f \left( \frac{t}{n} \right) - e^{\frac{t}{n}a} \| \sum_{m=0}^{n-1} M^\frac{m}{n} (e^{\frac{t}{n}|a|})^{n-1-m}.
\]

Since
\[ n(M^{\frac{1}{n}} - e^{\frac{t}{n}|a|}) \to \log M - t|a| \]

and
\[ n\| f \left( \frac{t}{n} \right) - e^{\frac{t}{n}a} \| \leq t\| f \left( \frac{t}{n} \right) - 1 \| + t\| f \left( \frac{t}{n} \right) - e^{\frac{t}{n}a} + 1 \| \to 0 \quad (n \to \infty), \]

we get
\[ \| f \left( \frac{t}{n} \right)^n - e^{ta} \| \to 0 \quad (n \to \infty). \]

This concludes the proof. \( \square \)

**Corollary 1.** For \( a_i \in X \quad (i = 1, \cdots, m) \)
\[ \| \left\{ \left( 1 + \frac{a_1}{n} \right) \cdots \left( 1 + \frac{a_m}{n} \right) \right\}^n - \exp(a_1 + \cdots + a_m) \| \to 0, \]
\[ \| \left( e^{\frac{a_1}{n}} \cdots e^{\frac{a_m}{n}} \right)^n - \exp(a_1 + \cdots + a_m) \| \to 0. \]

**Proof.** By setting \( f(t) = (1 + ta_1) \cdots (1 + ta_m) \) or \( f(t) = e^{ta_1} \cdots e^{ta_m} \), these follow from the theorem. \( \square \)

Let \( \phi(z) \) be a holomorphic function in a neighborhood \( |z - 1| < \delta \). Then for \( a \in X : \|a - 1\| < \delta \), \( \phi(a) \) is defined by
\[ \phi(a) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(1)}{n!} (a - 1)^n, \]
which converges in the norm. Thus for \( f(t) \) with the property set out in the theorem \( \phi(f(t)) \) is well-defined for sufficiently small \( t \). Since \( \phi(f(0)) = \phi(1) \) and the right norm
derivative of $\phi(f(t))$ at $t = 0$ is $\phi'(1)f'(0)$, we have

**Corollary 2.** If $\phi(z)$ is a scalar valued holomorphic function in a neighborhood of $z = 1$, with $\phi(1) = 1$, then for $f(t)$ which has the property set out in the theorem,

$$||\phi(f(t))^{n} - \exp(t\phi'(1)a)|| \rightarrow 0 \quad (n \rightarrow \infty) \quad for \quad 0 \leq t < \infty.$$ 

In particular, we have

**Corollary 3.**

$$||\{(1 + \frac{a_1}{n})^{\lambda_1} \cdots (1 + \frac{a_m}{n})^{\lambda_m}\}^{n} - \exp(\lambda_1 a_1 + \cdots + \lambda_m a_m)|| \rightarrow 0 \quad for \quad \lambda_i \in \mathbb{R}.$$ 

In the proof of Theorem 1 that the domain of $f$ is the right half real line is not essential. We can get the same result as above even if the domain of $f$ is a half line with end point 0 in $\mathbb{C}$. More generally we show

**Theorem 4.** Let $X$ be a unital Banach algebra, set $D = \{z \in \mathbb{C} : \alpha \leq \arg z \leq \beta, \ 0 \leq \alpha \leq 2\pi\}$. If a function $f : D \rightarrow X$ satisfies $f(0) = 1$ and $f'(0) = a$, that is,

$$||\frac{f(z) - f(0)}{z} - a|| \rightarrow 0 \quad (z \in D, z \rightarrow 0),$$

then for every $z \in D$, $||f(\frac{z}{n})^{n} - \exp za|| \rightarrow 0 \quad (n \rightarrow \infty)$. 

**Proof.** In the same way as the proof of Theorem 1 one can easily show that $||f(z)||_{\mathcal{H}}$ is bounded on a neighborhood of $0 \in D$, and that, for fixed $z \in D$,

$$||f(\frac{z}{n})^{n} - e^{za}|| \leq ||f(\frac{z}{n}) - e^{\frac{z}{n}a}|| \sum_{m=0}^{n-1} M_{\mathcal{H}}^{\frac{m}{n}}(e^{\frac{|z|}{n}}||a||)^{m-1},$$

from which the theorem follows. \(\square\)

**References**

Section 2.

Let $A$ and $B$ be bounded selfadjoint operators on a Hilbert space. The following celebrated inequality was found by Furutau in [4] and simply proved in [5].

$$ A \geq B \geq 0 \implies A^{(p+r)/q} \geq (A^{r/2}B^pA^{r/2})^{1/q} $$

(1)

for $p \geq 0, q \geq 1, r \geq 0$ such that $(1 + r)q \geq p + r$.

Ando [1] showed the following theorem in the case of $s = p = r$ with a splendid idea. Then Fujii, Furuta, Kamai [2], by making use of Ando's result, proved that $A \geq B$ implies (2).

**Theorem A.** $A \geq B$ implies that for $p \geq 0, r \geq s \geq 0$

$$ e^{sA} \geq (e^{r/2}A^{p}e^{r/2}A)^{1/(r+p)} $$

(2)

In [1] Ando also showed the converse:

**Theorem B.** If

$$ e^{tA} \geq (e^{t/2}A^{p}e^{t/2}A)^{1/(t+p)} \quad \text{for every } t > 0, $$

then $A \geq B$.

The aim of this note is to give a new way to get exponential inequalities from operator inequalities like (1), and to extend Theorems A, B.

We start with a quite simple proof of Theorem A. This technique seems to be very effective.
to study operator inequality.

Another proof of Theorem A. For sufficiently large $n$ we have $1 + \frac{A}{n} \geq 1 + \frac{B}{n} \geq 0$. By substituting $np$ and $nr$ to $p$ and $r$ of (1), respectively, we get,

$$(1 + \frac{A}{n})^{n(p+q)} \geq \{(1 + \frac{A}{n})^{n\|A\|} (1 + \frac{B}{n})^{np} (1 + \frac{A}{n})^{n\|A\|}\}^{1/q}, \text{ for } rq \geq p + r.$$ 

Since for selfadjoint operator $X$, $(1 + \frac{X}{n})^{n}$ converges to $e^{X}$ in the operator norm as $n \to \infty$, we gain (2) by setting $s = \frac{e^{+\pi}}{q}$. \hfill \Box

We slightly extend Theorem A by using itself.

Proposition 1. $A \geq B$ implies

$$e^{sA} \geq \{e^{\frac{t}{2}A}e^{(qA+pB)}e^{\frac{t}{2}A}\}^{\frac{s}{(p+q+r)}}$$

for $p, q, r, s$ with $0 \leq s \leq r, \ 0 \leq p, p + q$, and $0 < p + q + r$.

Proof. If $p + q = 0$, then $e^{(qA+pB)}$ is contractive, so that the above inequality follows. Therefore we assume that $p + q > 0$. Since

$$\frac{qA + pB}{q + p} \leq A,$$

by using (2), we gain (3). \hfill \Box

Now we extend Theorem B:

**Theorem 2.** If there are $p, q, r, s$ with $p > 0, p + q \geq 0, r \geq s > 0$ such that

$$e^{stA} \geq \{e^{\frac{t}{2}A}e^{(qA+pB)}e^{\frac{t}{2}A}\}^{\frac{s}{(p+q+r)}}$$

for every $t > 0$, then $A \geq B$.

Proof. If $p + q + r = s$, then the above inequality implies that $e^{t(qA+pB)}$ is contractive because of $p + q = 0$. Hence $A \geq B$. Suppose $p + q + r > s$. Set

$$f(t) = e^{-\frac{t}{2}A}e^{-t(qA+pB)}e^{\frac{t}{2}A}, \quad g(t) = e^{-stA}.$$ 

Then we get

$$(f(t)\overline{x}, x) \geq (g(t)x, x) \quad (\|x\| = 1, \ t > 0),$$

for $x \neq 0$.
from which it follows that

$$(f(t)x,x)^{\frac{4}{p+q+r}} \geq (g(t)x,x) \quad (t > 0)$$

because of Jensen's inequality. Since the values of both sides of the inequality above at $t = 0$ are 1, the right derivative of the left hand side at $t = 0$ is greater than or equal to the one of the right hand side. Since the norm derivative of $e^{tT}$ at $t = 0$ is generally $T$, we have

$$\frac{s}{p+q+r}((-\frac{r}{2}A-(qA+pB)-\frac{r}{2}A)x,x) \geq (-sAx,x).$$

Hence we gain $A \geq B$. \hfill \square

We end this note with referring to an exponential inequality which appeared in [3]:

If $A - B \geq \delta > 0$, then $e^{tA} - e^{tB} \geq \delta/2 > 0$ for some $t > 0$.

This seems to be useful, so that we give a more generalized result, which we can see by a simple calculation.

Let $f(t), g(t)$ be selfadjoint operator valued functions defined in a neighborhood of $t = 0$. If $f(0) = g(0)$ and $f'(0) - g'(0) \geq \delta > 0$, where the derivative is in the sense of norm, then $f(t) - g(t) \geq \delta/2$ for $t$ in a neighborhood of 0.

References


[4] T. Furuta, $A \geq B \geq 0$ assures $(B^rA^pB^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. A.M.S. 101(1987) 85–88.