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Quantum deformations of certain prehomogeneous vector spaces

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0 Notation

Let $\mathfrak{g}$ be a semisimple Lie algebra over the complex number field $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ and $W \subset GL(\mathfrak{h})$ be the root system and the Weyl group respectively. For each $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. We denote the set of positive roots by $\Delta^+$ and the set of simple roots by $\{\alpha_i\}_{i \in I_0}$, where $I_0$ is an index set. Set

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}.$$ 

For $i \in I_0$ let $h_i \in \mathfrak{h}$, $\varpi_i \in \mathfrak{h}^*$ and $s_i \in W$ be the simple coroot, the fundamental weight, the simple reflection corresponding to $i$ respectively. Take $e_i \in \mathfrak{g}_{\alpha_i}$, and $f_i \in \mathfrak{g}_{-\alpha_i}$, satisfying $[e_i, f_i] = h_i$. Let $(\cdot, \cdot): \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for short roots $\alpha$. We set

$$d_i = \frac{(\alpha_i, \alpha_i)}{2} \quad (i \in I_0), \quad a_{ij} = \alpha_j(h_i) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \quad (i, j \in I_0).$$

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For a subset $I$ of $I_0$ we set
\[
\Delta_I = \Delta \cap \sum_{i \in I} \mathbb{Z} \alpha_i, \quad W_I = \langle s_i | i \in I \rangle,
\]
\[
I_I = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha), \quad n^+_I = \oplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha, \quad n^-_I = \oplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_{-\alpha},
\]
\[
h^*_I = \oplus_{i \in I_0 \setminus I} \mathbb{C} \varpi_i \subset \mathfrak{h}^*, \quad h^*_{I, \mathbb{Z}} = \oplus_{i \in I_0 \setminus I} \mathbb{Z} \varpi_i \subset \mathfrak{h}^*.
\]

For a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the enveloping algebra of $\mathfrak{a}$.

1 Quantized enveloping algebras

The quantized enveloping algebra $U_q(\mathfrak{g})$ ([1], [7]) is an associative algebra over the rational function field $\mathbb{C}(q)$ generated by the elements $\{E_i, F_i, K_i, K_i^{-1}\}_{i \in I_0}$ satisfying the following relations:

\[
K_i K_j = K_j K_i,
\]
\[
K_i K_i^{-1} = K_i^{-1} K_i = 1,
\]
\[
K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j,
\]
\[
K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j,
\]
\[
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},
\]
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} K_i^{-k} E_j E_i^k = 0 \quad (i \neq j),
\]
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} F_i^{-k} F_j F_i^k = 0 \quad (i \neq j),
\]

where $q_i = q^{d_i}$, and

\[
[m]_t = \frac{t^m - t^{-m}}{t - t^{-1}}, \quad [m]_t! = \prod_{k=1}^m [k]_t, \quad \binom{m}{n}_t = \frac{[m]_t!}{[n]_t! [m-n]_t!} \quad (m \geq n \geq 0).
\]
We define the Hopf algebra structure on $U_q(\mathfrak{g})$ as follows. The comultiplication
\[ \Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \]
is the algebra homomorphism satisfying
\[ \Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i. \]
The counit $\epsilon : U_q(\mathfrak{g}) \to \mathbb{C}(q)$ is the algebra homomorphism satisfying
\[ \epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0. \]
The antipode $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the algebra antiautomorphism satisfying
\[ S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i, \quad S(F_i) = -K_i^{-1} F_i. \]
The adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ is defined as follows. For $x, y \in U_q(\mathfrak{g})$
write $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$ and set $(\text{ad}x)(y) = \sum_k x_k^1 y S(x_k^2)$. Then $\text{ad} : U_q(\mathfrak{g}) \to \text{End}_{\mathbb{C}(q)}(U_q(\mathfrak{g}))$ is an algebra homomorphism.

We define subalgebras $U_q(\mathfrak{n}^\pm), U_q(\mathfrak{h})$ and $U_q(\mathfrak{l}_I)$ for $I \subset I_0$ by
\[ U_q(\mathfrak{n}^+) = \langle E_i \mid i \in I_0 \rangle, \quad U_q(\mathfrak{n}^-) = \langle F_i \mid i \in I_0 \rangle, \]
\[ U_q(\mathfrak{h}) = \langle K_i^\pm \mid i \in I_0 \rangle, \quad U_q(\mathfrak{l}_I) = \langle K_i^\pm, E_j, F_j \mid i \in I_0, j \in I \rangle. \]
For $i \in I_0$ we define an algebra automorphism $T_i$ of $U_q(\mathfrak{g})$ (see [8]) by
\[ T_i(K_j) = K_j K_i^{-a_{ij}}, \]
\[ T_i(E_j) = \begin{cases} -F_i K_i & (i = j) \\ \sum_{k=0}^{-a_{ij}} (-q_i)^{-k} E_i^{(-a_{ij} - k)} E_j E_i^{(k)} & (i \neq j) \end{cases}, \]
\[ T_i(F_j) = \begin{cases} -K_i^{-1} E_i & (i = j) \\ \sum_{k=0}^{-a_{ij}} (-q_i)^k F_i^{(k)} F_j F_i^{(-a_{ij} - k)} & (i \neq j) \end{cases}, \]
where
\[ E_i^{(k)} = \frac{1}{[k]_{q_i}!} E_i^k, \quad F_i^{(k)} = \frac{1}{[k]_{q_i}!} F_i^k. \]
For $w \in W$ we choose a reduced expression $w = s_{i_1} \cdots s_{i_k}$ and set $T_w = T_{i_1} \cdots T_{i_k}$. It is known that $T_w$ does not depend on the choice of the reduced expression.

For $I \subset I_0$ let $w_I$ be the longest element of $W_I$ and set

$$U_q(n_I^-) = U_q(n^-) \cap T^{-1}_w U_q(n^-).$$

Let $w_0$ be the longest element of $W$ and take a reduced expression $w_{I}\!w_0 = s_{i_1} \cdots s_{i_r}$ of $w_{I}\!w_0$. We set

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \quad Y_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k})$$

for $k = 1, \ldots, r$. Then it is known that $\{\beta_k | 1 \leq k \leq r\} = \Delta^+ \setminus \Delta_I$, and that $\{Y_{d_1}^{\beta_1} \cdots Y_{d_r}^{\beta_r} | d_1, \ldots, d_r \in \mathbb{Z}_{\geq 0}\}$ is a basis of $U_q(n_I^-)$. This basis depends on the choice of the reduced expression of $w_{I}\!w_0$ in general.

**Proposition 1.1** (ad $U_q(I)$) $(U_q(n_I^-)) \subset U_q(n_I^-)$.

For $N \in \mathbb{Z}_{>0}$ we set $U_{q,N}(\mathfrak{g}) = \mathbb{C}(q^{1/N}) \otimes_{\mathbb{C}(q)} U(\mathfrak{g})$, and let $U_{q,N}(n^\pm), U_{q,N}(\mathfrak{h}), U_{q,N}(l_I), U_{q,N}(n_I^-)$ be the $\mathbb{C}(q)$-subalgebras of $U_{q,N}(\mathfrak{g})$ generated by $U_q(n^\pm), U_q(\mathfrak{h}), U_q(I), U_q(n_I^-)$ respectively.

For $\lambda \in \mathfrak{h}_I^*$ we define a $U(\mathfrak{g})$-module $M_I(\lambda)$ by

$$M_I(\lambda) = U(\mathfrak{g})/(\sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - \lambda(h)) + U(\mathfrak{g})n^+ + U(\mathfrak{g})(1 \cap n^-)).$$

It is a highest weight module with highest weight $\lambda$ and highest weight vector $m_{I,\lambda} = 1$, where $1$ denotes the element of $M_I(\lambda)$ corresponding to $1 \in U(\mathfrak{g})$. $M_I(\lambda)$ contains a unique maximal proper submodule $K_I(\lambda)$, and $L(\lambda) = M_I(\lambda)/K_I(\lambda)$ is a unique (up to an isomorphism) irreducible highest weight module with highest weight $\lambda$.

For $\lambda \in \mathfrak{h}_I^*/N$ we define a $U(\mathfrak{g})$-module $M_I(\lambda)$ by

$$M_{I,q,N}(\lambda) = U_{q,N}(\mathfrak{g})/(\sum_{i \in I_0} U_{q,N}(\mathfrak{g})(K_i - q_i^{\lambda(h_i)}) + \sum_{i \in I_0} U_{q,N}(\mathfrak{g})E_i + \sum_{j \in I} U_{q,N}(\mathfrak{g})F_j).$$
It is a highest weight module with highest weight $\lambda$ and highest weight vector $m_{I,\lambda,q,N} = I$. $M_{I}(\lambda)$ contains a unique maximal proper submodule $K_{I,q,N}(\lambda)$, and $L_{q,N}(\lambda) = M_{I,q,N}(\lambda)/K_{I,q,N}(\lambda)$ is a unique irreducible highest weight module with highest weight $\lambda$.

2 Main result

In the rest of this note we fix $I \subset I_{0}$ satisfying $n_{I}^{+} \neq \{0\}$ and $[n_{I}^{+}, n_{I}^{+}] = \{0\}$. This is equivalent to the following condition:

$$I = I_{0} \setminus \{i_{0}\} \text{ with } m_{i_{0}} = 1,$$

where $\theta = \sum_{i \in I_{0}} m_{i} \alpha_{i}$ is the highest root (see [14]).

We set $l = I_{f}, m^{\pm} = n_{I}^{\pm}$ for simplicity.

**Proposition 2.1** The element $Y_{\beta} \in U_{q}(m^{-})$ for $\beta \in \Delta^{+} \setminus \Delta_{I}$ does not depend on the choice of a reduced expression of $w_{I}w_{0}$.

Fix a reduced expression $w_{I}w_{0} = s_{i_{1}} \ldots s_{i_{r}}$ and set $\beta_{p} = s_{i_{1}} \ldots s_{i_{p-1}}(\alpha_{i_{p}})$. We set

$$U_{q}(m^{-})^{m} = \sum_{\gamma \in m_{\alpha_{0}} + Q_{I}^{+}} \mathbb{C}(q)Y_{\beta_{1}}^{m_{r_{1}}} \ldots Y_{\beta_{r}}^{m_{r}} = \bigoplus_{\gamma \in m_{\alpha_{0}} + Q_{I}^{+}} U_{q}(m^{-})_{-\gamma}.$$

Here $U_{q}(m^{-})_{-\gamma}$ is the weight space with respect to the adjoint action of $U_{q}(\mathfrak{h})$ on $U_{q}(m^{-})$, and $Q_{I}^{+} = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_{i}$. 

**Lemma 2.2** We have

$$U_{q}(m^{-}) = \bigoplus_{m=0}^{\infty} U_{q}(m^{-})^{m},$$

$$U_{q}(m^{-})^{m} = \bigoplus_{\sum_{p} m_{p} = m} \mathbb{C}(q)Y_{\beta_{1}}^{m_{1}} \ldots Y_{\beta_{r}}^{m_{r}} = \bigoplus_{\gamma \in m_{\alpha_{0}} + Q_{I}^{+}} U_{q}(m^{-})_{-\gamma}.$$
By Lemma 2.2 we can write

\[ Y_{\beta_{p_{1}}}Y_{\beta_{p_{2}}} = \sum_{s_{1} \leq s_{2}} a_{s_{1}, s_{2}}^{p_{1}, p_{2}} Y_{s_{1}} Y_{s_{2}} \quad (a_{s_{1}, s_{2}}^{p_{1}, p_{2}} \in \mathbb{C}(q)) \]  

(1)

for \( p_{1} > p_{2} \).

**Proposition 2.3** The \( \mathbb{C}(q) \)-algebra \( U_{q}(m^{-}) \) is generated by the elements \( \{Y_{\beta_{p}}|1 \leq p \leq r\} \) satisfying the fundamental relations (1) for \( p_{1} > p_{2} \).

By the commutativity of \( m^{-} \), \( U(m^{-}) \) is isomorphic to the symmetric algebra \( S(m^{-}) \). Since \( m^{-} \) is identified with \( (m^{+})^{*} \) via the Killing form of \( g \), \( S(m^{-}) \) is isomorphic to the algebra \( \mathbb{C}[m^{+}] \) of polynomial functions on \( m^{+} \). Hence we have an identification \( U(m^{-}) = \mathbb{C}[m^{+}] \). We denote by \( \mathbb{C}[m^{+}]^{m} (m \in \mathbb{Z}_{\geq 0}) \) the subspace of \( \mathbb{C}[m^{+}] \) consisting of homogeneous elements with degree \( m \). We set \( \mathfrak{h}_{\mathbb{Z}}^{*}(I, +) = \{\lambda \in \mathfrak{h}^{*}|\lambda(h_{i}) \in \mathbb{Z}_{\geq 0} (i \in I)\} \). For \( \lambda \in \mathfrak{h}_{\mathbb{Z}}^{*}(I, +) \) we denote the finite dimensional irreducible \( U(\mathfrak{g})\)-module (resp. \( U_{q}(\mathfrak{g})\)-module) with highest weight \( \lambda \) by \( V(\lambda) \) (resp. \( V_{q}(\lambda) \)). We can decompose the finite dimensional \( \mathfrak{g}\)-module \( \mathbb{C}[m^{+}]^{m} \) into a direct sum of submodules isomorphic to \( V(\lambda) \) for some \( \lambda \in \mathfrak{h}_{\mathbb{Z}}^{*}(I, +) \). It is known that

\[ \mathbb{C}[m^{+}] \simeq \bigoplus_{\lambda \in \Gamma^{m}} V(\lambda) \]

for finite subset \( \Gamma^{m} \) of \( \mathfrak{h}_{\mathbb{Z}}^{*}(I, +) \) satisfying \( \Gamma^{m} \cap \Gamma^{m'} = \emptyset \) for \( m \neq m' \) (see [11], [12], [6]). On the other hand, since \( U_{q}(m^{-})^{m} \) is a finite dimensional \( U_{q}(\mathfrak{g})\)-module whose character is the same as that of \( \mathbb{C}[m^{+}]^{m} \), we have

\[ U_{q}(m^{-})^{m} \simeq \bigoplus_{\lambda \in \Gamma^{m}} V_{q}(\lambda). \]

Let \( \mathfrak{l} \) be the algebraic group corresponding to \( \mathfrak{g} \). It is known that \( m^{+} \) consists of finitely many \( \mathfrak{l}\)-orbits, and that the orbits can be labeled by

\[ \{L\text{-orbits on } m^{+}\} = \{C_{0}, C_{1}, \ldots, C_{t}\}, \quad \{0\} = C_{0} \subset C_{1} \subset \cdots \subset C_{t} = m^{+}. \]
We set
\[ \mathcal{I}(\overline{C_p}) = \{ f \in \mathbb{C}[m^+] \mid f(\overline{C_p}) = 0 \}. \]

Since \( \mathcal{I}(\overline{C_p}) \) is an \( \mathfrak{l} \)-submodule of \( \mathbb{C}[m^+] \), we have
\[ \mathcal{I}(\overline{C_p}) = \bigoplus_{m} \mathcal{I}^{m}(\overline{C_p}), \quad \mathcal{I}^{m}(\overline{C_p}) = \mathcal{I}(\overline{C_p}) \cap \mathbb{C}[m^+]^{m} \cong \bigoplus_{\lambda \in \Gamma_{p}^{m}} V(\lambda) \]
for a subset \( \Gamma_{p}^{m} \) of \( \Gamma^{m} \). The following facts are known (see, for example, [14]):

**Proposition 2.4** Let \( p = 0, \ldots, t-1 \).

(i) \( \mathcal{I}^{m}(\overline{C_p}) = 0 \) for \( m \leq p \).

(ii) \( \mathcal{I}^{p+1}(\overline{C_p}) \) is an irreducible \( \mathfrak{l} \)-module.

(iii) \( \mathcal{I}(\overline{C_p}) \) is generated by \( \mathcal{I}^{p+1}(\overline{C_p}) \) as an ideal of \( \mathbb{C}[m^+] \).

**Proposition 2.5** For \( p = 0, \ldots, t-1 \) there exists a unique \( \lambda_{p} \in \mathfrak{h}_{\mathfrak{l}}^{*} \) such that \( K_{I}(\lambda_{p}) = \mathcal{I}(\overline{C_p})m_{I, \lambda_{p}} \). Moreover, we have \( \lambda_{p} \in \mathfrak{h}_{\mathfrak{l}, \mathbb{Z}}^{*}/2 \).

We set
\[ \mathcal{I}_{q}^{m}(\overline{C_p}) = \bigoplus_{\lambda \in \Gamma_{p}^{m}} V_{q}(\lambda) \subset U_{q}(m^{-})^{m}, \]
\[ \mathcal{I}_{q}(\overline{C_p}) = \bigoplus_{m} \mathcal{I}_{q}^{m}(\overline{C_p}) \subset U_{q}(m^{-}), \]
\[ \mathcal{I}_{q,N}^{m}(\overline{C_p}) = \mathbb{C}(q^{1/N}) \otimes \mathbb{C}(q) \mathcal{I}_{q}^{m}(\overline{C_p}) \subset U_{q,N}(m^{-})^{m}, \]
\[ \mathcal{I}_{q,N}(\overline{C_p}) = \bigoplus_{m} \mathcal{I}_{q,N}^{m}(\overline{C_p}) \subset U_{q,N}(m^{-}). \]

Here we identify \( U_{q}(m^{-})^{m} \) with \( \bigoplus_{\lambda \in \Gamma^{m}} V_{q}(\lambda) \).

**Proposition 2.6** ([15]) For \( p = 0, \ldots, t-1 \) we have
\[ \mathrm{ch}(L_{q,2}(\lambda_{p})) = \mathrm{ch}(L(\lambda_{p})), \quad K_{I,q,2}(\lambda_{p}) = U_{q,2}(m^{-})I_{q,2}(\overline{C_p})m_{I, \lambda_{p}, q, 2}. \]
By Proposition 2.6 we have the main result.

**Theorem 2.7** ([15]) *We have*

\[ \mathcal{I}_q(C_p) = U_q(m^-)I_{p+1}(C_p) = I_{p+1}(C_p)U_q(m^-) \]

*for \( p = 0, \ldots, t-1 \).*

## 3 Examples

We shall give an explicit description of \( I_{p+1}(C_p) \) in each individual case. (see [16], [17])

### 3.1 Type \( A_n \)

We label the vertices of the Dynkin diagram as follows.

```
      1 2  k-1  k  k+1  n-1  n
      \bullet-1-2
```

Hence we have \( I_0 = \{1, \ldots, n\} \). Set \( I = I_0 \setminus \{i_0\} \), where \( i_0 = k \ (k - 1 \leq n - k) \).

We fix a reduced expression

\[ w_Iw_0 = (s_{k}s_{k+1} \cdots s_n)(s_{k-1}s_k \cdots s_n-1) \cdots (s_1s_2 \cdots s_{n-k+1}). \]

We set

\[ Y_{i,j} = (-1)^{k-i}(T_kT_{k+1} \cdots T_n)(T_{k-1}T_k \cdots T_{n-1}) \cdots (T_{i+1}T_{i+2} \cdots T_{n-k+i+1}) \]

\[ T_iT_{i+1} \cdots T_{i+j-2}(F_{i+j-1}) \]

\( (1 \leq i \leq k, 1 \leq j \leq n + 1 - k) \).

Set

\[ \beta_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_k + \cdots + \alpha_{k+j}. \]
We have $Y_{i,j} \in U_q(m^-)_{-\beta_i,j}$.

Then we have the following fundamental relations of $U_q(m^-)$.

$$Y_{i,j}Y_{l,m} = \begin{cases} 
q Y_{l,m}Y_{i,j} & (i = l, j < m \text{ or } i > l, j = m) \\
Y_{l,m}Y_{i,j} & (i > l, j > m) \\
Y_{l,m}Y_{i,j} + (q - q^{-1})Y_{i,m}Y_{l,j} & (i > l, j < m).
\end{cases}$$

We label $k + 1$ $L$-orbits on $m^+$ as in Section 2. For $p = 0, 1, \ldots, k - 1$ we have

$$I_{q}^{p+1}(C_p) = \sum \mathbb{C}(q) \begin{pmatrix} i_1 & i_2 & \cdots & i_{p+1} \\
j_1 & j_2 & \cdots & j_{p+1} \end{pmatrix}$$

where we sum over all the sequences $\{i_1, i_2, \ldots, i_{p+1}\}, \{j_1, j_2, \ldots, j_{p+1}\} \subset \mathbb{N}$ satisfying

$$1 \leq i_1 < i_2 < \cdots < i_{p+1} \leq k, \quad 1 \leq j_1 < j_2 < \cdots < j_{p+1} \leq n + 1 - k,$$

and set

$$\left( \begin{array}{cccc} i_1 & i_2 & \cdots & i_{p+1} \\
j_1 & j_2 & \cdots & j_{p+1} \end{array} \right) = \sum_{\sigma \in S_{p+1}} (-q)^{l(\sigma)} Y_{i_1,j_{\sigma(1)}} Y_{i_2,j_{\sigma(2)}} \cdots Y_{i_{p+1},j_{\sigma(p+1)}},$$

$$l(\sigma) = \#\{(i,j) | i < j, \sigma(i) > \sigma(j)\}.$$

### 3.2 Type $C_n$

We label the vertices of the Dynkin diagram as follows.

```
1 — 2 — 3 — 4 — 5 — 6 — 7
```

Hence we have $I_0 = \{1, \ldots, n\}$. Set $I = I_0 \setminus \{i_0\}$, where $i_0 = n$. We fix a reduced expression

$$w_I w_0 = (s_n s_{n-1} \cdots s_1)(s_n s_{n-1} \cdots s_2) \cdots (s_n s_{n-1}) s_n.$$
We set
\[
Y_{i,j} = c_{i,j}(T_{n}T_{n-1} \cdots T_1)(T_{n}T_{n-1} \cdots T_2) \cdots (T_{n}T_{n-1} \cdots T_{n-j})
\]
\[
T_{n}T_{n-1} \cdots T_{n-j+i+1}(F_{n-j+i})
\]
\[(1 \leq i \leq j \leq n),\]
where
\[
c_{i,j} = \begin{cases} 
(q + q^{-1}) & (1 \leq i = j \leq n) \\
(-1)^{j-i} & (1 \leq i < j \leq n).
\end{cases}
\]
Set
\[
\beta_{i,j} = \alpha_{i} + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2\alpha_{j} + \cdots + 2\alpha_{n-1} + \alpha_{n}.
\]
We have $Y_{i,j} \in U_{q}(m^{-})_{-\beta_{i,j}}$.

Then we have the following fundamental relations of $U_{q}(m^{-})$.

We label $n + 1$ $L$-orbits on $m^{+}$ as in Section 2. For $p = 0, 1, \ldots, n - 1$ the highest weight vector of $\mathbb{T}_{q}^{p+1}(C_{p})$ is
\[
\sum_{\sigma \in S_{p+1}} (-q^{-1})^{l(\sigma)}Y_{i_{1},j_{\sigma(1)}}Y_{i_{2},j_{\sigma(2)}} \cdots Y_{i_{p+1},j_{\sigma(p+1)}}
\]
where \( i_1 = j_1 = n-p, \ i_2 = j_2 = n-p+1, \ldots, \ i_{p+1} = j_{p+1} = n \) and \( Y_{j,i} = q^{-2}Y_{i,j} \) \((i < j)\).

### 3.3 Type \( B_n \)

We label the vertices of the Dynkin diagram as follows.

```
1 — 2 — n-2 — n-1 — n
```

Hence we have \( I_0 = \{1, \ldots, n\} \). Set \( I = I_0 \setminus \{i_0\} \), where \( i_0 = 1 \). We fix a reduced expression

\[
w_Iw_0 = s_1s_2 \cdots s_{n-1}s_ns_{n-1}s_{n-2} \cdots s_2s_1.
\]

We set

\[
Y_i = \begin{cases} 
T_1T_2 \cdots T_{i-1}(F_i) & (1 \leq i \leq n) \\
T_1T_2 \cdots T_{n-1}T_nT_{n-1}T_{n-2} \cdots T_{2n-i+1}(F_{2n-i}) & (n+1 \leq i \leq 2n-1).
\end{cases}
\]

Then we have the following fundamental relations of \( U_q(\mathrm{m}^{-}) \).

\[
Y_iY_j = \begin{cases} 
q^{-2}Y_jY_i & (i > j, i+j \neq 2n) \\
Y_jY_i + \frac{q^{-2} - 1}{q + q^{-1}}Y_n^2 & (i = n+1, j = n-1) \\
Y_jY_i + (q^{-2} - q^2) \sum_{l=1}^{i-n-1} (-q^2)^{i-l-1}Y_{j+l}Y_{i-l} - (-q^2)^{i-n-1} \frac{q^{-2} - 1}{q + q^{-1}}Y_n^2 & (j \leq n-2, i+j = 2n)
\end{cases}
\]

We label 3 \( L \)-orbits on \( \mathrm{m}^{+} \) as in Section 2. For \( p = 0,1 \) we have

\[
\mathcal{I}_q^1(\overline{C_0}) = \sum_{i=1}^{2n-1} \mathbb{C}(q)Y_i, \\
\mathcal{I}_q^2(\overline{C_1}) = \mathbb{C}(q)\psi
\]

where \( \psi = Y_nY_n - (q + q^{-1})(1 + q^{-2}) \sum_{i=1}^{n-1} (-q^{-2})^{i-1}Y_{n-i}Y_{n+i} \).
3.4 Type $D_n$

We have the following two cases.

Case 1

\[
I_0 = \{1, \ldots, n\}, i_0 = 1
\]

Case 2

\[
I_0 = \{1, \ldots, n\}, i_0 = n
\]

In case 1 we fix a reduced expression

\[
w_Iw_0 = s_1s_2\cdots s_{n-1}s_ns_{n-2}s_{n-3}\cdots s_2s_1.
\]

Set

\[
Y_i = \begin{cases} T_1T_2\cdots T_{i-1}(F_i) & (1 \leq i \leq n) \\ T_1T_2\cdots T_{n-1}T_nT_{n-2}T_{n-3}\cdots T_{2n-i}(F_{2n-i-1}) & (n + 1 \leq i \leq 2n - 2). \end{cases}
\]

Then we have the following fundamental relations of $U_q(\mathfrak{m}^-)$.

\[
Y_iY_j = \begin{cases} q^{-1}Y_jY_i & (i > j, i + j \neq 2n - 1) \\ Y_jY_i & (i = n, j = n - 1) \\ Y_jY_i - (q - q^{-1})\sum_{l=1}^{i-n}(-q)^{l-1}Y_{j+l}Y_{i-l} & (j \leq n - 2, i + j = 2n - 1) \end{cases}
\]

We label 3 $L$-orbits on $\mathfrak{m}^+$ as in Section 2. For $p = 0, 1$ we have

\[
\mathcal{I}_q^1(C_0) = \sum_{i=1}^{2n-2} \mathbb{C}(q)Y_i,
\]

\[
\mathcal{I}_q^2(C_1) = \mathbb{C}(q)\psi
\]
where $\psi = \sum_{i=1}^{n-1} (-q^{-1})^{i-1} Y_{n-i} Y_{n+i-1}$.

In case 2 we fix a reduced expression

$$w_1 w_0 = (s_{\tau(1)} s_{n-2} \cdots s_1) (s_{\tau(2)} s_{n-2} \cdots s_2) \cdots (s_{\tau(n-2)} s_{n-2}) s_{\tau(n-1)},$$

where

$$\tau(i) = \begin{cases} 
  n & (i : \text{odd}) \\
  n - 1 & (i : \text{even}). 
\end{cases}$$

We set

$$Y_{i,j} = (-1)^{i+j-1} (T_{\tau(1)} T_{n-2} \cdots T_1) (T_{\tau(2)} T_{n-2} \cdots T_2) \cdots (T_{\tau(n-j)} T_{n-2} \cdots T_{n-j}) T_{\tau(n-j+1)} T_{n-2} \cdots T_{n-j+i+1} (F_{n-j+i})$$

$$(1 \leq i < j \leq n).$$

We have $Y_{i,j} \in U_q(m^-)_{-\beta_{i,j}}$, where

$$\beta_{i,j} = \begin{cases} 
  \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n & (j \leq n-1) \\
  \alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-2} + \alpha_n & (j = n). 
\end{cases}$$

Then we have the following fundamental relations of $U_q(m^-)$.

$$Y_{i,j} Y_{i,m} = \begin{cases} 
  qY_{i,m} Y_{i,j} & (l < i < m = j \text{ or } l < i = m < j \text{ or } l = i < m < j) \\
  Y_{i,m} Y_{i,j} & (i < l < m < j) \\
  Y_{i,m} Y_{i,j} + (q - q^{-1}) Y_{i,m} Y_{i,j} & (l < i < m < j) \\
  Y_{i,m} Y_{i,j} & (l < m < i < j) \\
  + (q - q^{-1}) \{ Y_{i,s} Y_{m,j} - q^{-1} Y_{m,s} Y_{i,j} \} & (l < m < i < j) 
\end{cases}$$
We label \([n/2] + 1\) \(L\)-orbits on \(m^+\) as in Section 2. For \(p = 0, 1, \ldots, [(n - 2)/2]\) we have

\[
\mathcal{I}_q^{p+1}(\overline{C}_p) = \sum \mathbb{C}(q)\left( i_1 \ i_2 \ \ldots \ i_{2p+2} \right)
\]

where we sum over all the sequence \(\{i_1, i_2, \ldots, i_{2p+2}\} \subset \mathbb{N}\) satisfying \(1 \leq i_1 < i_2 < \ldots < i_{2p+2} \leq n\), and set

\[
\left( i_1 \ i_2 \ \ldots \ i_{2p+2} \right) = \sum_{\sigma \in \tilde{S}_{2p+2}} (-q^{-1})^{l(\sigma)} Y_{i_{\sigma(1)}, i_{\sigma(3)}, \ldots, i_{\sigma(2p+1), i_{\sigma(2p+3)}}},
\]

\(\tilde{S}_{2p+2} = \{\sigma \in S_{2p+2} | \sigma(2k-1) < \sigma(2k), \sigma(2k-1) < \sigma(2k)\}\).

### 3.5 Type \(E_6\)

We label the vertices of the Dynkin diagram as follows.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 5 & 6 \\
\end{array}
\]

Hence we have \(I_0 = \{1, 2, 3, 4, 5, 6\}\). Set \(i_0 = 1\), \(\Lambda = \{1, 2, \ldots, 16\}\). We fix a reduced expression

\[
w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_5 s_3 s_2 s_4 s_3 s_5 s_6.
\]

and set \(Y_i = Y_{\beta_i}\) for \(i \in \Lambda\) (see Section 1).

Define \(A(n) = (i_1^n, i_2^n, i_3^n, j_1^n, j_2^n, j_3^n) \in \Lambda^8 (1 \leq n \leq 10)\) as follows:

- \(A(1) = (1, 2, 3, 4, 5, 6, 7, 8)\), \(A(2) = (1, 2, 3, 4, 9, 10, 11, 12)\),
- \(A(3) = (1, 2, 5, 6, 9, 10, 13, 14)\), \(A(4) = (1, 3, 5, 7, 9, 11, 13, 15)\),
- \(A(5) = (2, 3, 5, 8, 9, 12, 14, 15)\), \(A(6) = (1, 4, 6, 7, 10, 11, 13, 16)\),
- \(A(7) = (2, 4, 6, 8, 10, 12, 14, 16)\), \(A(8) = (3, 4, 7, 8, 11, 12, 15, 16)\),
- \(A(9) = (5, 6, 7, 8, 13, 14, 15, 16)\), \(A(10) = (9, 10, 11, 12, 13, 14, 15, 16)\).
For $1 \leq i < j \leq 16$ we have the following fundamental relations of $U_q(\mathfrak{m}^-)$.

\[
Y_i Y_j = \begin{cases} 
Y_j Y_i & \text{if there exist } n \text{ such that } i = i_1^n, j = j_1^n \\
Y_{j_2^n} Y_{i_2^n} + (q - q^{-1}) Y_{i_1^n} Y_{j_1^n} & \text{if there exist } n \text{ such that } i = i_2^n, j = j_2^n \\
Y_{i_m^n} Y_{j_m^n} + q Y_{j_m^{-1}} Y_{i_m^{-1}} - q^{-1} Y_{i_m} Y_{j_m} & \text{if there exist } n, m = 3, 4 \text{ such that } i = i_m^n, j = j_m^n \\
q Y_j Y_i & \text{otherwise.} 
\end{cases}
\]

We label 3 $L$-orbits on $\mathfrak{m}^+$ as in Section 2. For $p = 0, 1$ we have

\[
\mathcal{I}_q^1(C_0) = \sum_{i=1}^{16} \mathbb{C}(q) Y_i, \\
\mathcal{I}_q^2(C_1) = \sum_{n=1}^{10} \mathbb{C}(q) \psi_n
\]

where $\psi_n = Y_{i_2^n} Y_{j_2^n} - q Y_{i_3^n} Y_{j_3^n} + q^2 Y_{i_2^n} Y_{j_2^n} - q^3 Y_{i_1^n} Y_{j_3^n}$.

### 3.6 Type $E_7$

We label the vertices of the Dynkin diagram as follows.

\[
1 \quad 2 \quad 3 \quad 4 \quad 6 \quad 7 \\
\bullet \quad \quad \quad \quad \quad \bullet \\
\quad \quad \quad \quad \quad \quad \bullet \\
\quad \quad \quad \quad \quad \quad \bullet
\]

Hence we have $I_0 = \{1, 2, 3, 4, 5, 6, 7\}$. Set $i_0 = 1$, $\Lambda = \{1, 2, \ldots, 27\}$. We fix a reduced expression

\[
w_I w_0 = s_1 s_2 s_3 s_4 s_6 s_8 s_2 s_1 s_7 s_6 s_4 s_3 s_5 s_4 s_6 s_7 s_2 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1.
\]

and set $Y_i = Y_{\beta_i}$ for $i \in \Lambda$ (see Section 1).
Define $B(n) = (i_5^n, i_4^n, i_3^n, i_2^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n) \in \Lambda^{10} (1 \leq n \leq 27)$ as follows:

$B(1) = (10, 19, 20, 21, 23, 22, 24, 25, 26, 27)$, 
$B(2) = (9, 14, 16, 17, 23, 18, 24, 25, 26, 27)$, 
$B(3) = (8, 13, 15, 17, 21, 18, 22, 25, 26, 27)$, 
$B(4) = (7, 12, 15, 16, 20, 18, 22, 24, 26, 27)$, 
$B(5) = (6, 11, 15, 16, 20, 17, 20, 23, 26, 27)$, 
$B(6) = (5, 12, 13, 14, 19, 18, 22, 24, 25, 27)$, 
$B(7) = (4, 11, 13, 14, 19, 17, 21, 23, 25, 27)$, 
$B(8) = (3, 11, 12, 14, 19, 16, 20, 23, 24, 27)$, 
$B(9) = (2, 11, 12, 13, 19, 15, 20, 21, 22, 27)$, 
$B(10) = (1, 11, 12, 13, 14, 15, 16, 17, 18, 27)$, 
$B(11) = (5, 7, 8, 9, 10, 15, 22, 24, 25, 26)$, 
$B(12) = (4, 6, 8, 9, 10, 17, 20, 23, 25, 26)$, 
$B(13) = (3, 6, 7, 9, 10, 16, 20, 23, 24, 26)$, 
$B(14) = (2, 6, 7, 8, 10, 17, 21, 23, 25, 26)$, 
$B(15) = (3, 4, 5, 9, 10, 14, 19, 22, 24, 25)$, 
$B(16) = (2, 4, 5, 8, 10, 13, 19, 21, 22, 25)$, 
$B(17) = (2, 3, 5, 7, 10, 12, 19, 20, 22, 24)$, 
$B(18) = (2, 3, 4, 6, 10, 11, 19, 20, 21, 23)$, 
$B(19) = (1, 6, 7, 8, 9, 15, 16, 17, 18, 26)$, 
$B(20) = (1, 4, 5, 8, 9, 13, 14, 17, 18, 25)$, 
$B(21) = (1, 3, 5, 7, 9, 12, 14, 16, 18, 24)$, 
$B(22) = (1, 3, 4, 6, 9, 11, 14, 16, 17, 23)$, 
$B(23) = (1, 2, 5, 7, 8, 12, 13, 15, 18, 22)$, 
$B(24) = (1, 2, 4, 6, 8, 11, 13, 15, 17, 21)$, 
$B(25) = (1, 2, 3, 6, 7, 11, 12, 15, 16, 20)$, 
$B(26) = (1, 2, 3, 4, 5, 11, 12, 13, 14, 19)$, 
$B(27) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$.

For $1 \leq i < j \leq 27$ we have the following fundamental relations of $U_q(m^-)$.

\[
Y_i Y_j = \begin{cases} 
Y_j Y_i & \text{if there exist } n \text{ such that } \{i, j\} = \{i_1^n, j_1^n\} \\
Y_{i_2^n} Y_{j_2^n} + (q - q^{-1})Y_{i_1^n} Y_{j_1^n} & \text{if there exist } n \text{ such that } i = i_2^n, j = j_2^n \\
Y_{j_m} Y_{i_m} + qY_{j_{m-1}} Y_{i_{m-1}} - q^{-1}Y_{i_{m-1}} Y_{j_{m-1}} & \text{if there exist } n, m \in \{3, 4, 5\} \text{ such that } i = i_m^n, j = j_m^n \\
qY_j Y_i & \text{otherwise.}
\end{cases}
\]
Set
\[ \psi_n = Y_{i_5} Y_{j_5} - q Y_{i_4} Y_{j_4} + q^2 Y_{i_3} Y_{j_3} - q^3 Y_{i_2} Y_{j_2} + q^4 Y_{i_1} Y_{j_1}, \]
\[ \varphi = \sum_{n \in \Lambda} (-q)^{|\beta_n| - 1} Y_n \psi_n, \]
where \(|\beta| = \sum_{i \in I_0} m_i (\beta = \sum_{i \in I_0} m_i \alpha_i)\).

We label 4 \(L\)-orbits on \(m^+\) as in Section 2. For \(p = 0, 1, 2\) we have
\[ I_q^1(C_0) = \sum_{i=1}^{27} C(q) Y_i, \]
\[ I_q^2(C_1) = \sum_{n \in \Lambda} C(q) \psi_n \]
\[ I_q^3(C_2) = C(q) \varphi. \]

References


