Quantum deformations of certain prehomogeneous vector spaces (Representation Theory and Noncommutative Harmonic Analysis)

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Quantum deformations of certain prehomogeneous vector spaces

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0 Notation

Let $\mathfrak{g}$ be a semisimple Lie algebra over the complex number field $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ and $W \subset GL(\mathfrak{h})$ be the root system and the Weyl group respectively. For each $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. We denote the set of positive roots by $\Delta^+$ and the set of simple roots by $\{\alpha_i\}_{i \in I_0}$, where $I_0$ is an index set. Set

$$ n^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad n^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}. $$

For $i \in I_0$ let $h_i \in \mathfrak{h}$, $\varpi_i \in \mathfrak{h}^*$ and $s_i \in W$ be the simple coroot, the fundamental weight, the simple reflection corresponding to $i$ respectively. Take $e_i \in \mathfrak{g}_{\alpha_i}$, and $f_i \in \mathfrak{g}_{-\alpha_i}$, satisfying $[e_i, f_i] = h_i$. Let $(\ , \ ) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ be the invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for short roots $\alpha$. We set

$$ d_i = \frac{(\alpha_i, \alpha_i)}{2} \quad (i \in I_0), \quad a_{ij} = \alpha_j(h_i) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \quad (i, j \in I_0). $$

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For a subset $I$ of $I_0$ we set
\[
\Delta_I = \Delta \cap \sum_{i \in I} \mathbb{Z} \alpha_i, \quad W_I = \langle s_i \mid i \in I \rangle,
\]
\[
I_I = \mathfrak{h} \oplus \left( \oplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha \right), \quad n_I^+ = \oplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha, \quad n_I^- = \oplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_{-\alpha},
\]
\[
h_I^* = \oplus_{i \in I_0 \setminus I} \mathbb{C} \varpi_i \subset \mathfrak{g}^*, \quad h_{I, Z}^* = \oplus_{i \in I_0 \setminus I} \mathbb{Z} \varpi_i \subset \mathfrak{g}^*.
\]

For a Lie algebra $a$ we denote by $U(a)$ the enveloping algebra of $a$.

## 1 Quantized enveloping algebras

The quantized enveloping algebra $U_q(\mathfrak{g})$ ([1], [7]) is an associative algebra over the rational function field $\mathbb{C}(q)$ generated by the elements $\{E_i, F_i, K_i, K_i^{-1}\}_{i \in I_0}$ satisfying the following relations:

\[
K_i K_j = K_j K_i, \\
K_i K_i^{-1} = K_i^{-1} K_i = 1, \\
K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \\
K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j, \\
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \left( \frac{1 - a_{ij}}{k} \right) E_i^{1-a_{ij} - k} E_j E_i^k = 0 \quad (i \neq j), \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \left( \frac{1 - a_{ij}}{k} \right) F_i^{1-a_{ij} - k} F_j F_i^k = 0 \quad (i \neq j),
\]

where $q_i = q^{d_i}$, and

\[
[m]_t = \frac{t^m - t^{-m}}{t - t^{-1}}, \quad [m]_t! = \prod_{k=1}^{m} [k]_t, \quad \binom{m}{n}_t = \frac{[m]_t!}{[n]_t![m-n]_t!} \quad (m \geq n \geq 0).
\]
We define the Hopf algebra structure on $U_q(\mathfrak{g})$ as follows. The comultiplication
\[ \Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \] is the algebra homomorphism satisfying
\[ \Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i. \]

The counit $\epsilon : U_q(\mathfrak{g}) \to \mathbb{C}(q)$ is the algebra homomorphism satisfying
\[ \epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0. \]

The antipode $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the algebra antiautomorphism satisfying
\[ S(K_i) = K_i^{-1}, \quad S(E_i) = -E_iK_i, \quad S(F_i) = -K_i^{-1}F_i. \]

The adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ is defined as follows. For $x, y \in U_q(\mathfrak{g})$ write $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$ and set $(\text{ad}x)(y) = \sum_k x_k^1yS(x_k^2)$. Then $\text{ad} : U_q(\mathfrak{g}) \to \text{End}_{\mathbb{C}(q)}(U_q(\mathfrak{g}))$ is an algebra homomorphism.

We define subalgebras $U_q(n^\pm), U_q(h)$ and $U_q(l_I)$ for $I \subset I_0$ by
\[ U_q(n^+) = \langle E_i | i \in I_0 \rangle, \quad U_q(n^-) = \langle F_i | i \in I_0 \rangle, \]
\[ U_q(h) = \langle K_i^\pm | i \in I_0 \rangle, \quad U_q(l_I) = \langle K_i^\pm, E_j, F_j | i \in I_0, j \in I \rangle. \]

For $i \in I_0$ we define an algebra automorphism $T_i$ of $U_q(\mathfrak{g})$ (see [8]) by
\[ T_i(K_j) = K_jK_i^{-a_{ij}}, \]
\[ T_i(E_j) = \begin{cases} -F_iK_i & (i = j) \\ \sum_{k=0}^{a_{ij}}(-q_i)^{-k}E_i^{(-a_{ij}-k)}E_jE_i^{(k)} & (i \neq j) \end{cases} \]
\[ T_i(F_j) = \begin{cases} -K_i^{-1}E_i & (i = j) \\ \sum_{k=0}^{a_{ij}}(-q_i)^kF_jF_i^{(-a_{ij}-k)} & (i \neq j) \end{cases} \]
where
\[ E_i^{(k)} = \frac{1}{[k]_{q_i}!}E_i^k, \quad F_i^{(k)} = \frac{1}{[k]_{q_i}!}F_i^k. \]
For $w \in W$ we choose a reduced expression $w = s_{i_1} \cdots s_{i_k}$ and set $T_w = T_{i_1} \cdots T_{i_k}$. It is known that $T_w$ does not depend on the choice of the reduced expression.

For $I \subset I_0$ let $w_I$ be the longest element of $W_I$ and set

$$U_q(n_I^-) = U_q(n^-) \cap T_{w_I}^{-1} U_q(n^-).$$

Let $w_0$ be the longest element of $W$ and take a reduced expression $w_I w_0 = s_{i_{1}} \cdots s_{i_{r}}$ of $w_I w_0$. We set

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \quad Y_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k})$$

for $k = 1, \ldots, r$. Then it is known that $\{\beta_k | 1 \leq k \leq r\} = \triangle^+ \setminus \triangle_I$, and that

$$\{Y_{\beta_1}^d \cdots Y_{\beta_r}^{d_r} | d_1, \ldots, d_r \in \mathbb{Z}_{\geq 0}\}$$

is a basis of $U_q(n_I^-)$. This basis depends on the choice of the reduced expression of $w_I w_0$ in general.

**Proposition 1.1** (ad $U_q(I)$)

$$(U_q(n_I^-)) \subset U_q(n_I^-).$$

For $N \in \mathbb{Z}_{>0}$ we set $U_{q,N}(g) = \mathbb{C}(q^{1/N}) \otimes_{\mathbb{C}(q)} U_q(g)$, and let $U_{q,N}(n^\pm)$, $U_{q,N}(b)$, $U_{q,N}(I)$, $U_{q,N}(n_I^-)$ be the $\mathbb{C}(q)$-subalgebras of $U_q(g)$ generated by $U_q(n^\pm)$, $U_q(b)$, $U_q(I)$, $U_q(n_I^-)$ respectively.

For $\lambda \in h_I^*$ we define a $U(g)$-module $M_I(\lambda)$ by

$$M_I(\lambda) = U(g)/\left( \sum_{h \in h} U(g)(h - \lambda(h)) + U(g)n^+ + U(g)(1 \cap n^-) \right).$$

It is a highest weight module with highest weight $\lambda$ and highest weight vector $m_{I,\lambda} = 1$, where $\bar{1}$ denotes the element of $M_I(\lambda)$ corresponding to $1 \in U(g)$. $M_I(\lambda)$ contains a unique maximal proper submodule $K_I(\lambda)$, and $L(\lambda) = M_I(\lambda)/K_I(\lambda)$ is a unique (up to an isomorphism) irreducible highest weight module with highest weight $\lambda$.

For $\lambda \in h_{I,Z}^*$ we define a $U(g)$-module $M_I(\lambda)$ by

$$M_{I,q,N}(\lambda) = U_{q,N}(g)/\left( \sum_{i \in I_0} U_{q,N}(g)(K_I - q_i^{\lambda(h_i)}) + \sum_{i \in I_0} U_{q,N}(g)E_i + \sum_{j \in I} U_{q,N}(g)F_j \right).$$
It is a highest weight module with highest weight $\lambda$ and highest weight vector $m_{I, \lambda, q, N} = \overline{1}$. $M_{I}(\lambda)$ contains a unique maximal proper submodule $K_{I, q, N}(\lambda)$, and $L_{q, N}(\lambda) = M_{I, q, N}(\lambda)/K_{I, q, N}(\lambda)$ is a unique irreducible highest weight module with highest weight $\lambda$.

2 Main result

In the rest of this note we fix $I \subset I_0$ satisfying $\mathfrak{n}^{+}_I \neq \{0\}$ and $[\mathfrak{n}^{+}_I, \mathfrak{n}^{+}_I] = \{0\}$. This is equivalent to the following condition:

$I = I_0 \setminus \{i_0\}$ with $m_{i_0} = 1$,

where $\theta = \sum_{i \in I_0} m_i \alpha_i$ is the highest root (see [14]).

We set $I = I, m^\pm = n^\pm_I$ for simplicity.

**Proposition 2.1** The element $Y_\beta \in U_q(\mathfrak{m}^-)$ for $\beta \in \Delta^+ \setminus \Delta_I$ does not depend on the choice of a reduced expression of $w_I w_0$.

Fix a reduced expression $w_I w_0 = s_{i_1} \ldots s_{i_r}$ and set $\beta_p = s_{i_1} \ldots s_{i_p-1}(\alpha_{i_p})$. We set

$U_q(\mathfrak{m}^-)^m = \sum_{=,p_{1},\ldots,p_{m}} \mathbb{C}(q) Y_{\beta_{p_{1}}} \cdots Y_{\beta_{p_{m}}} \quad (m \geq 0)$.

**Lemma 2.2** We have

$U_q(\mathfrak{m}^-) = \bigoplus_{m=0}^{\infty} U_q(\mathfrak{m}^-)^m,$

$U_q(\mathfrak{m}^-)^m = \bigoplus_{\sum_{p} m_{p} = m} \mathbb{C}(q) Y_{\beta_{p_{1}}}^{m_{1}} \cdots Y_{\beta_{p_{r}}}^{m_{r}} = \bigoplus_{\gamma \in m a_{i_0} + Q_I^+} U_q(\mathfrak{m}^-)_{-\gamma}.$

Here $U_q(\mathfrak{m}^-)_{-\gamma}$ is the weight space with respect to the adjoint action of $U_q(\mathfrak{h})$ on $U_q(\mathfrak{m}^-)$, and $Q_I^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. 


By Lemma 2.2 we can write

$$Y_{\beta_{p_{1}}}Y_{\beta_{p_{2}}} = \sum_{\substack{s_{1} \leq s_{2} \\ \beta_{p_{1}} + \beta_{p_{2}} = \beta_{s_{1}} + \beta_{s_{2}}}} \alpha_{s_{1},s_{2}}^{p_{1},p_{2}} Y_{\beta_{s_{1}}}Y_{\beta_{s_{2}}} \quad (\alpha_{s_{1},s_{2}}^{p_{1},p_{2}} \in \mathbb{C}(q))$$

(1)

for $p_{1} > p_{2}$.

**Proposition 2.3** The $\mathbb{C}(q)$-algebra $U_{q}(m^{-})$ is generated by the elements $\{Y_{\beta_{p}}|1 \leq p \leq r\}$ satisfying the fundamental relations (1) for $p_{1} > p_{2}$.

By the commutativity of $m^{-}$, $U(m^{-})$ is isomorphic to the symmetric algebra $S(m^{-})$. Since $m^{-}$ is identified with $(m^{+})^{*}$ via the Killing form of $g$, $S(m^{-})$ is isomorphic to the algebra $\mathbb{C}[m^{+}]$ of polynomial functions on $m^{+}$. Hence we have an identification $U(m^{-}) = \mathbb{C}[m^{+}]$. We denote by $\mathbb{C}[m^{+}]^{m}$ ($m \in \mathbb{Z}_{\geq 0}$) the subspace of $\mathbb{C}[m^{+}]$ consisting of homogeneous elements with degree $m$. We set $\mathfrak{h}_{\mathbb{Z}}(I, +) = \{\lambda \in \mathfrak{h}^{*} | \lambda(h_{i}) \in \mathbb{Z}_{\geq 0} (i \in I)\}$. For $\lambda \in \mathfrak{h}_{\mathbb{Z}}^{*}(I, +)$ we denote the finite dimensional irreducible $U(1)$-module (resp. $U_{q}(1)$-module) with highest weight $\lambda$ by $V(\lambda)$ (resp. $V_{q}(\lambda)$). We can decompose the finite dimensional $l$-module $\mathbb{C}[m^{+}]^{m}$ into a direct sum of submodules isomorphic to $V(\lambda)$ for some $\lambda \in \mathfrak{h}_{\mathbb{Z}}^{*}(I, +)$. It is known that

$$\mathbb{C}[m^{+}] \simeq \bigoplus_{\lambda \in \Gamma^{m}} V(\lambda)$$

for finite subset $\Gamma^{m}$ of $\mathfrak{h}_{\mathbb{Z}}^{*}(I, +)$ satisfying $\Gamma^{m} \cap \Gamma^{m'} = \emptyset$ for $m \neq m'$ (see [11], [12], [6]). On the other hand, since $U_{q}(m^{-})^{m}$ is a finite dimensional $U_{q}(1)$-module whose character is the same as that of $\mathbb{C}[m^{+}]^{m}$, we have

$$U_{q}(m^{-})^{m} \simeq \bigoplus_{\lambda \in \Gamma^{m}} V_{q}(\lambda).$$

Let $L$ be the algebraic group corresponding to $l$. It is known that $m^{+}$ consists of finitely many $L$-orbits, and that the orbits can be labeled by

$$\{\text{L-orbits on } m^{+}\} = \{C_{0}, C_{1}, \ldots, C_{t}\}, \quad \{0\} = C_{0} \subset C_{1} \subset \cdots \subset C_{t} = m^{+}.$$
We set

\[ \mathcal{I}(\overline{C_p}) = \{ f \in \mathbb{C}[\mathrm{m}^+] | f(\overline{C_p}) = 0 \}. \]

Since \( \mathcal{I}(\overline{C_p}) \) is an \( \mathfrak{l} \)-submodule of \( \mathbb{C}[\mathrm{m}^+] \), we have

\[ \mathcal{I}(\overline{C_p}) = \bigoplus_m \mathcal{I}^m(\overline{C_p}), \quad \mathcal{I}^m(\overline{C_p}) = \mathcal{I}(\overline{C_p}) \cap \mathbb{C}[\mathrm{m}^+]^m \cong \bigoplus_{\lambda \in \Gamma^m_p} V(\lambda) \]

for a subset \( \Gamma^m_p \) of \( \Gamma^m \). The following facts are known (see, for example, [14]):

**Proposition 2.4** Let \( p = 0, \ldots, t-1 \).

(i) \( \mathcal{I}^m(\overline{C_p}) = 0 \) for \( m \leq p \).

(ii) \( \mathcal{I}^{p+1}(\overline{C_p}) \) is an irreducible \( \mathfrak{l} \)-module.

(iii) \( \mathcal{I}(\overline{C_p}) \) is generated by \( \mathcal{I}^{p+1}(\overline{C_p}) \) as an ideal of \( \mathbb{C}[\mathrm{m}^+] \).

**Proposition 2.5** For \( p = 0, \ldots, t-1 \) there exists a unique \( \lambda_p \in \mathfrak{h}^*_I \) such that \( K_I(\lambda_p) = \mathcal{I}(\overline{C_p})m_{I,\lambda_p} \). Moreover, we have \( \lambda_p \in \mathfrak{h}^{*_I,\mathbb{Z}}/2 \).

We set

\[ \mathcal{I}^m_q(\overline{C_p}) = \bigoplus_{\lambda \in \Gamma_p^m} V_q(\lambda) \subset U_q(\mathrm{m}^-)^m, \]
\[ \mathcal{I}_q(\overline{C_p}) = \bigoplus_m \mathcal{I}^m_q(\overline{C_p}) \subset U_q(\mathrm{m}^-), \]
\[ \mathcal{I}^m_{q,N}(\overline{C_p}) = \mathbb{C}(q^{1/N}) \otimes_{\mathbb{C}(q)} \mathcal{I}^m_q(\overline{C_p}) \subset U_{q,N}(\mathrm{m}^-)^m, \]
\[ \mathcal{I}_{q,N}(\overline{C_p}) = \bigoplus_m \mathcal{I}^m_{q,N}(\overline{C_p}) \subset U_{q,N}(\mathrm{m}^-). \]

Here we identify \( U_q(\mathrm{m}^-)^m \) with \( \bigoplus_{\lambda \in \Gamma^m} V_q(\lambda) \).

**Proposition 2.6** ([15]) For \( p = 0, \ldots, t-1 \) we have

\[ \mathrm{ch}(L_{q,2}(\lambda_p)) = \mathrm{ch}(L(\lambda_p)), \quad K_{I,q,2}(\lambda_p) = U_{q,2}(\mathrm{m}^-)I_{q,2}^{p+1}(\overline{C_p})m_{I,\lambda_p,q,2}. \]
By Proposition 2.6 we have the main result.

**Theorem 2.7 ([15])** We have

\[ \mathcal{I}_q(\overline{C_p}) = U_q(m^-)\mathcal{I}^{p+1}(\overline{C_p}) = \mathcal{I}^{p+1}(\overline{C_p})U_q(m^-) \]

for \( p = 0, \ldots, t - 1 \).

## 3 Examples

We shall give an explicit description of \( \mathcal{I}^{p+1}(\overline{C_p}) \) in each individual case. (see [16], [17])

### 3.1 Type \( A_n \)

We label the vertices of the Dynkin diagram as follows.

\[
\begin{array}{cccccccccc}
1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n-1 & n \\
\end{array}
\]

Hence we have \( I_0 = \{1, \ldots, n\} \). Set \( I = I_0 \setminus \{i_0\} \), where \( i_0 = k \) \((k - 1 \leq n - k)\).

We fix a reduced expression

\[ w_I w_0 = (s_k s_{k+1} \cdots s_n)(s_{k-1} s_k \cdots s_{n-1}) \cdots (s_1 s_2 \cdots s_{n-k+1}) \]

We set

\[
Y_{i,j} = (-1)^{k-i}(T_k T_{k+1} \cdots T_n)(T_{k-1} T_k \cdots T_{n-1}) \cdots (T_{i+1} T_{i+2} \cdots T_{n-k+i+1})
\]

\[
T_i T_{i+1} \cdots T_{i+j-2} (F_{i+j-1})
\]

\((1 \leq i \leq k, 1 \leq j \leq n + 1 - k)\).

Set

\[ \beta_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_k + \cdots + \alpha_{k+j} \].
We have \( Y_{i,j} \in U_q(m^-)_{-\beta_{i,j}} \).

Then we have the following fundamental relations of \( U_q(m^-) \).

\[
Y_{i,j}Y_{l,m} = \begin{cases} 
q Y_{l,m}Y_{i,j} & (i = l, j < m \text{ or } i > l, j = m) \\
Y_{l,m}Y_{i,j} & (i > l, j > m) \\
Y_{l,m}Y_{i,j} + (q - q^{-1})Y_{i,m}Y_{i,j} & (i > l, j < m).
\end{cases}
\]

We label \( k + 1 \) \( L \)-orbits on \( m^+ \) as in Section 2. For \( p = 0, 1, \ldots, k - 1 \) we have

\[
I_{q}^{p+1}(\overline{C_p}) = \sum \mathbb{C}(q) \left( \begin{array}{cccc} i_1 & i_2 & \ldots & i_{p+1} \\ j_1 & j_2 & \ldots & j_{p+1} \end{array} \right)
\]

where we sum over all the sequences \( \{i_1, i_2, \ldots, i_{p+1}\}, \{j_1, j_2, \ldots, j_{p+1}\} \subset \mathbb{N} \) satisfying

\[
1 \leq i_1 < i_2 < \cdots < i_{p+1} \leq k, \quad 1 \leq j_1 < j_2 < \cdots < j_{p+1} \leq n + 1 - k,
\]

and set

\[
\left( \begin{array}{cccc} i_1 & i_2 & \ldots & i_{p+1} \\ j_1 & j_2 & \ldots & j_{p+1} \end{array} \right) = \sum_{\sigma \in S_{p+1}} (-q)^{l(\sigma)} Y_{i_1,j_{\sigma(1)}} Y_{i_2,j_{\sigma(2)}} \cdots Y_{i_{p+1},j_{\sigma(p+1)}},
\]

\[
l(\sigma) = \# \{(i,j) | i < j, \sigma(i) > \sigma(j)\}.
\]

### 3.2 Type \( C_n \)

We label the vertices of the Dynkin diagram as follows.

```
1 2 \ldots n-2 n-1 \rightarrow n
```

Hence we have \( I_0 = \{1, \ldots, n\} \). Set \( I = I_0 \setminus \{i_0\} \), where \( i_0 = n \). We fix a reduced expression

\[
w_Iw_0 = (s_n s_{n-1} \cdots s_1)(s_n s_{n-1} \cdots s_2) \cdots (s_n s_{n-1}) s_n.
\]
We set
\[ Y_{i,j} = c_{i,j}(T_1\cdots T_n - T_n\cdots T_1)(T_n\cdots T_2)\cdots(T_n T_{n-1}\cdots T_{n-i+1}(F_{n-j+i}) \]
\[ (1 \leq i \leq j \leq n), \]
where
\[ c_{i,j} = \begin{cases} (q + q^{-1}) & (1 \leq i = j \leq n) \\ (-1)^{j-i} & (1 \leq i < j \leq n). \end{cases} \]

Set
\[ \beta_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n. \]

We have \( Y_{i,j} \in U_q(\mathfrak{m}^-)_{-\beta_{i,j}} \).

Then we have the following fundamental relations of \( U_q(\mathfrak{m}^-) \).

We label \( n + 1 \ L \)-orbits on \( \mathfrak{m}^+ \) as in Section 2. For \( p = 0, 1, \ldots, n - 1 \) the highest weight vector of \( T_q^{p+1}(\overline{C}_p) \) is
\[ \sum_{\sigma \in S_{p+1}} (-q^{-1})^{l(\sigma)} Y_{i_1, j_{\sigma(1)}} Y_{i_2, j_{\sigma(2)}} \cdots Y_{i_{p+1}, j_{\sigma(p+1)}} \]
where $i_1 = j_1 = n-p$, $i_2 = j_2 = n-p+1$, \ldots, $i_{p+1} = j_{p+1} = n$ and $Y_{j,i} = q^{-2}Y_{i,j}$ ($i < j$).

### 3.3 Type $B_n$

We label the vertices of the Dynkin diagram as follows.

```
1 - 2 - \ldots - n-2 - n-1 - n
```

Hence we have $I_0 = \{1, \ldots, n\}$. Set $I = I_0 \setminus \{i_0\}$, where $i_0 = 1$. We fix a reduced expression

$$w_I w_0 = s_{1} s_{2} \cdots s_{n-1} s_{n}^{-1} s_{n-2} \cdots s_{2} s_{1}.$$

We set

$$Y_i = \begin{cases} T_1 T_2 \cdots T_{i-1} (F_i) & (1 \leq i \leq n) \\ T_1 T_2 \cdots T_{n-1} T_n T_{n-1} T_{n-2} \cdots T_{2n-i+1} (F_{2n-i}) & (n+1 \leq i \leq 2n-1). \end{cases}$$

Then we have the following fundamental relations of $U_q(m^-)$.

$$Y_i Y_j = \begin{cases} q^{-2} Y_j Y_i & (i > j, i+j \neq 2n) \\ Y_j Y_i + \frac{q^{-2} - 1}{q+q^{-1}} Y_{n}^2 & (i = n+1, j = n-1) \\ Y_j Y_i + (q^{-2} - q^2) \sum_{l=1}^{i-n-1} (-q^2)^{l-1} Y_{j+l} Y_{i-l} \\ \quad - (-q^2)^{i-n-1} \frac{q^{-2} - 1}{q+q^{-1}} Y_{n}^2 & (j \leq n-2, i+j = 2n) \end{cases}$$

We label 3 $L$-orbits on $m^+$ as in Section 2. For $p = 0, 1$ we have

$$I_q^1(C_0) = \sum_{i=1}^{2n-1} C(q) Y_i,$$

$$I_q^2(C_1) = C(q) \psi,$$

where $\psi = Y_n Y_{n} - (q + q^{-1})(1 + q^{-2}) \sum_{i=1}^{n-1} (-q^2)^{i-1} Y_{n-i} Y_{n+i}$. 

3.4 Type $D_n$

We have the following two cases.

Case 1

![Diagram]

$I_0 = \{1, \ldots, n\}, i_0 = 1$

Case 2

![Diagram]

$I_0 = \{1, \ldots, n\}, i_0 = n$

In case 1 we fix a reduced expression

$$w_Iw_0 = s_1s_2\cdots s_{n-1}s_ns_{n-2}s_{n-3}\cdots s_2s_1.$$ 

Set

$$Y_i = \begin{cases} 
T_1T_2\cdots T_{i-1}(F_i) & (1 \leq i \leq n) \\
T_1T_2\cdots T_{n-1}T_nT_{n-2}T_{n-3}\cdots T_{2n-i}(F_{2n-i-1}) & (n + 1 \leq i \leq 2n - 2).
\end{cases}$$

Then we have the following fundamental relations of $U_q(m^-)$.

$$Y_iY_j = \begin{cases} 
q^{-1}Y_jY_i & (i > j, i + j \neq 2n - 1) \\
Y_jY_i & (i = n, j = n - 1) \\
Y_jY_i - (q - q^{-1})\sum_{l=1}^{n-i}(-q)^{l-1}Y_{j+l}Y_{l-i} & (j \leq n - 2, i + j = 2n - 1)
\end{cases}$$

We label 3 $L$-orbits on $m^+$ as in Section 2. For $p = 0, 1$ we have

$$\mathcal{I}_q^1(C_0) = \sum_{i=1}^{2n-2} C(q)Y_i,$$

$$\mathcal{I}_q^2(C_1) = C(q)\psi$$
where $\psi = \sum_{i=1}^{n-1} (-q^{-1})^{i-1} Y_{n-i} Y_{n+i-1}.$

In case 2 we fix a reduced expression

$$w_I w_0 = (s_{\tau(1)} s_{n-2} \cdots s_1) (s_{\tau(2)} s_{n-2} \cdots s_2) \cdots (s_{\tau(n-2)} s_{n-2}) s_{\tau(n-1)},$$

where

$$\tau(i) = \begin{cases} n & (i \text{ odd}) \\ n - 1 & (i \text{ even}). \end{cases}$$

We set

$$Y_{i,j} = (-1)^{i+j-1} (T_{\tau(1)} T_{n-2} \cdots T_1) (T_{\tau(2)} T_{n-2} \cdots T_2) \cdots (T_{\tau(n-j)} T_{n-2} \cdots T_n)$$

$$T_{\tau(n-j+1)} T_{n-2} \cdots T_{n-j+i+1} (F_{n-j+i})$$

$$(1 \leq i < j \leq n).$$

We have $Y_{i,j} \in U_q(m^-)_{-\beta_{i,j}},$ where

$$\beta_{i,j} = \begin{cases} \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n & (j \leq n - 1) \\ \alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-2} + \alpha_n & (j = n). \end{cases}$$

Then we have the following fundamental relations of $U_q(m^-)$.

$$Y_{i,j} Y_{i,m} = \begin{cases} q Y_{i,m} Y_{i,j} & (l < i < m = j \text{ or } l < i < m < j) \\ Y_{i,m} Y_{i,j} & (i < l < m < j) \\ Y_{i,m} Y_{i,j} + (q - q^{-1}) Y_{i,m} Y_{i,j} & (l < i < m < j) \\ Y_{i,m} Y_{i,j} & (l < m < i < j) \\ + (q - q^{-1}) \{ Y_{i,t} Y_{m,j} - q^{-1} Y_{m,i} Y_{i,j} \} & (l < m < i < j) \end{cases}$$
We label \([n/2] + 1\) \(L\)-orbits on \(\mathfrak{m}^+\) as in Section 2. For \(p = 0, 1, \ldots, [(n - 2)/2]\) we have

\[
\mathcal{I}_{q}^{p+1}(\overline{C_p}) = \sum \mathbb{C}(q)(i_1 \ i_2 \ \ldots \ i_{2p+2})
\]

where we sum over all the sequence \(\{i_1, i_2, \ldots, i_{2p+2}\} \subset \mathbb{N}\) satisfying \(1 \leq i_1 < i_2 < \ldots < i_{2p+2} \leq n\), and set

\[
\tilde{S}_{2p+2} = \{\sigma \in S_{2p+2} | \sigma(2k-1) < \sigma(2k+1), \sigma(2k-1) < \sigma(2k)\}.
\]

3.5 Type \(E_6\)

We label the vertices of the Dynkin diagram as follows.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 5 & 6 \\
\end{array}
\]

Hence we have \(I_0 = \{1, 2, 3, 4, 5, 6\}\). Set \(i_0 = 1\), \(\Lambda = \{1, 2, \ldots, 16\}\). We fix a reduced expression

\[
w_I \omega_0 = s_1 s_2 s_3 s_4 s_5 s_2 s_1 s_6 s_5 s_2 s_4 s_3 s_5 s_6.
\]

and set \(Y_i = Y_{\beta_i}\) for \(i \in \Lambda\) (see Section 1).

Define \(A(n) = (i_4^n, i_3^n, i_2^n, i_1^n, j_2^n, j_1^n, j_3^n, j_4^n) \in \Lambda^8 (1 \leq n \leq 10)\) as follows:

\[
\begin{align*}
A(1) &= (1, 2, 3, 4, 5, 6, 7, 8), & A(2) &= (1, 2, 3, 4, 9, 10, 11, 12), \\
A(3) &= (1, 2, 5, 6, 9, 10, 13, 14), & A(4) &= (1, 3, 5, 7, 9, 11, 13, 15), \\
A(5) &= (2, 3, 5, 8, 9, 12, 14, 15), & A(6) &= (1, 4, 6, 7, 10, 11, 13, 16), \\
A(7) &= (2, 4, 6, 8, 10, 12, 14, 16), & A(8) &= (3, 4, 7, 8, 11, 12, 15, 16), \\
A(9) &= (5, 6, 7, 8, 13, 14, 15, 16), & A(10) &= (9, 10, 11, 12, 13, 14, 15, 16).
\end{align*}
\]
For $1 \leq i < j \leq 16$ we have the following fundamental relations of $U_q(m^-)$.

\[
Y_i Y_j = \begin{cases} 
Y_j Y_i & \text{if there exist } n \text{ such that } i = i_1^n, j = j_1^n \\
Y_j Y_i + (q - q^{-1})Y_i Y_j & \text{if there exist } n \text{ such that } i = i_2^n, j = j_2^n \\
Y_{j_m} Y_{i_n} + q Y_{j_{m-1}} Y_{i_{m-1}} - q^{-1} Y_{i_n} Y_{j_{m-1}} & \text{if there exist } n, m = 3, 4 \text{ such that } i = i_m^n, j = j_m^n \\
q Y_j Y_i & \text{otherwise.} 
\end{cases}
\]

We label 3 $L$-orbits on $m^+$ as in Section 2. For $p = 0, 1$ we have

\[
\mathcal{I}^1_q(C_0) = \sum_{i=1}^{16} C(q) Y_i, \\
\mathcal{I}^2_q(C_1) = \sum_{n=1}^{10} C(q) \psi_n
\]

where $\psi_n = Y_{i_1^n} Y_{j_1^n} - q Y_{i_2^n} Y_{j_2^n} + q^2 Y_{i_3^n} Y_{j_3^n} - q^3 Y_{i_4^n} Y_{j_4^n}$.

### 3.6 Type $E_7$

We label the vertices of the Dynkin diagram as follows.

\[
\begin{array}{ccccccccc}
& & & & & & & & & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array}
\]

Hence we have $I_0 = \{1, 2, 3, 4, 5, 6, 7\}$. Set $i_0 = 1$, $\Lambda = \{1, 2, \ldots, 27\}$. We fix a reduced expression

\[
\psi_1 \psi_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_6 s_5 s_4 s_3 s_2 s_1 s_7 s_6 s_4 s_3 s_4 s_5 s_4 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1.
\]

and set $Y_i = Y_{\beta_i}$ for $i \in \Lambda$ (see Section 1).
Define $\mathbf{B}(n) = (i_5^n, i_4^n, i_3^n, i_2^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n) \in \Lambda^{10} (1 \leq n \leq 27)$ as follows:

$\mathbf{B}(1) = (10, 19, 20, 21, 23, 22, 24, 25, 26, 27)$, $\mathbf{B}(2) = (9, 14, 16, 17, 23, 18, 24, 25, 26, 27)$, $\mathbf{B}(3) = (8, 13, 15, 17, 21, 18, 22, 25, 26, 27)$, $\mathbf{B}(4) = (7, 12, 15, 16, 20, 18, 22, 24, 26, 27)$, $\mathbf{B}(5) = (6, 11, 15, 16, 20, 17, 23, 26, 27)$, $\mathbf{B}(6) = (5, 12, 13, 14, 18, 22, 24, 25, 27)$, $\mathbf{B}(7) = (4, 11, 13, 14, 19, 17, 21, 23, 25, 27)$, $\mathbf{B}(8) = (3, 11, 12, 14, 19, 16, 20, 23, 24, 27)$, $\mathbf{B}(9) = (2, 11, 12, 13, 19, 15, 20, 21, 22, 27)$, $\mathbf{B}(10) = (1, 11, 12, 13, 14, 15, 16, 17, 18, 27)$, $\mathbf{B}(11) = (5, 7, 8, 9, 10, 15, 22, 24, 25, 26)$, $\mathbf{B}(12) = (4, 6, 8, 9, 10, 17, 20, 23, 25, 26)$, $\mathbf{B}(13) = (3, 6, 7, 9, 10, 16, 20, 23, 24, 26)$, $\mathbf{B}(14) = (2, 6, 7, 8, 10, 17, 21, 23, 25, 26)$, $\mathbf{B}(15) = (3, 4, 5, 9, 10, 14, 19, 23, 24, 25)$, $\mathbf{B}(16) = (2, 4, 5, 8, 10, 13, 19, 21, 22, 25)$, $\mathbf{B}(17) = (2, 3, 5, 7, 10, 12, 19, 20, 22, 24)$, $\mathbf{B}(18) = (2, 3, 4, 6, 10, 11, 19, 20, 21, 23)$, $\mathbf{B}(19) = (1, 6, 7, 8, 9, 15, 16, 17, 18, 26)$, $\mathbf{B}(20) = (1, 4, 5, 8, 9, 13, 14, 17, 18, 25)$, $\mathbf{B}(21) = (1, 3, 5, 7, 9, 12, 14, 16, 18, 24)$, $\mathbf{B}(22) = (1, 3, 4, 6, 9, 11, 14, 16, 17, 23)$, $\mathbf{B}(23) = (1, 2, 5, 7, 8, 12, 13, 15, 18, 22)$, $\mathbf{B}(24) = (1, 2, 4, 6, 8, 11, 13, 15, 17, 21)$, $\mathbf{B}(25) = (1, 2, 3, 6, 7, 11, 12, 15, 16, 20)$, $\mathbf{B}(26) = (1, 2, 3, 4, 5, 11, 12, 13, 14, 19)$, $\mathbf{B}(27) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$.

For $1 \leq i < j \leq 27$ we have the following fundamental relations of $U_q(m^-)$.

\[
Y_i Y_j = \begin{cases} 
Y_i Y_j & \text{if there exist } n \text{ such that } \{i, j\} = \{i_5^n, j_5^n\} \\
Y_j Y_i + (q - q^{-1})Y_i Y_j & \text{if there exist } n \text{ such that } i = i_2^n, j = j_2^n \\
Y_{j_m} Y_{i_m} + q Y_{j_{m-1}} Y_{i_{m-1}} - q^{-1} Y_{i_{m-1}} Y_{j_{m-1}} & \text{if there exist } n, m \in \{3, 4, 5\} \text{ such that } i = i_m^n, j = j_m^n \\
q Y_j Y_i & \text{otherwise.}
\end{cases}
\]
Set
\[ \psi_n = Y_{i_5^{n}j_5}Y_{i_4^{n}}Y_{j_4^{n}} - q^{2}Y_{i_3^{n}}Y_{j_3^{n}} - q^{3}Y_{i_2}Y_{j_2} + q^{4}Y_{i_1}Y_{j_1} \]
\[ \varphi = \sum_{n \in \Lambda} (-q)^{|\beta_n|} \psi_n \]

where \( |\beta| = \sum_{i \in I_0} m_i (\beta = \sum_{i \in I_0} m_i \alpha_i) \).

We label 4 \( L \)-orbits on \( \mathfrak{m}^+ \) as in Section 2. For \( p = 0, 1, 2 \) we have
\[ I_q^1(\overline{C_0}) = \sum_{i=1}^{27} \mathbb{C}(q)Y_i, \]
\[ I_q^2(\overline{C_1}) = \sum_{n \in \Lambda} \mathbb{C}(q)\psi_n \]
\[ I_q^3(\overline{C_2}) = \mathbb{C}(q)\varphi. \]

References


