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<tr>
<td>Author(s)</td>
<td>Uchiyama, Atsushi</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1999), 1080: 67-70</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1999-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62712">http://hdl.handle.net/2433/62712</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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An Example of a $p$-Quasihyponormal Operator

Introduction. A bounded linear operator $T$ on a Hilbert space $\mathcal{H}$ is called $p$-hyponormal if $(T^*T)^p \geq (TT^*)^p$ for $p > 0$, and $T$ is called $p$-quasihyponormal if $T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0$ for $p > 0$. $T$ is called paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$. It is well-known by Ando[3] that every $p$-hyponormal operator is paranormal. M. Lee and S. Lee showed that every $p$-quasihyponormal operator $T = U|T|$ is $q$-hyponormal for all $q \in (0, p)$ by Heinz's inequality and its generalized Aluthge transform $T(s, t) = |T|^sU|T|^t$ for $s, t > 0$ is a $q$-hyponormal for some $q = q(s, t, p) > 0$. (See [1],[2],[6],[7] and [13]). But the assertions that $p$-quasihyponormal is $q$-quasihyponormal if $0 < q < p$ and the generalized Aluthge transform $T(s, t) = |T|^sU|T|^t$ for $s, t > 0$ of a $p$-quasihyponormal operator $T = U|T|$ is a $q$-quasihyponormal for some $q = q(s, t, p) > 0$ are not true.

In this paper, we give a $p$-quasihyponormal operator $T = U|T|$ such that (i) $T$ is not $q$-quasihyponormal for all $q \in (0, p)$, (ii) $|T|^sU|T|^t$ for $s, t > 0$ is not $q$-quasihyponormal for all $q \in (0, \infty)$ and (iii) $T$ is a $p$-quasihyponormal for a $p > 1$, but is not paranormal.

Lemma 1. (Hölder-McCarthy Inequality[9]) For any positive operator $A$ and $x \in \mathcal{H}$,

$$(1) \quad (A^r x, x) \leq \|x\|^{2(1-r)} (Ax, x)^r \quad \text{(if } 0 < r \leq 1\text{)},$$

$$(2) \quad (A^r x, x) \geq \|x\|^{2(1-r)} (Ax, x)^r \quad \text{(if } r \geq 1\text{)}.$$  

Using above lemma, M. Lee and H. Lee obtained the following.
**Theorem 1.** (M. Lee and H. Lee[10]) If $T$ is a $p$-quasihyponormal operator such as $0 < p \leq 1$, then $T$ is paranormal.

Here, we construct an example of $p$-quasihyponormal operator which satisfies the conditions(i)-(iii) in the introduction.

Let $\{\varepsilon_n; n \in \mathbb{Z}\}$ be the canonical orthonormal basis of $\ell^2(\mathbb{Z})$ and $p_n$ the projection of $\ell^2(\mathbb{Z})$ to $\mathbb{C}\varepsilon_n$. Using the shift operator $S$ on $\ell^2(\mathbb{Z})$ with $S\varepsilon_n = \varepsilon_{n+1}$ and positive $2 \times 2$ Hermitian matrices $A$ and $B$, we define operators $H$ and $T$ on $\mathbb{C}^2 \otimes \ell^2(\mathbb{Z})$ by

$$H = \sum_{n<0} A \otimes p_n + \sum_{n \geq 0} B \otimes p_n$$

and

$$T = (1 \otimes S)H.$$

$T = U|T|$, where $U = 1 \otimes S$ and $|T| = H$. Since $|T^*| = U|T|U^* = \Sigma_{n \leq 0} A \otimes p_n + \Sigma_{n > 0} B \otimes p_n$, it is easy to see that

$$T^* (|T|^{2p} - |T^*|^{2p})T = A(B^{2p} - A^{2p})A \otimes p_{-1}$$

for $p > 0$. Hence we have the following.

**Lemma 2.** $T$ is $p$-quasihyponormal if and only if $A(B^{2p} - A^{2p})A \geq 0$.

In what follows we assume that $A$ and $B$ are of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

respectively, here $\alpha > 0$. Let $f$ be a function on the half interval $(0, \infty)$ defined by

$$f(p) = \left(\frac{9^p + 1}{2}\right)^{\frac{1}{2p}}.$$

Then it is strictly increasing.

**Theorem 2.**

1. $T$ is $p$-quasihyponormal if and only if $\alpha \leq f(p)$.

2. If $\alpha = f(p)$, then $T$ is not $q$-quasihyponormal for $q \in (0, p)$, but $q$-quasihyponormal for $q \in [p, \infty)$. Hence $T$ satisfies the condition(i).

**Proof.** (1) Since

$$B^{2p} = \frac{1}{2} \begin{pmatrix} 9^p + 1 & 9^p - 1 \\ 9^p - 1 & 9^p + 1 \end{pmatrix},$$
it is easy to see that \( T \) is \( p \)-quasihyponormal if and only if 
\[
(9^p + 1)/2 - \alpha^{2p} \geq 0.
\]
(2) It is immediate from (1). QED

**Theorem 3.** Let \( T(s, t) = |T|^{sU}T^t \) for \( s, t > 0 \).

(1) If \( T(s, t) \) is \( p \)-quasihyponormal, then \( \alpha \leq f(s) \).

(2) If \( \alpha = f(p) \) and \( s \in (0, p) \), then \( T(s, t) \) is not \( q \)-quasihyponormal for all \( q > 0 \). Hence \( T \) satisfies the condition(ii).

**Proof.**

(1) Since
\[
T(s, t)^*(|T(s, t)|^{2p} - |T(s, t)^*|^{2p})T(s, t)
= A^{s+t} \{(A^t B^{2s} A^t)^p - A^{2(s+t)p}\} A^{s+t} \otimes p_{-2}
+ A^t B^{s} \{B^{2(s+t)p} - (B^s A^{2t} B^s)^p\} B^s A^t \otimes p_{-1},
\]
\( T(s, t) \) is \( p \)-quasihyponormal if and only if
\[
(A^t B^{2s} A^t)^p - A^{2(s+t)p} \geq 0, \quad \text{and} \quad A^t B^{s} \{B^{2(s+t)p} - (B^s A^{2t} B^s)^p\} B^s A^t \geq 0.
\]
The former inequality implies that \( \alpha \leq f(s) \).

(2) It is immediate from (1). QED

**Theorem 4.** \( T \) is paranormal if and only if \( \alpha \leq \sqrt{5} = f(1) \).

**Proof.** It is well-known by Ando[3] that an operator \( S \) is a paranormal if and only if \( S^{*2}S^2 - 2kS^{*}S + k^2 \geq 0 \) for all \( k \in \mathbb{R} \).

Since
\[
T^{*2}T^2 - 2kT^{*}T + k^2 = \sum_{n<-1} (A^2 - k)^2 \otimes p_n + (AB^2 A - 2k A^2 + k^2) \otimes p_{-1}
+ \sum_{n\geq0} (B^2 - k)^2 \otimes p_n.
\]
\( T \) is a paranormal \( \Leftrightarrow AB^2 A - 2k A^2 + k^2 \geq 0 \ \forall k \in \mathbb{R} \)
\( \Leftrightarrow 5\alpha^2 - 2k\alpha^2 + k^2 \geq 0 \ \forall k \in \mathbb{R} \)
\( \Leftrightarrow \alpha^4 - 5\alpha^2 \leq 0 \)
\( \Leftrightarrow \alpha \leq \sqrt{5} = f(1) \) (since \( \alpha > 0 \)). QED

**Remark.** If \( \alpha = f(p) \) for \( p > 1 \), then \( T \) is a \( p \)-quasihyponormal by Theorem 2, but \( T \) is not paranormal by Theorem 4. Hence \( T \) satisfies the condition(iii).
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