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Quantum deformations of certain prehomogeneous vector spaces

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0 Notation

Let $\mathfrak{g}$ be a semisimple Lie algebra over the complex number field $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ and $W \subset GL(\mathfrak{h})$ be the root system and the Weyl group respectively. For each $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. We denote the set of positive roots by $\Delta^+$ and the set of simple roots by $\{\alpha_i\}_{i \in I_0}$, where $I_0$ is an index set. Set

$$n^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad n^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}.$$ 

For $i \in I_0$ let $h_i \in \mathfrak{h}$, $\varpi_i \in \mathfrak{h}^*$ and $s_i \in W$ be the simple coroot, the fundamental weight, the simple reflection corresponding to $i$ respectively. Take $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$, satisfying $[e_i, f_i] = h_i$. Let $(\ , \) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for short roots $\alpha$. We set

$$d_i = \frac{(\alpha_i, \alpha_i)}{2} \quad (i \in I_0), \quad a_{ij} = \alpha_j(h_i) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \quad (i, j \in I_0).$$

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For a subset $I$ of $I_0$ we set

$$\Delta_I = \Delta \cap \sum_{i \in I} \mathbb{Z} \alpha_i, \quad W_I = \langle s_i | i \in I \rangle,$$

$$I_I = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta_I \setminus \Delta} \mathfrak{g}_\alpha), \quad n_I^+ = \oplus_{\alpha \in \Delta_I \setminus \Delta} \mathfrak{g}_\alpha, \quad n_I^- = \oplus_{\alpha \in \Delta_I \setminus \Delta} \mathfrak{g}_{-\alpha},$$

$$\mathfrak{h}_I^* = \oplus_{i \in I_0 \setminus I} \mathbb{C} \varpi_i \subset \mathfrak{h}^*, \quad \mathfrak{n}_I = \oplus_{i \in I_0 \setminus I} \mathbb{Z} \varpi_i \subset \mathfrak{g}^*.$$

For a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the enveloping algebra of $\mathfrak{a}$.

1 Quantized enveloping algebras

The quantized enveloping algebra $U_q(\mathfrak{g})$ ([1], [7]) is an associative algebra over the rational function field $\mathbb{C}(q)$ generated by the elements $\{E_i, F_i, K_i, K_i^{-1}\}_{i \in I_0}$ satisfying the following relations:

$$K_i K_j = K_j K_i,$$

$$K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$K_i E_j K_i^{-1} = q_{ij} E_j,$$

$$K_i F_j K_i^{-1} = q_{ij}^{-1} F_j,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} \frac{E_i^{1-a_{ij}-k} E_j E_i^k}{q_i} = 0 \quad (i \neq j),$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} \frac{F_i^{1-a_{ij}-k} F_j F_i^k}{q_i} = 0 \quad (i \neq j),$$

where $q_i = q^{d_i}$, and

$$[m]_t = \frac{t^m - t^{-m}}{t - t^{-1}}, \quad \frac{[m]_t!}{[n]_t!} = \prod_{k=1}^{m} [k]_t, \quad \frac{[m]_t!}{[n]_t! [m-n]_t!} \quad (m \geq n \geq 0).$$
We define the Hopf algebra structure on $U_q(g)$ as follows. The comultiplication
$\Delta : U_q(g) \to U_q(g) \otimes U_q(g)$ is the algebra homomorphism satisfying

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i.$$  

The counit $\epsilon : U_q(g) \to \mathbb{C}(q)$ is the algebra homomorphism satisfying

$$\epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0.$$  

The antipode $S : U_q(g) \to U_q(g)$ is the algebra antiautomorphism satisfying

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i, \quad S(F_i) = -K_i^{-1} F_i.$$  

The adjoint action of $U_q(g)$ on $U_q(g)$ is defined as follows. For $x, y \in U_q(g)$ write
$\Delta(x) = \sum_k x_k^1 \otimes x_k^2$ and set $(\text{ad} x)(y) = \sum_k x_k^1 y S(x_k^2_k)$. Then
$\text{ad} : U_q(g) \to \text{End}_{\mathbb{C}(g)}(U_q(g))$ is an algebra homomorphism.

We define subalgebras $U_q(n^\pm), U_q(h)$ and $U_q(l_I)$ for $I \subset I_0$ by

$$U_q(n^+) = \langle E_i | i \in I_0 \rangle, \quad U_q(n^-) = \langle F_i | i \in I_0 \rangle,$$

$$U_q(h) = \langle K_i^\pm | i \in I_0 \rangle, \quad U_q(l_I) = \langle K_i^\pm, E_j, F_j | i \in I_0, j \in I \rangle.$$  

For $i \in I_0$ we define an algebra automorphism $T_i$ of $U_q(g)$ (see [8]) by

$$T_i(K_j) = K_j K_i^{-a_{ij}},$$

$$T_i(E_j) = \begin{cases} -F_i K_i & (i = j) \\
                         \sum_{k=0}^{-a_{ij}} (-q_i)^{-k} E_i E_j^{(-a_{ij}-k)} & (i \neq j), \end{cases}$$

$$T_i(F_j) = \begin{cases} -K_i^{-1} E_i & (i = j) \\
                         \sum_{k=0}^{-a_{ij}} (-q_i)^k F_i F_j^{(-a_{ij}-k)} & (i \neq j), \end{cases}$$

where

$$E_i^{(k)} = \frac{1}{[k]_{q_i}} E_i^k, \quad F_i^{(k)} = \frac{1}{[k]_{q_i}} F_i^k.$$
For $w \in W$ we choose a reduced expression $w = s_{i_1} \cdots s_{i_k}$ and set $T_w = T_{i_1} \cdots T_{i_k}$. It is known that $T_w$ does not depend on the choice of the reduced expression.

For $I \subset I_0$ let $w_I$ be the longest element of $W_I$ and set

$$U_q(n_I^-) = U_q(n^-) \cap T_{w_I}^{-1}U_q(n^-).$$

Let $w_0$ be the longest element of $W$ and take a reduced expression $w_I w_0 = S_{i_1} \cdots S_{i_r}$ of $w_I w_0$. We set

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \quad Y_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k})$$

for $k = 1, \ldots, r$. Then it is known that $\{\beta_k|1 \leq k \leq r\} = \Delta^+ \setminus \Delta_I$, and that $\{Y_{\beta_1}^{d_1} \cdots Y_{\beta_r}^{d_r}|d_1, \ldots, d_r \in \mathbb{Z}_{\geq 0}\}$ is a basis of $U_q(n_I^-)$. This basis depends on the choice of the reduced expression of $w_I w_0$ in general.

**Proposition 1.1** (ad $U_q(\mathfrak{t}_I)$)

$(U_q(n_I^-)) \subset U_q(n_I^-)$.

For $N \in \mathbb{Z}_{>0}$ we set $U_{q,N}(\mathfrak{g}) = \mathbb{C}(q^{1/N}) \otimes_{\mathbb{C}(q)} U_q(\mathfrak{g})$, and let $U_{q,N}(\mathfrak{n}^\pm), U_{q,N}(\mathfrak{h}), U_{q,N}(1_I), U_{q,N}(\mathfrak{n}_I^-)$ be the $\mathbb{C}(q)$-subalgebras of $U_{q,N}(\mathfrak{g})$ generated by $U_q(\mathfrak{n}^\pm), U_q(\mathfrak{h}), U_q(1_I), U_q(\mathfrak{n}_I^-)$ respectively.

For $\lambda \in \mathfrak{h}_I^*$ we define a $U(\mathfrak{g})$-module $M_I(\lambda)$ by

$$M_I(\lambda) = U(\mathfrak{g})/(\sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - \lambda(h)) + U(\mathfrak{g})n^+ + U(\mathfrak{g})(1 \cap n^-)).$$

It is a highest weight module with highest weight $\lambda$ and highest weight vector $m_{I,\lambda} = \bar{1}$, where $\bar{1}$ denotes the element of $M_I(\lambda)$ corresponding to $1 \in U(\mathfrak{g})$. $M_I(\lambda)$ contains a unique maximal proper submodule $K_I(\lambda)$, and $L(\lambda) = M_I(\lambda)/K_I(\lambda)$ is a unique (up to an isomorphism) irreducible highest weight module with highest weight $\lambda$.

For $\lambda \in \mathfrak{h}_{I,\mathbb{Z}}^* / N$ we define a $U(\mathfrak{g})$-module $M_I(\lambda)$ by

$$M_{I,q,N}(\lambda) = U_{q,N}(\mathfrak{g})/(\sum_{i \in I_0} U_{q,N}(\mathfrak{g})(K_i - q_i^{\lambda(h_i)}) + \sum_{i \in I_0} U_{q,N}(\mathfrak{g})E_i + \sum_{j \in I} U_{q,N}(\mathfrak{g})F_j).$$
It is a highest weight module with highest weight $\lambda$ and highest weight vector $m_{I,\lambda,q,N} = \overline{1}$. $M_I(\lambda)$ contains a unique maximal proper submodule $K_{I,q,N}(\lambda)$, and $L_{q,N}(\lambda) = M_{I,q,N}(\lambda)/K_{I,q,N}(\lambda)$ is a unique irreducible highest weight module with highest weight $\lambda$.

2 Main result

In the rest of this note we fix $I \subset I_0$ satisfying $n_I^+ \neq \{0\}$ and $[n_I^+, n_I^+] = \{0\}$. This is equivalent to the following condition:

$$I = I_0 \setminus \{i_0\} \text{ with } m_{i_0} = 1,$$

where $\theta = \sum_{i \in I_0} m_i \alpha_i$ is the highest root (see [14]).

We set $l = l_I$, $m^\pm = n_I^\pm$ for simplicity.

**Proposition 2.1** The element $Y_\beta \in U_q(m^-)$ for $\beta \in \Delta^+ \setminus \Delta_I$ dose not depend on the choice of a reduced expression of $w_I w_0$.

Fix a reduced expression $w_I w_0 = s_{i_1} \ldots s_{i_r}$ and set $\beta_p = s_{i_1} \ldots s_{i_{p-1}}(\alpha_{i_p})$. We set

$$U_q(m^-)^m = \sum_{\beta_1, \ldots, \beta_m} \mathbb{C}(q) Y_{\beta_1} \cdots Y_{\beta_m} \quad (m \geq 0).$$

**Lemma 2.2** We have

$$U_q(m^-) = \bigoplus_{m=0}^{\infty} U_q(m^-)^m,$$

$$U_q(m^-)^m = \bigoplus_{\sum_p m_p = m} \mathbb{C}(q) Y_{\beta_1}^{m_1} \cdots Y_{\beta_r}^{m_r} = \bigoplus_{\gamma \in m a_{i_0} + Q_I^+} U_q(m^-)_{-\gamma}.$$

Here $U_q(m^-)_{-\gamma}$ is the weight space with respect to the adjoint action of $U_q(\mathfrak{h})$ on $U_q(m^-)$, and $Q_I^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. 
By Lemma 2.2 we can write

\[ Y_{\beta_{p_{1}}}Y_{\beta_{p_{2}}} = \sum_{s_{1} \leq s_{2}} \alpha^{p_{1},p_{2}}_{s_{1},s_{2}} Y_{\beta_{s_{1}}}Y_{\beta_{s_{2}}} \quad (\alpha^{p_{1},p_{2}}_{s_{1},s_{2}} \in \mathbb{C}(q)) \]  

for \( p_{1} > p_{2} \).

**Proposition 2.3** The \( \mathbb{C}(q) \)-algebra \( U_{q}(m^{-}) \) is generated by the elements \( \{Y_{\beta_{p}}|1 \leq p \leq r\} \) satisfying the fundamental relations (1) for \( p_{1} > p_{2} \).

By the commutativity of \( m^{-} \), \( U(m^{-}) \) is isomorphic to the symmetric algebra \( S(m^{-}) \). Since \( m^{-} \) is identified with \( (m^{+})^{*} \) via the Killing form of \( g \), \( S(m^{-}) \) is isomorphic to the algebra \( \mathbb{C}[m^{+}] \) of polynomial functions on \( m^{+} \). Hence we have an identification \( U(m^{-}) = \mathbb{C}[m^{+}] \). We denote by \( \mathbb{C}[m^{+}]^{m} (m \in \mathbb{Z}_{\geq 0}) \) the subspace of \( \mathbb{C}[m^{+}] \) consisting of homogeneous elements with degree \( m \). We set \( \mathfrak{h}_{m}^{\ast}(I, +) = \{ \lambda \in \mathfrak{h}^{\ast} | \lambda(h_{i}) \in \mathbb{Z}, \lambda(h_{i}) \in \mathbb{Z}_{\geq 0} (i \in I) \} \). For \( \lambda \in \mathfrak{h}_{m}^{\ast}(I, +) \) we denote the finite dimensional irreducible \( U(l)-\)module (resp. \( U_{q}(l)-\)module) with highest weight \( \lambda \) by \( V(\lambda) \) (resp. \( V_{q}(\lambda) \)). We can decompose the finite dimensional \( l \)-module \( \mathbb{C}[m^{+}]^{m} \) into a direct sum of submodules isomorphic to \( V(\lambda) \) for some \( \lambda \in \mathfrak{h}_{m}^{\ast}(I, +) \). It is known that

\[ \mathbb{C}[m^{+}] \simeq \bigoplus_{\lambda \in \Gamma_{m}^{m}} V(\lambda) \]

for finite subset \( \Gamma_{m}^{m} \) of \( \mathfrak{h}_{m}^{\ast}(I, +) \) satisfying \( \Gamma_{m}^{m} \cap \Gamma_{m'}^{m} = \emptyset \) for \( m \neq m' \) (see [11], [12], [6]). On the other hand, since \( U_{q}(m^{-})^{m} \) is a finite dimensional \( U_{q}(l) \)-module whose character is the same as that of \( \mathbb{C}[m^{+}]^{m} \), we have

\[ U_{q}(m^{-})^{m} \simeq \bigoplus_{\lambda \in \Gamma_{m}^{m}} V_{q}(\lambda). \]

Let \( L \) be the algebraic group corresponding to \( l \). It is known that \( m^{+} \) consists of finitely many \( L \)-orbits, and that the orbits can be labeled by

\[ \{L \text{-orbits on } m^{+}\} = \{C_{0}, C_{1}, \ldots, C_{t}\}, \quad \{0\} = C_{0} \subset C_{1} \subset \cdots \subset C_{t} = m^{+}. \]
We set
\[ \mathcal{I}(\overline{C_{p}}) = \{ f \in \mathbb{C}[m^+] | f(\overline{C_{p}}) = 0 \}. \]

Since \( \mathcal{I}(\overline{C_{p}}) \) is an \( l \)-submodule of \( \mathbb{C}[m^+] \), we have
\[ \mathcal{I}(\overline{C_{p}}) = \bigoplus_{m} \mathcal{I}^{m}(\overline{C_{p}}), \quad \mathcal{I}^{m}(\overline{C_{p}}) = \mathcal{I}(\overline{C_{p}}) \cap \mathbb{C}[m^+]^{m} \cong \bigoplus_{\lambda \in \Gamma^{m}_{p}} V(\lambda) \]
for a subset \( \Gamma^{m}_{p} \) of \( \Gamma^{m} \). The following facts are known (see, for example, [14]):

**Proposition 2.4** Let \( p = 0, \ldots, t - 1 \).

(i) \( \mathcal{I}^{m}(\overline{C_{p}}) = 0 \) for \( m \leq p \).

(ii) \( \mathcal{I}^{p+1}(\overline{C_{p}}) \) is an irreducible \( l \)-module.

(iii) \( \mathcal{I}(\overline{C_{p}}) \) is generated by \( \mathcal{I}^{p+1}(\overline{C_{p}}) \) as an ideal of \( \mathbb{C}[m^+] \).

**Proposition 2.5** For \( p = 0, \ldots, t - 1 \) there exists a unique \( \lambda_{p} \in \mathfrak{h}_{I}^{*} \) such that \( K_{I,\mathcal{I}(\overline{C_{p}})} = \mathcal{I}(\overline{C_{p}})m_{I,\lambda_{p}}. \) Moreover, we have \( \lambda_{p} \in \mathfrak{h}_{I,\mathbb{Z}}^{*}/2 \).

We set
\[ \mathcal{I}_{q}^{m}(\overline{C_{p}}) = \bigoplus_{\lambda \in \Gamma^{m}_{p}} V_{q}(\lambda) \subset U_{q}^{m}(m^{-}), \]
\[ \mathcal{I}_{q}(\overline{C_{p}}) = \bigoplus_{m} \mathcal{I}_{q}^{m}(\overline{C_{p}}) \subset U_{q}(m^{-}), \]
\[ \mathcal{I}_{q,N}^{m}(\overline{C_{p}}) = \mathbb{C}(q^{1/N}) \otimes \mathbb{C}(q) \mathcal{I}_{q,p}^{m}(\overline{C_{p}}) \subset U_{q,N}(m^{-})^{m}, \]
\[ \mathcal{I}_{q,N}(\overline{C_{p}}) = \bigoplus_{m} \mathcal{I}_{q,N}^{m}(\overline{C_{p}}) \subset U_{q,N}(m^{-}). \]

Here we identify \( U_{q}(m^{-})^{m} \) with \( \bigoplus_{\lambda \in \Gamma^{m}} V_{q}(\lambda) \).

**Proposition 2.6** ([15]) For \( p = 0, \ldots, t - 1 \) we have
\[ \text{ch}(L_{q,2}(\lambda_{p})) = \text{ch}(L(\lambda_{p})), \quad K_{I,q,2}(\lambda_{p}) = U_{q,2}(m^{-})\mathcal{I}_{q,2}^{p+1}(\overline{C_{p}})m_{I,\lambda_{p},q,2}. \]
By Proposition 2.6 we have the main result.

**Theorem 2.7 ([15])** We have

\[ \mathcal{I}_q(C_p) = U_q(m^-)\mathcal{I}_q^{p+1}(C_p) = \mathcal{I}_q^{p+1}(C_p)U_q(m^-) \]

for \( p = 0, \ldots, t-1 \).

3 Examples

We shall give an explicit description of \( \mathcal{I}_q^{p+1}(C_p) \) in each individual case. (see [16], [17])

3.1 Type \( A_n \)

We label the vertices of the Dynkin diagram as follows.

```
1 2 k-1 k k+1 ... n-1 n
```

Hence we have \( I_0 = \{1, \ldots, n\} \). Set \( I = I_0 \setminus \{i_0\} \), where \( i_0 = k \) \((k-1 \leq n-k)\).

We fix a reduced expression

\[ w_Iw_0 = (s_ks_{k+1} \cdots s_n)(s_{k-1}s_k \cdots s_{n-1}) \cdots (s_1s_2 \cdots s_{n-k+1}) \]

We set

\[ Y_{i,j} = (-1)^{k-i}(T_kT_{k+1} \cdots T_n)(T_{k-1}T_k \cdots T_{n-1}) \cdots (T_{i+1}T_{i+2} \cdots T_{n-k+i+1}) \]

\[ T_iT_{i+1} \cdots T_{i+j-2}(F_{i+j-1}) \]

\((1 \leq i \leq k, 1 \leq j \leq n+1-k)\).

Set

\[ \beta_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_k + \cdots + \alpha_{k+j}. \]
We have $Y_{i,j} \in U_q(m^-)_{\beta_{i,j}}$.

Then we have the following fundamental relations of $U_q(m^-)$.

\[
Y_{i,j}Y_{l,m} = \begin{cases} 
qY_{l,m}Y_{i,j} & (i = l, j < m \text{ or } i > l, j = m) \\
Y_{l,m}Y_{i,j} & (i > l, j > m) \\
Y_{l,m}Y_{i,j} + (q-q^{-1})Y_{i,m}Y_{l,j} & (i > l, j < m).
\end{cases}
\]

We label $k+1$ $L$-orbits on $m^+$ as in Section 2. For $p = 0, 1, \ldots, k-1$ we have

\[
\mathcal{I}_q^{p+1}(C_p) = \sum \mathbb{C}(q)(i_1 \ldots j_{p+1})
\]

where we sum over all the sequences $\{i_1, i_2, \ldots, i_{p+1}\}, \{j_1, j_2, \ldots, j_{p+1}\} \subset \mathbb{N}$ satisfying

\[
1 \leq i_1 < i_2 < \cdots < i_{p+1} \leq k, \quad 1 \leq j_1 < j_2 < \cdots < j_{p+1} \leq n+1-k,
\]

and set

\[
\left( \begin{array}{cccc}
i_1 & i_2 & \cdots & i_{p+1} \\
j_1 & j_2 & \cdots & j_{p+1}
\end{array} \right) = \sum_{\sigma \in S_{p+1}} (-q)^l(\sigma)Y_{i_1,j_{\sigma(1)}}Y_{i_2,j_{\sigma(2)}} \cdots Y_{i_{p+1},j_{\sigma(p+1)}},
\]

\[
l(\sigma) = \# \{(i,j) | i < j, \sigma(i) > \sigma(j)\}.
\]

### 3.2 Type $C_n$

We label the vertices of the Dynkin diagram as follows.

![Dynkin diagram](image)

Hence we have $I_0 = \{1, \ldots, n\}$. Set $I = I_0 \setminus \{i_0\}$, where $i_0 = n$. We fix a reduced expression

\[
w_Iw_0 = (s_n s_{n-1} \cdots s_1)(s_n s_{n-1} \cdots s_2) \cdots (s_n s_{n-1}) s_n.
\]
We set
\[ Y_{i,j} = c_{i,j}(T_n T_{n-1} \cdots T_1)(T_n T_{n-1} \cdots T_2) \cdots (T_n T_{n-1} \cdots T_{n-j}) \]
\[ \cdots (T_n T_{n-1} \cdots T_{n-i+1})(F_{n-j+i}) \]
where
\[ c_{i,j} = \begin{cases} 
(q + q^{-1}) & (1 \leq i = j \leq n) \\
(-1)^{j-i} & (1 \leq i < j \leq n).
\end{cases} \]
Set
\[ \beta_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n. \]
We have \( Y_{i,j} \in U_q(m^{-})_{-\beta_{i,j}} \).

Then we have the following fundamental relations of \( U_q(m^{-}) \).
\[
Y_{i,j} Y_{i,m} = \begin{cases} 
q_{n-j+i} Y_{i,m} Y_{i,j} & (j = m, i > l) \\
Y_{i,m} Y_{i,j} & (j > m, i < l) \\
q_{n-m+i} Y_{i,m} Y_{i,j} & (j > m, i = l) \\
Y_{i,m} Y_{i,j} + (q - q^{-1}) Y_{i,m} Y_{i,j} & (l < i < m < j) \\
q Y_{i,m} Y_{i,j} + (q - q^{-1}) Y_{i,m} Y_{i,j} & (l < i = m < j) \\
Y_{i,m} Y_{i,j} + (q^2 - q^{-2}) Y_{i,m} Y_{i,j} & (l = m < i < j) \\
Y_{i,m} Y_{i,j} + (q - q^{-1}) \{ Y_{i,m} Y_{i,j} - qY_{m,i} Y_{i,j} \} & (l < m < i < j) \\
Y_{i,m} Y_{i,j} + q^{-1}(q^2 - q^{-2}) Y_{i,j}^2 & (l = m < i = j) \\
Y_{i,m} Y_{i,j} + (q^2 - q^{-2}) Y_{i,j} Y_{m,i} & (l < m < i = j)
\end{cases}
\]
We label \( n + 1 \) \( L \)-orbits on \( m^+ \) as in Section 2. For \( p = 0, 1, \ldots, n - 1 \) the highest weight vector of \( T_{q+1}(C_p) \) is
\[
\sum_{\sigma \in S_{p+1}} (-q^{-1})^{|l(\sigma)|} Y_{i_{j_{1}(\sigma)}(1)} Y_{i_{j_{2}(\sigma)}(2)} \cdots Y_{i_{j_{p+1}(\sigma)(p+1)}}
\]
where \(i_1 = j_1 = n - p, \ i_2 = j_2 = n - p + 1, \ldots, \ i_{p+1} = j_{p+1} = n\) and \(Y_{i,j} = q^{-2}Y_{j,i} \ (i < j)\).

### 3.3 Type \(B_n\)

We label the vertices of the Dynkin diagram as follows.

![Dynkin Diagram](image)

Hence we have \(I_0 = \{1, \ldots, n\}\). Set \(I = I_0 \setminus \{i_0\}\), where \(i_0 = 1\). We fix a reduced expression

\[
w_I w_0 = s_1 s_2 \cdots s_{n-1} s_n s_{n-1} s_{n-2} \cdots s_2 s_1.
\]

We set

\[
Y_i = \begin{cases} 
T_1 T_2 \cdots T_{i-1}(F_i) & (1 \leq i \leq n) \\
T_1 T_2 \cdots T_{n-1} T_n T_{n-1} T_{n-2} \cdots T_{2n-i+1}(F_{2n-i}) & (n + 1 \leq i \leq 2n - 1).
\end{cases}
\]

Then we have the following fundamental relations of \(U_q(m^-)\).

\[
Y_i Y_j = \begin{cases} 
q^{-2}Y_j Y_i & (i > j, i + j \neq 2n) \\
Y_j Y_i + \frac{q^{-2} - 1}{q + q^{-1}} Y_n^2 & (i = n + 1, j = n - 1) \\
Y_j Y_i + (q^{-2} - q^2) \sum_{l=1}^{i+n-1} (-q^2)^{l-1} Y_{j+l} Y_{i-l} - (-q^2)^{i+n-1} \frac{q^{-2} - 1}{q + q^{-1}} Y_n^2 & (j \leq n - 2, i + j = 2n)
\end{cases}
\]

We label 3 \(L\)-orbits on \(m^+\) as in Section 2. For \(p = 0, 1\) we have

\[
I^1_q(C_0) = \sum_{i=1}^{2n-1} C(q)Y_i, \\
I^2_q(C_1) = C(q)\psi
\]

where \(\psi = Y_n Y_n - (q + q^{-1})(1 + q^{-2}) \sum_{i=1}^{n-1} (-q^{-2})^{i-1} Y_{n-i} Y_{n+i} \).
3.4 Type $D_n$

We have the following two cases.

Case 1

\[ I_0 = \{1, \ldots, n\}, i_0 = 1 \]

Case 2

\[ I_0 = \{1, \ldots, n\}, i_0 = n \]

In case 1 we fix a reduced expression

\[ w_Iw_0 = s_1s_2 \cdots s_{n-1}s_ns_{n-2} \cdots s_2s_1. \]

Set

\[ Y_i = \begin{cases} 
T_1T_2 \cdots T_{i-1}(F_i) & (1 \leq i \leq n) \\
T_1T_2 \cdots T_{n-1}T_nT_{n-2}T_{n-3} \cdots T_{2n-i}(F_{2n-i-1}) & (n + 1 \leq i \leq 2n - 2). 
\end{cases} \]

Then we have the following fundamental relations of $U_q(m^-)$.

\[ Y_i Y_j = \begin{cases} 
q^{-1}Y_jY_i & (i > j, i + j \neq 2n - 1) \\
Y_jY_i & (i = n, j = n - 1) \\
Y_jY_i - (q - q^{-1}) \sum_{l=1}^{i-1}(-q)^{l-1} Y_{j+l}Y_{i-l} & (j \leq n - 2, i + j = 2n - 1) 
\end{cases} \]

We label 3 $L$-orbits on $m^+$ as in Section 2. For $p = 0, 1$ we have

\[ I_q^1(C_0) = \sum_{i=1}^{2n-2} C(q)Y_i, \]

\[ I_q^2(C_1) = C(q)\psi \]
where $\psi = \sum_{i=1}^{n-1} (-q^{-1})^{i-1} Y_{n-i} Y_{n+i-1}$.

In case 2 we fix a reduced expression

$$w_Iw_0 = (s_{\tau(1)}s_{n-2} \cdots s_1)(s_{\tau(2)}s_{n-2} \cdots s_2) \cdots (s_{\tau(n-2)}s_{n-2})s_{\tau(n-1)},$$

where

$$\tau(i) = \begin{cases} 
  n & \text{if } i \text{ is odd} \\
  n-1 & \text{if } i \text{ is even.}
\end{cases}$$

We set

$$Y_{i,j} = (-1)^{i+j-1} (T_{\tau(1)}T_{n-2} \cdots T_1)(T_{\tau(2)}T_{n-2} \cdots T_2) \cdots (T_{\tau(n-j)}T_{n-2} \cdots T_{n-j})$$

$$T_{\tau(n-j+1)}T_{n-2} \cdots T_{n-j+i+1}(F_{n-j+i})$$

$$(1 \leq i < j \leq n).$$

We have $Y_{i,j} \in U_q(m^-)_{-\beta_{i,j}}$, where

$$\beta_{i,j} = \begin{cases} 
  \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n & (j \leq n-1) \\
  \alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-2} + \alpha_n & (j = n).
\end{cases}$$

Then we have the following fundamental relations of $U_q(m^-)$.

$$Y_{i,j}Y_{i,m} = \begin{cases} 
  qY_{i,m}Y_{i,j} & (l < i < m = j \text{ or} l < i = m < j) \\
  Y_{i,m}Y_{i,j} & (i < l < m < j) \\
  Y_{i,m}Y_{i,j} + (q - q^{-1})Y_{i,m}Y_{i,j} & (l < i < m < j) \\
  Y_{i,m}Y_{i,j} & \\
  +(q - q^{-1})\{Y_{i,m}Y_{m,j} - q^{-1}Y_{m,i}Y_{i,j}\} & (l < m < i < j)
\end{cases}$$
We label \([n/2] + 1\) \(L\)-orbits on \(m^+\) as in Section 2. For \(p = 0, 1, \ldots, [(n-2)/2]\) we have

\[
\mathcal{I}_q^{p+1}(C_p) = \sum C(q) \left( i_1 \ i_2 \ \ldots \ i_{2p+2} \right)
\]

where we sum over all the sequence \(\{i_1, i_2, \ldots, i_{2p+2}\} \subset \mathbb{N}\) satisfying \(1 \leq i_1 < i_2 < \ldots < i_{2p+2} \leq n\), and set

\[
\tilde{S}_{2p+2} = \{\sigma \in S_{2p+2} | \sigma(2k-1) < \sigma(2k+1), \sigma(2k-1) < \sigma(2k)\}.
\]

### 3.5 Type \(E_6\)

We label the vertices of the Dynkin diagram as follows.

![Dynkin diagram](image)

Hence we have \(I_0 = \{1, 2, 3, 4, 5, 6\}\). Set \(i_0 = 1\), \(\Lambda = \{1, 2, \ldots, 16\}\). We fix a reduced expression

\[
w_Iw_0 = s_1s_2s_3s_4s_5s_6s_1s_6s_5s_2s_4s_3s_5s_6.
\]

and set \(Y_i = Y_{\beta_i}\) for \(i \in \Lambda\) (see Section 1).

Define \(A(n) = (i_1^n, i_2^n, i_3^n, i_4^n, i_5^n, i_6^n, j_1^n, j_2^n, j_3^n, j_4^n) \in \Lambda^8 (1 \leq n \leq 10)\) as follows:

- \(A(1) = (1, 2, 3, 4, 5, 6, 7, 8)\), \(A(2) = (1, 2, 3, 4, 9, 10, 11, 12)\),
- \(A(3) = (1, 2, 5, 6, 9, 10, 13, 14)\), \(A(4) = (1, 3, 5, 7, 9, 11, 13, 15)\),
- \(A(5) = (2, 3, 5, 8, 9, 12, 14, 15)\), \(A(6) = (1, 4, 6, 7, 10, 11, 13, 16)\),
- \(A(7) = (2, 4, 6, 8, 10, 12, 14, 16)\), \(A(8) = (3, 4, 7, 8, 11, 12, 15, 16)\),
- \(A(9) = (5, 6, 7, 8, 13, 14, 15, 16)\), \(A(10) = (9, 10, 11, 12, 13, 14, 15, 16)\).
For $1 \leq i < j \leq 16$ we have the following fundamental relations of $U_q(m^-)$.

\[
Y_i Y_j = \begin{cases} 
Y_j Y_i & \text{if there exist } n \text{ such that } i = i_1^n, j = j_1^n \\
Y_{i_2^n} + (q - q^{-1}) Y_{i_1^n} Y_{j_1^n} & \text{if there exist } n \text{ such that } i = i_2^n, j = j_2^n \\
Y_{j_m^n} Y_{i_m^n} + q Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q^{-1} Y_{i_m^n} Y_{j_{m-1}^n} & \text{if there exist } n, m = 3, 4 \text{ such that } i = i_m^n, j = j_m^n \\
q Y_i Y_j & \text{otherwise.}
\end{cases}
\]

We label 3 $L$-orbits on $m^+$ as in Section 2. For $p = 0, 1$ we have

\[
\mathcal{I}^1_q(C_0) = \sum_{i=1}^{16} \mathbb{C}(q) Y_i,
\]

\[
\mathcal{I}^2_q(C_1) = \sum_{n=1}^{10} \mathbb{C}(q) \psi_n
\]

where $\psi_n = Y_{i_2^n} Y_{j_2^n} - q Y_{i_3^n} Y_{j_3^n} + q^2 Y_{i_2^n} Y_{j_2^n} - q^3 Y_{i_3^n} Y_{j_3^n}$.

3.6 Type \textbf{E}_7

We label the vertices of the Dynkin diagram as follows.

\[\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 6 & 7 \\
5
\end{array}\]

Hence we have $I_0 = \{1, 2, 3, 4, 5, 6, 7\}$. Set $i_0 = 1$, $\Lambda = \{1, 2, \ldots, 27\}$. We fix a reduced expression

\[w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11} s_{12} s_{13} s_{14} s_{15} s_{16} s_{17} s_{18} s_{19} s_{20} s_{21} s_{22} s_{23} s_{24} s_{25} s_{26} s_{27} s_{28} s_{29} s_{30} s_{31} \]

and set $Y_i = Y_{\beta_i}$ for $i \in \Lambda$ (see Section 1).
Define $\mathbf{B}(n) = (i_5^n, i_4^n, i_3^n, i_2^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n) \in \Lambda^{10} \ (1 \leq n \leq 27)$ as follows:

$\mathbf{B}(1) = (10, 19, 20, 21, 23, 22, 24, 25, 26, 27)$, $\mathbf{B}(2) = (9, 14, 16, 17, 23, 18, 24, 25, 26, 27)$,

$\mathbf{B}(3) = (8, 15, 17, 21, 18, 22, 24, 25, 26, 27)$, $\mathbf{B}(4) = (7, 12, 15, 16, 20, 18, 22, 24, 26, 27)$,

$\mathbf{B}(5) = (6, 11, 15, 16, 20, 17, 23, 26, 27)$, $\mathbf{B}(6) = (5, 12, 13, 14, 19, 18, 22, 24, 25, 27)$,

$\mathbf{B}(7) = (4, 11, 13, 14, 19, 17, 21, 23, 25, 27)$, $\mathbf{B}(8) = (3, 11, 12, 14, 19, 16, 20, 23, 24, 27)$,

$\mathbf{B}(9) = (2, 11, 12, 13, 19, 15, 20, 21, 22, 27)$, $\mathbf{B}(10) = (1, 11, 12, 13, 14, 15, 16, 17, 18, 27)$,

$\mathbf{B}(11) = (5, 7, 8, 9, 10, 15, 22, 24, 25, 26)$, $\mathbf{B}(12) = (4, 6, 8, 9, 10, 17, 20, 23, 25, 26)$,

$\mathbf{B}(13) = (3, 6, 7, 9, 10, 16, 20, 23, 24, 26)$, $\mathbf{B}(14) = (2, 6, 7, 8, 10, 17, 21, 23, 25, 26)$,

$\mathbf{B}(15) = (3, 4, 5, 9, 10, 14, 19, 23, 24, 25)$, $\mathbf{B}(16) = (2, 4, 5, 8, 10, 13, 19, 21, 22, 25)$,

$\mathbf{B}(17) = (2, 3, 5, 7, 10, 12, 19, 20, 22, 24)$, $\mathbf{B}(18) = (2, 3, 4, 6, 10, 11, 19, 20, 21, 23)$,

$\mathbf{B}(19) = (1, 6, 7, 8, 9, 15, 16, 17, 18, 26)$, $\mathbf{B}(20) = (1, 4, 5, 8, 9, 13, 14, 17, 18, 25)$,

$\mathbf{B}(21) = (1, 3, 5, 7, 9, 12, 14, 16, 18, 24)$, $\mathbf{B}(22) = (1, 3, 4, 6, 9, 11, 14, 16, 17, 23)$,

$\mathbf{B}(23) = (1, 2, 5, 7, 8, 12, 13, 15, 18, 22)$, $\mathbf{B}(24) = (1, 2, 4, 6, 8, 11, 13, 15, 17, 21)$,

$\mathbf{B}(25) = (1, 2, 3, 6, 7, 11, 12, 15, 16, 20)$, $\mathbf{B}(26) = (1, 2, 3, 4, 5, 11, 12, 13, 14, 19)$,

$\mathbf{B}(27) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$.

For $1 \leq i < j \leq 27$ we have the following fundamental relations of $U_q(m^-)$.

\[
Y_i Y_j = \begin{cases} 
Y_j Y_i & \text{if there exist } n \text{ such that } \{i, j\} = \{i_1^n, j_1^n\} \\
Y_i Y_j + (q - q^{-1})Y_i^n Y_j^n & \text{if there exist } n \text{ such that } i = i_2^n, j = j_2^n \\
Y_{j_m^n} Y_{i_m^n} + q Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q^{-1} Y_{i_{m-1}^n} Y_{j_{m-1}^n} & \text{if there exist } n, m \in \{3, 4, 5\} \text{ such that } i = i_m^n, j = j_m^n \\
q Y_j Y_i & \text{otherwise.} 
\end{cases}
\]
Set

\[ \psi_n = Y_{i_2} Y_{j_2} - q Y_{i_4} Y_{j_4} + q^2 Y_{i_3} Y_{j_3} - q^3 Y_{i_2} Y_{j_2} + q^4 Y_{i_1} Y_{j_1}, \]
\[ \varphi = \sum_{n \in \Lambda} (-q)^{|\beta| - 1} \psi_n, \]

where \( |\beta| = \sum_{i \in I_0} m_i (\beta = \sum_{i \in I_0} m_i \alpha_i) \).

We label \( 4 \) \( L \)-orbits on \( \mathfrak{m}^+ \) as in Section 2. For \( p = 0, 1, 2 \) we have

\[ I_q^1(C_0) = \sum_{i=1}^{27} \mathbb{C}(q) Y_i, \]
\[ I_q^2(C_1) = \sum_{n \in \Lambda} \mathbb{C}(q) \psi_n \]
\[ I_q^2(C_2) = \mathbb{C}(q) \varphi. \]

References


