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On a problem of Fell and Doran

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Let X be a real or complex locally convex space and $\mathcal{A}$ – an algebra over the same field of scalars as X. A representation $\mathbf{T}$ of $\mathcal{A}$ on X is an algebra homomorphism $a \to T_a$ of $\mathcal{A}$ into the algebra $L(X)$ of all continuous endomorphisms of the space X. We say that $\mathbf{T}$ is irreducible if there is no proper closed linear subspace of X which is invariant for all operators $T_a, a \in \mathcal{A}$. Thus $\mathbf{T}$ is irreducible if and only if for each non-zero element x in X the orbit $\mathcal{O}(\mathbf{T}; x) = \{T_a x : a \in \mathcal{A}\}$ is dense in X. A representation $\mathbf{T}$ is said totally irreducible (cf [2]), if for each natural n and for each n-tuple $x_1, \ldots, x_n$ of linearly independent elements of X the multiple orbit $\mathcal{O}(\mathbf{T}; x_1, \ldots, x_n) = \{(T_a x_1, \ldots, T_a x_n) \in X^n : a \in \mathcal{A}\}$ is dense in X^n in the product topology. Thus $\mathbf{T}$ is totally irreducible if and only if the algebra $\mathbf{T}\mathcal{A} = \{T_a : a \in \mathcal{A}\}$ is dense in $L(X)$ in the strong operator topology (the topology of poinwise convergence on X). The commutant of $\mathbf{T}$ is defined as $\mathbf{T}' = \{T \in L(X) : TT_a = T_a T \forall a \in \mathcal{A}\}$. W say that $\mathbf{T}'$ is trivial if it consists of scalar multiples of the odentity operator only. Fell and Doran asked in [2] (Problem II, p. 329) the following question.

Problem. Let X be a locally convex space and $\mathbf{T}$ an irreducible representation on X with a trivial commutant of an algebra $\mathcal{A}$ over the same field of scalars as X. Does it follow that $\mathbf{T}$ is totally irreducible ?

The above problem asks whether the topological version of the celebrated Jacobson Density Theorem (cf. below) holds true. Let X be a real or complex vector space. An algebraic representation of an algebra $\mathcal{A}$ on X is a homomorphism into the algebra of all homomorphisms of X and it is said (algebraically) irreducible if all orbits for a non-zero element $x \in X$ are equal to X, in particular each non-zero proper subspace of X cannot be invariant for all operators T_a. Generally speaking, an irreducible representation on a locally convex space does not need be algebraically irreducible. The following result is a consequence of the Jacobson Density Theorem (cf. [1], p. 123, Theorem 10; [2], pp. 283-286; [3], pp. 271-274).

Theorem J. Let X be a real or complex vector space and let $\mathcal{A}$ be an algebra over the same field of scalars as X. Let $\mathbf{T}$ be an algebraically irreducible representation of $\mathcal{A}$ on X such that every endomorphism of X commuting with all operators T_a is a scalar multiple of the identity operator. Then for each linearly inependent n-tuple $x_1, \ldots, x_n$ of

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elements of X the orbit $\mathcal{O}(\mathbf{T}; x_1, \ldots, x_n)$ coincides with X^n.

The above result does not imply a positive answer to the above Problem even when $\mathbf{T}$ is algebraically irreducible, since the triviality of $\mathbf{T}'$ does not necessarily mean that the endomorphisms of X commuting with all operators T_a must coincide with the scalar multiples of the identity. A positive answer to this problem in the case of an algebraically irreducible representation on a space of type F was given in [4]. There is no known example of an infinite dimensional Banach space X for which the answer to the Problem is in positive, nor is there an example of a locally convex space for which the answer is negative, though the general belief is that such an example should exist. In this talk we shall discuss the following topics.

I. A positive answer to the Problem of Fell and Doran in the case when $X = (s)$ – the (completely metrizable) locally convex space of all scalar valued sequences, or, more generally, when X is an arbitrary product of the field of scalars ([7]). Those are the only known examples of the positive solution of the Problem in case when the considered representations are not automatically algebraically irreducible.

II. Necessary and sufficient conditions in order that a representation satisfying the assumptions of the Problem satisfies also the conclusion ([5]). We do not know whether these conditions are satisfied automatically. If so, they give a way of attacking the Problem, if not, they offer an additional con- dition under which the answer is positive. The conditions are expressed in terms of intertwinning operators.

II. A concept of "order" of a strongly closed proper subalgebra of L(X). The existence of a subalgebra of order 3 implies immediately the existence of a counterexample to the Problem ([6]).

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