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作用素不等式二題

Exponential operator inequalities

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Section 1.

Let $X$ be a unital Banach algebra over $\mathbb{R}$ or $\mathbb{C}$, that is, a complete normed algebra with a unit 1 such that $\|1\| = 1$.
The aim of this note is, roughly speaking, to show that if $f : [0, \infty) \to X$ satisfies $f(0) = 1$, $f'(0) = a$, then $f(\frac{t}{n})^n$ converges to $e^{ta}$ as $n \to \infty$, where

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

by definition.
If $X = \mathbb{R}$, this assertion clearly follows from the L'hospital theorem. Since a set of all bounded operators on a Banach space is a unital Banach algebra, for a bounded operator $A$, $e^A$ is defined as above. In this case for bounded operators $A, B$ the Lie product formula:

\[ \exp(A + B) = (n) \lim_{n \to \infty} \{ \exp(\frac{A}{n}) \exp(\frac{B}{n}) \}^n \]

is well-known, where $(n)$ means that the limit is in the sense of the (operator) norm topology. This implies that the above assertion holds for $f(t) = \exp(tA) \exp(tB)$ as well. The above definition $e^x$ is not useful for unbounded operator. However it is well-known that if $A$ is a generator of $(C_0)$ contractive semigroup, then

\[ e^{tA} = (s) \lim_{n \to \infty} (1 - \frac{t}{n}A)^{-n} \text{ for } t > 0, \]

where $(s)$ means that the limit is in the sense of strong topology. The Lie product formula was extended to the case of unbounded operators on a Banach space in [2][4].
Chernoff [1] showed a product formula in a more general form as follows:

Let \( f(t) \) be a strongly continuous function from \([0, \infty)\) to the linear contractions on a Banach space. Suppose that \( f(0) = 1 \) and the strong derivative \( f'(0) \) has a closure \( A \) which is a generator of a \((C_{0})\) contractive semigroup. Then \( f(t/n)^{n} \) strongly converges to \( e^{tA} \).

In the proof of this theorem the condition \( ||f(t)|| \leq 1 \) plays an important role, so it is not easy to relax it. However we encounter many cases where \( f(t) \) is not a contraction and the derivative \( A \) is bounded: in this case

\[
\frac{f(t)}{||f(t)||}
\]

is a contraction, but may not be differentiable at \( t = 0 \); so we can not use the Chernoff's theorem. Therefore we need to make a new product formula for bounded operators. See [3] for product formulas.

**Theorem 1.** Let \( X \) be a unital Banach algebra, and let \( f(t) \) be a function from an interval \( 0 \leq t < \zeta \) to \( X \). If \( f(0) = 1 \) and \( f(t) \) has a norm right derivative \( a \) at \( t = 0 \), then

\[
||f(t/n)^{n} - \exp(ta)|| \rightarrow 0 \quad (n \rightarrow \infty) \quad for \quad 0 \leq t < \infty.
\]

**Proof.** For every \( t : 0 \leq t < \infty \), \( f(t/n) \) is defined for sufficiently large \( n \), so we may assume \( f \) is defined on \([0, \infty)\). We claim that there is \( r > 0 \) such that \( ||f(t)||^{1/t} \) is bounded on \((0, r)\).

To see this we may show that \( \frac{1}{t} \log ||f(t)|| \) is bounded above on \( 0 < t < r \).

Since

\[
||\frac{f(t) - 1}{t} - a|| \rightarrow 0 \quad (t \rightarrow +0),
\]

\( \frac{1}{t}(||f(t)|| - 1) \) is bounded, and \( ||f(t)|| \rightarrow 1 \quad (t \rightarrow +0) \). Thus

\[
\frac{\log ||f(t)||}{t} = \left\{ \begin{array}{ll} \frac{\log ||f(t)|| - \log 1}{||f(t)|| - 1} & (||f(t)|| \neq 1) \\
0 & (||f(t)|| = 1) \end{array} \right.
\]

is bounded on some interval \((0, r)\).

Now take an arbitrary \( t : 0 < t < \infty \), and fix it. By the claim above, we can see that \( \{||f(t/n)||^{n}\}_{n} \) is bounded. Thus there is \( M > 0 \) such that

\[
e^{\|\|a\|\|} \leq M, \quad ||f(t/n)||^{n} \leq M \quad for \quad every \quad n.
\]
From
\[ f\left(\frac{t}{n}\right)^n - e^{ta} = \sum_{m=0}^{n-1} f\left(\frac{t}{n}\right)^m \{f\left(\frac{t}{n}\right) - e^{\frac{t}{n}a}\} \{e^{\frac{t}{n}a} - e^{ta}\}^{n-1-m}, \]
it follows that
\[ ||f\left(\frac{t}{n}\right)^n - e^{ta}|| \leq ||f\left(\frac{t}{n}\right)^m - e^{\frac{t}{n}a}|| \sum_{m=0}^{n-1} M\frac{m}{n} \{e^{\frac{t}{n}||a||}\}^{n-1-m} \]
\[ = n||f\left(\frac{t}{n}\right) - e^{\frac{t}{n}a}|| \cdot \frac{M - e^{t||a||}}{n(M^\frac{1}{n} - e^{\frac{t}{n}||a||})}. \]

Since
\[ n(M^\frac{1}{n} - e^{\frac{t}{n}||a||}) \to \log M - t||a|| \]
and
\[ n||f\left(\frac{t}{n}\right) - e^{\frac{t}{n}a}|| \leq t||\frac{n}{t}\{f\left(\frac{t}{n}\right) - 1\} - a|| + t||\frac{n}{t}\{-e^{\frac{t}{n}a} + 1\} + a|| \to 0 \ (n \to \infty), \]
we get
\[ ||f\left(\frac{t}{n}\right)^n - e^{ta}|| \to 0 \ (n \to \infty). \]

This concludes the proof. \(\square\)

**Corollary 1.** For \(a_i \in X \ (i = 1, \cdots, m)\)
\[ ||\left\{(1 + \frac{a_1}{n}) \cdots (1 + \frac{a_m}{n})\right\}^n - \exp(a_1 + \cdots + a_m)|| \to 0, \]
\[ ||\left\{e^{\frac{a_1}{n}} \cdots e^{\frac{a_m}{n}}\right\}^n - \exp(a_1 + \cdots + a_m)|| \to 0. \]

**Proof.** By setting \(f(t) = (1 + ta_1) \cdots (1 + ta_m)\) or \(f(t) = e^{ta_1} \cdots e^{ta_m}\), these follows from the theorem. \(\square\)

Let \(\phi(z)\) be a holomorphic function in a neighborhood \(|z - 1| < \delta\). Then for \(a \in X : ||a - 1|| < \delta\), \(\phi(a)\) is defined by
\[ \phi(a) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(1)}{n!} (a - 1)^n, \]
which converges in the norm. Thus for \(f(t)\) with the property set out in the theorem \(\phi(f(t))\) is well-defined for sufficiently small \(t\). Since \(\phi(f(0)) = \phi(1)\) and the right norm
derivative of $\phi(f(t))$ at $t = 0$ is $\phi'(1)f'(0)$, we have

**Corollary 2.** If $\phi(z)$ is a scalar valued holomorphic function in a neighborhood of $z = 1$, with $\phi(1) = 1$, then for $f(t)$ which has the property set out in the theorem,

$$\|\phi(f(t)) - \exp(t\phi'(1)a)\| \to 0 \quad (n \to \infty) \quad \text{for} \quad 0 \leq t < \infty.$$

In particular, we have

**Corollary 3.**

$$\|\{(1 + \frac{a_1}{n})^{\lambda_1} \cdots (1 + \frac{a_m}{n})^{\lambda_m}\}^n - \exp(\lambda_1a_1 + \cdots + \lambda_ma_m)\| \to 0 \quad \text{for} \quad \lambda_i \in \mathbb{R}.$$

In the proof of Theorem 1 that the domain of $f$ is the right half real line is not essential. We can get the same result as above even if the domain of $f$ is a half line with end point 0 in $\mathbb{C}$. More generally we show

**Theorem 4.** Let $X$ be a unital Banach algebra, set $D = \{z \in \mathbb{C}: \alpha \leq \arg z \leq \beta, \quad 0 \leq \alpha \leq 2\pi\}$. If a function $f : D \to X$ satisfies $f(0) = 1$ and $f'(0) = a$, that is,

$$\left\|\frac{f(z) - f(0)}{z} - a\right\| \to 0 \quad (z \in D, \ z \to 0),$$

then for every $z \in D$, $\|f(\frac{z}{n})^n - \exp(za)\| \to 0 \quad (n \to \infty)$.

**Proof.** In the same way as the proof of Theorem 1 one can easily show that $\|f(z)\|^\frac{1}{|z|}$ is bounded on a neighborhood of $0 \in D$, and that, for fixed $z \in D$,

$$\|f(\frac{z}{n})^n - e^{za}\| \leq \|f(\frac{z}{n}) - e^{\frac{z}{n}a}\| \sum_{m=0}^{n-1} M^\frac{a}{n} (e^{\frac{|z|}{n}}\|a\|)^{h-1-m},$$

from which the theorem follows. \hfill \square

**References**

Section 2.

Let $A$ and $B$ be bounded selfadjoint operators on a Hilbert space. The following celebrated inequality was found by Furuta in [4] and simply proved in [5].

\[ A \geq B \geq 0 \implies A^{(p+r)/q} \geq (A^{r/2}B^p A^{r/2})^{1/q} \]  

(1)

for $p \geq 0, q \geq 1, r \geq 0$ such that $(1 + r)q \geq p + r$.

Ando [1] showed the following theorem in the case of $s = p = r$ with a splendid idea. Then Fujii, Furuta, Kamai [2], by making use of Ando's result, proved that $A \geq B$ implies (2).

**Theorem A.** $A \geq B$ implies that for $p \geq 0, r \geq s \geq 0$

\[ e^{sA} \geq (e^{\frac{r}{2}A} e^{pB} e^{\frac{r}{2}A})^{\frac{1}{r+p}}. \]

(2)

In [1] Ando also showed the converse:

**Theorem B.** If

\[ e^{tA} \geq (e^{\frac{r}{2}A} e^{pB} e^{\frac{r}{2}A})^{\frac{1}{r+p}} \quad \text{for every} \quad t > 0, \]

then $A \geq B$.

The aim of this note is to give a new way to get exponential inequalities from operator inequalities like (1), and to extend Theorems A, B.

We start with a quite simple proof of Theorem A. This technique seems to be very effective.
to study operator inequality.

**Another proof of Theorem A.** For sufficiently large \( n \) we have \( 1 + \frac{A}{n} \geq 1 + \frac{B}{n} \geq 0 \). By substituting \( np \) and \( nr \) to \( p \) and \( r \) of (1), respectively, we get,

\[
(1 + \frac{A}{n})^{\frac{np+rq}{q}} \geq \{(1 + \frac{A}{n})^{\frac{np}{q}}(1 + \frac{B}{n})^{np}(1 + \frac{A}{n})^{\frac{rq}{q}}\}^{1/q}, \text{ for } rq \geq p + r.
\]

Since for selfadjoint operator \( X \), \( (1 + \frac{X}{n})^{n} \) converges to \( e^{X} \) in the operator norm as \( n \to \infty \), we gain (2) by setting \( s = \frac{p+q+r}{q} \).

We slightly extend Theorem A by using itself.

**Proposition 1.** \( A \geq B \) implies

\[
e^{sA} \geq \{e^{\frac{sA}{2}}e^{(qA+pB)}e^{\frac{sA}{2}}\}^{\frac{s}{2(p+q+r)}}
\]  

(3)

for \( p, q, r, s \) with \( 0 \leq s \leq r \), \( 0 \leq p, p + q \), and \( 0 < p + q + r \).

**Proof.** If \( p + q = 0 \), then \( e^{(qA+pB)} \) is contractive, so that the above inequality follows. Therefore we assume that \( p + q > 0 \). Since

\[
\frac{qA+pB}{q+p} \leq A,
\]

by using (2), we gain (3).

Now we extend Theorem B:

**Theorem 2.** If there are \( p, q, r, s \) with \( p > 0, p + q \geq 0, r \geq s > 0 \) such that

\[
e^{sA} \geq \{e^{\frac{sA}{2}}e^{(qA+pB)}e^{\frac{sA}{2}}\}^{\frac{s}{2(p+q+r)}}
\]  

for every \( t > 0 \), then \( A \geq B \).

**Proof.** If \( p + q + r = s \), then the above inequality implies that \( e^{t(qA+pB)} \) is contractive because of \( p + q = 0 \). Hence \( A \geq B \). Suppose \( p + q + r > s \). Set

\[
f(t) = e^{\frac{-tA}{2}}e^{-t(qA+pB)}e^{\frac{tA}{2}}, \quad g(t) = e^{-stA}.
\]

Then we get

\[
(f(t)_{\frac{s}{p+q+r}}x, x) \geq (g(t)x, x) \quad (||x|| = 1, \quad t > 0),
\]
from which it follows that

\[(f(t)x, x)^{(\frac{4}{p+q+r})} \geq (g(t)x, x) \quad (t > 0)\]

because of Jensen's inequality. Since the values of both sides of the inequality above at \(t = 0\) are 1, the right derivative of the left hand side at \(t = 0\) is greater than or equal to the one of the right hand side. Since the norm derivative of \(e^{tT}\) at \(t = 0\) is generally \(T\), we have

\[\frac{s}{(p + q + r)} ((-\frac{r}{2}A - (qA + pB) - \frac{r}{2}A)x, x) \geq (-sAx, x).\]

Hence we gain \(A \geq B\).

We end this note with referring to an exponential inequality which appeared in [3]:

*If \(A - B \geq \delta > 0\), then \(e^{tA} - e^{tB} \geq \delta/2 > 0\) for some \(t > 0\).*

This seems to be useful, so that we give a more generalized result, which we can see by a simple calculation.

*Let \(f(t), g(t)\) be selfadjoint operator valued functions defined in a neighborhood of \(t = 0\). If \(f(0) = g(0)\) and \(f'(0) - g'(0) \geq \delta > 0\), where the derivative is in the sense of norm, then \(f(t) - g(t) \geq \delta/2\) for \(t\) in a neighborhood of 0.*

**References**


[4] T. Furuta, \(A \geq B \geq 0\) assures \((B^{r}A^{p}B^{r})^{1/q} \geq B^{(p+2r)/q}\) for \(r \geq 0, p \geq 0, q \geq 1\) with \((1 + 2r)q \geq p + 2r\), Proc. A.M.S. 101(1987) 85–88.