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THE BEHAVIOR OF INVARIANT INTEGRALS ON SEMISIMPLE SYMMETRIC SPACES AND ITS APPLICATION

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§0. Introduction

Let $X = G/H$ be a semisimple symmetric space. We assume that G is a connected reductive Lie group, and H is a subgroup of G defined by

$$H = G^{\sigma} = \{g \in G \mid \sigma(g) = g\},$$

where σ is an involution of G.

Example 0.1. We give a few examples of semisimple symmetric spaces $X = G/H$ satisfying the above assumptions:

$$U(p,q)/\,(U(r) \times U(p-r,q)), \qquad U(n,n)/GL(n,\boldsymbol{C}),$$
$$U(2,2)/\,((U(1,1) \times U(1,1)), \qquad GL(m+n,\boldsymbol{R})/\,(GL(m,\boldsymbol{R}) \times GL(n,\boldsymbol{R})).$$

Let $\tilde{\sigma}$ be the natural imbedding $X \hookrightarrow G$ given by $\tilde{\sigma}(gH) = g\sigma(g^{-1})$. We identify X with the submanifold $\tilde{\sigma}(X)$ of G.

Let X' be the set of regular semisimple elements in X, which is H-invariant, open dense in X. We choose a complete system $\{J_l \mid l \in L\}$ of representatives of H-conjugacy classes of Cartan subspaces of X. For any subset S of X, we write $S' = S \cap X'$. It is well-known that $X' = \bigsqcup_{l \in L} H.J_l'$.

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Definition 0.1. Let f be a function in $C_c^\infty(X)$. By the *invariant integral* F_f of f, we mean a function F_f on $\bigsqcup_{l \in L} J'_l$ defined by

$$F_f(x) = {}_X F_f(x) := \int_{H/Z_H(\tilde{\sigma}(x))} f(h.x)\, d\tilde{h},$$

where $d\tilde{h}$ is a suitably normalized H-invariant measure on $H/Z_H(\tilde{\sigma}(x))$.

Let x_0 be a singular (that is, not regular) semisimple element of X, and let $X_{\{x_0\}}$ be the the symmetric subspace attached to x_0 (cf. Definition 1.1). We are interested in examining the asymptotic behavior of invariant integrals around x_0. The objective of the first part (§1∼§3) of this article is to show that this behavior is reduced to that of invariant integrals on $X_{\{x_0\}}$ around x_0 (Proposition 1.1 and Proposition 1.2). In particular, when x_0 is a semiregular semisimple element (cf. Definition 1.2), this fact shows us that one has only to examine invariant integrals for a semisimple symmetric space of rank one, which is well investigated by many authors (for example, see Faraut [6] for $U(p,q;\boldsymbol{F})/(U(r;\boldsymbol{F})\times U(p-r,q;\boldsymbol{F}))$ $(\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H})$).

In the case where X is a group manifold $(G \times G)/\triangle G$, invariant integrals on X are studied in detail in early stages by Harish-Chandra and Hirai and so on. Our approach is based on that of Hirai [7] for group manifolds. We also make use of a result of Matsuki [10] for the Jordan decomposition for semisimple symmetric spaces.

In the second part (§4, §5) of this article, we apply our results on invariant integrals to the theory of invariant eigendistributions (IED's) on X (cf. Definition 4.1). We discuss the following problem : *What are the conditions for an IED on X' to be extensible to an IED on X?* In order to answer this problem, we first find the conditions (local matching conditions) that such an IED satisfies around each semiregular semisimple element of X, and we derive an answer to the problem by gathering these local matching conditions. After a few reviews on IED's in §4, we shall describe the above procedure in further detail in the case of $X = GL(4,\boldsymbol{R})/(GL(2,\boldsymbol{R}) \times GL(2,\boldsymbol{R}))$. In particular, we explain how the local matching conditions are deduced from our results obtained on invariant integrals on this space X. In case of $X = GL(4,\boldsymbol{R})/(GL(2,\boldsymbol{R}) \times GL(2,\boldsymbol{R}))$, as the semisimple part of the symmetric subspaces $X_{\{x_0\}}$ attached to a semiregular semisimple element x_0, appears the symmetric space $GL(2,\boldsymbol{R})/(GL(1,\boldsymbol{R}) \times GL(1,\boldsymbol{R}))$. This space belongs to a series $X = GL(n+1, \boldsymbol{R})/ (GL(1, \boldsymbol{R}) \times GL(n, \boldsymbol{R}))$ of rank one semisimple symmetric spaces, which are non-isotropic for $n > 1$ (cf. Kosters [9]). We note that the above problem has already been discussed in [1], [2], [3], [4] and [5] for all symmetric spaces enumerated in Example 1.1 but $X = GL(m+n, \boldsymbol{R})/ (GL(m, \boldsymbol{R}) \times GL(n, \boldsymbol{R}))$.

Contents

§ 1. Main Results

Definition 1.1. For an element $x_0 \in \bigcup_{l \in L} J_l$, let G_1 and H_1 be the centralizers $Z_G(\tilde{\sigma}(x_0))$ and $Z_H(\tilde{\sigma}(x_0))$ of $\tilde{\sigma}(x_0)$ in G and H respectively. Then, the symmetric space $X_{\{x_0\}} = G_1/H_1$ is considered as a subspace of $X = G/H$ by the natural imbedding $X_{\{x_0\}} \hookrightarrow X$. We call the space $X_{\{x_0\}}$ *the symmetric subspace attached to* x_0.

We note that a Cartan subspace J_l of X with $x_0 \in J_l$ is also a Cartan subspace of $X_{\{x_0\}}$. Let f be a function in $C_c^\infty(X_{\{x_0\}})$. We define the invariant integral of f by

$$_{X_{\{x_0\}}}F_f(x) = \int_{H_1/Z_{H_1}(\tilde{\sigma}(x))} f(h.x)\, d\tilde{h} \qquad (x \in \bigsqcup_{x \in J_l} J_l'),$$

where $d\tilde{h}$ is a suitably normalized H_1-invariant measure on $H_1/Z_{H_1}(\tilde{\sigma}(x))$. We remark that $Z_{H_1}(\tilde{\sigma}(x)) = Z_{H_1}(\tilde{\sigma}(J_l))$ for each $x \in J_l'$.

Now we shall state the main results of this article.

Proposition 1.1. *Fix an element x_0 of $\bigcup_{l \in L} J_l$. For any $l \in L$ with $x_0 \in J_l$, there exists an open neighborhood $\mathcal{O}_l$ of x_0 in J_l satisfying the following condition : For any function $f \in C_c^\infty(X)$, there exists a function $g \in C_c^\infty(X_{\{x_0\}})$ such that*

$$
\begin{gathered}
{}_X F_f|_{\mathcal{O}'} = {}_{X_{\{x_0\}}} F_g\Big|_{\mathcal{O}'} \\
\operatorname{supp} g \subset (H.\operatorname{supp} f) \cap X_{\{x_0\}},
\end{gathered}
$$

where we write $\mathcal{O}' = \bigsqcup_{x_0 \in J_l} \mathcal{O}'_l$.

Proposition 1.2. *Fix an element x_0 of $\bigcup_{l \in L} J_l$. We set $\mathcal{J}' = \bigsqcup_{x_0 \in J_l} J'_l$. Then we can take an open neighborhood U of x_0 in $X_{\{x_0\}}$ satisfying the following condition: For any function $g \in C_c^\infty(U)$, there exists a function $f \in C_c^\infty(X)$ such that*

$$
\begin{gathered}
{}_X F_f|_{\mathcal{J}' \cap U} = {}_{X_{\{x_0\}}} F_g\Big|_{\mathcal{J}' \cap U}, \\
\operatorname{supp} f \subset H.\operatorname{supp} g.
\end{gathered}
$$

Proposition 1.1 and 1.2 show us that the behavior of invariant integrals on X around x_0 is reduced to that of invariant integrals on $X_{\{x_0\}}$ around x_0. In the following discussion, it is important for us to consider the case where x_0 is semiregular semisimple.

Definition 1.2. $x \in X$ is called a *semiregular* semisimple element, if it satisfies the conditions:

(1) x : semisimple.
(2) x is not regular semisimple: $x \notin X'$.
(3) There is a neighborhood V of x in X such that

$$
y : \text{semisimple and } y \in V \setminus X' \implies X_{\{y\}} = X_{\{x\}}.
$$

That is, a semiregular semisimple element x has regularity next to the regular elements in a neighborhood of x.

Let x be a semiregular semisimple element of X. We find easily that

$$\begin{aligned} X_{\{x\}} \simeq X^S_{\{x\}} \times X^T_{\{x\}} \qquad &\text{(locally isomorphic)} \\ X^S_{\{x\}} &= \text{a semisimple symmetric space of rank one,} \\ X^T_{\{x\}} &= \boldsymbol{R}^r \times \boldsymbol{T}^s \text{ with } r+s = \operatorname{rank} X - 1. \end{aligned}$$

In case of rank one semisimple symmetric spaces, the behavior of invariant integrals is well-known. Hence, we know the behavior of invariant integrals on $X_{\{x_0\}}$, in case where x_0 is a semiregular semisimple. Thus we deduce the behavior of invariant integrals on X around x_0, thanks to Proposition 1.1 and 1.2.

We shall prove the above propositions in §2, §3. Our approach is based upon the method by [Hirai, 7] and so on, which is used in proving the corresponding results for semisimple Lie groups .

§2. Proof of Proposition 1.1

We first quote the following statement for semisimple Lie groups, which is known as "a theorem of compacity"

Lemma 2.1(A theorem of compacity, Harish-Chandra). *Let G be a semisimple Lie group and let γ_0 be an element in a Cartan subgroup $\tilde{J}$ of G. Let $Z_G(\gamma_0)$ be a centralizer of γ_0 in G, $x \mapsto \dot{x}$ the natural mapping of G onto the space $G/Z_G(\gamma_0)$. Then there exists an open neighborhood $\tilde{\mathcal{O}}_{\tilde{J}}(\gamma_0)$ of γ_0 in $\tilde{J}$ with the following property: Given any compact subset $\tilde{\omega}$ in G, there exists a compact subset $\tilde{\Omega} = \tilde{\Omega}(\tilde{\omega}, \tilde{\mathcal{O}}_{\tilde{J}}(\gamma_0))$ in $G/Z_G(\gamma_0)$ such that*

$$\gamma \in \tilde{\mathcal{O}}_{\tilde{J}}(\gamma_0) \ \text{ and } \ x\gamma x^{-1} \in \tilde{\omega} \implies \dot{x} \in \tilde{\Omega}.$$

For a proof of the lemma, refer to Warner[11, Theorem 8.1.4.1 (p.75)] for instance. The theorem of compacity for the case of semisimple Lie groups is generalized to the case of semisimple symmetric spaces as follows:

Lemma 2.2 (A theorem of compacity for semisimple symmetric spaces). *Let $X = G/H$ be a semisimple symmetric space and let x_0 be an element in a Cartan subspace J of X. We set $X_{\{x_0\}} = G_1/H_1$ as before (Definition 1.1). Then there exists an open neighborhood $\mathcal{O}_J(x_0)$ of x_0 in J with the following property: Given any compact subset ω in X, there exists a compact subset $\Omega = \Omega(\omega, \mathcal{O}_J(x_0))$ in H/H_1 such that*

$$x \in \mathcal{O}_J(x_0) \ \ \textit{and} \ \ h.x \in \omega \implies \bar{h} = h.H_1 \in \Omega.$$

Outline of the proof of Lemma 2.2. (See [Aoki-Kato, 1] for the detail of the proof.) For a Cartan subspace J with $x_0 \in J$, let $\tilde{J}$ be a Cartan subgroup of G including $\tilde{\sigma}(J)$, and put $\gamma_0 = \tilde{\sigma}(x_0)$. It is clear that $\gamma_0 \in \tilde{J}$. Applying Lemma 2.1 for γ_0 and $\tilde{J}$, we take an open neighborhood $\tilde{\mathcal{O}}_{\tilde{J}}(\gamma_0)$ of γ_0 in $\tilde{J}$ with the property as in the lemma 2.1. Let $\mathcal{O}_J(x_0)$ be an open neighborhood of x_0 in J defined by $\mathcal{O}_J(x_0) = \left(\tilde{\sigma}_{|J}\right)^{-1}\left(\tilde{\mathcal{O}}_{\tilde{J}}(\gamma_0)\right)$. Then, by an elementary observation, we can verify that $\mathcal{O}_J(x_0)$ has the property in Lemma 2.2. Q.E.D.

Let us turn now to the proof of Proposition 1.1. Let x_0 be a semisimple element of a Cartan subspace J_l. Then, by Lemma 2.2, there exists an open neighborhood $\mathcal{O}_{J_l}(x_0)$ of x_0 in J_l with the property as in the lemma. Put $\mathcal{O}_l = \mathcal{O}_{J_l}(x_0)$. We shall show that $\mathcal{O}_l$ admits the condition in Proposition 1.1. Fix a function $f \in C_c^\infty(X)$. We choose the support of f as a compact set ω in X in Lemma 2.2: $\omega = \operatorname{supp} f$. Thus, according to Lemma 2.2, there exists a compact subset $\Omega_l(f) := \Omega(\operatorname{supp} f, \mathcal{O}_l)$ in H/H_1 such that

$$x \in \mathcal{O}_l \ \text{ and } \ h.x \in \operatorname{supp} f \implies \bar{h} := h.H_1 \in \Omega_l(f).$$

Put $\Omega_f = \bigcup_{x_0 \in J_l} \Omega_l(f)$, and $\mathcal{O} = \bigcup_{l \in L} \mathcal{O}_l$. It is clear that
(1) Ω_f is a compact subset of H/H_1.
(2) $f(h.x) \neq 0 \quad (h \in H,\, x \in \mathcal{O}) \implies \bar{h} \in \Omega_f$.

In order to define $g = g_f \in C_c^\infty(X_{\{x_0\}})$ for each $f \in C_c^\infty(X)$, we take an auxiliary function $\alpha = \alpha_{\Omega_f} \in C_c^\infty(H)$ satisfying the condition:

$$\int_{H_1} \alpha(h.\xi)\, d\xi \equiv 1 \quad \text{for any } h \in H \text{ with } \bar{h} \in \Omega_f.$$

Define a function $g_f = g_{f,\alpha} \in C_c^\infty(X_{\{x_0\}})$ by

$$g_f(\eta) := \int_H \alpha(h) f(h.\eta)\, dh.$$

As is seen easily, we have

$$\operatorname{supp} g_f \subset H.\operatorname{supp} f \cap X_{\{x_0\}}.$$

Let x be an element of $\mathcal{O}' = \bigcup_{l \in L} \mathcal{O}'_l$, say $x \in \mathcal{O}'_l \subset \mathcal{O}_l \quad (l \in L)$. We normalize the H-invariant measure $d\bar{h}$ on H/H_1 by the requirement

$$\begin{aligned} {}_X F_f(x) &= \int_{H/Z_H(J_l)} f(h.x)\, d\tilde{h} \\ &= \int_{H/H_1} \left(\int_{H_1/Z_{H_1}(J_l)} f(h.\xi.x)\, d\tilde{\xi} \right) d\bar{h}. \end{aligned}$$

We note that $Z_H(J_l) = Z_{H_1}(J_l)$. Then it follows that

$$\begin{aligned} {}_X F_f(x) &= \int_{H/H_1} \left(\int_{H_1/Z_{H_1}(J_l)} f(h.\xi.x)\, d\tilde{\xi} \right) d\bar{h} \\ &= \int_{H/H_1} \int_{H_1} \alpha(h.\zeta)\, d\zeta \left(\int_{H_1/Z_{H_1}(J_l)} f(h.\xi.x)\, d\tilde{\xi} \right) d\bar{h} \\ &= \int_H \alpha(h) \left(\int_{H_1/Z_{H_1}(J_l)} f(h.\xi.x)\, d\tilde{\xi} \right) dh \\ &= \int_{H_1/Z_{H_1}(J_l)} \left(\int_H \alpha(h) f(h.\xi.x)\, dh \right) d\tilde{\xi} \\ &= \int_{H_1/Z_{H_1}(J_l)} g_f(\xi.x)\, d\tilde{\xi} \\ &= {}_{X_{\{x_0\}}} F_{g_f}(x). \end{aligned}$$

Thus, we have completed the proof of Proposition 1.1.

§3. Proof of Proposition 1.2

Retain the previous notations. In particular, for any $x_0 \in X$, we identify $X_{\{x_0\}}$ with the subspace $\iota(X_{\{x_0\}})$ of X, by the natural embedding:

$$\iota : X_{\{x_0\}} = G_1/H_1 \hookrightarrow G/H = X.$$

Before giving the proof of Proposition 1.2, we shall review the result for the Jordan decomposition for semisimple symmetric spaces $X = G/H$ due to [Matsuki 10, Proposition 2 (p.52)].

Lemma 3.1 (The Jordan decomposition for semisimple symmetric spaces, Matsuki). *Let $\mathfrak{g}$ be the Lie algebra of a reductive Lie group G, and $\mathfrak{g}_s(= [\mathfrak{g}, \mathfrak{g}])$ the semisimple part of $\mathfrak{g}$. Let σ be an involution of G. Then we have the following:*

(1) *Every element of G can be uniquely written as*

$$g = (\exp X_u)g_s.$$

Here g_s is an element of G such that $Ad(\tilde{\sigma}(g_s))$ is semisimple, and X_u is a nilpotent element of $\mathfrak{g}_s$ such that $\sigma X_u = Ad(g_s)\sigma Ad(g_s)^{-1}X_u = -X_u$.

(2) *Let g' be another element of G and write*

$$g' = (\exp X'_u)g'_s$$

as in (1). Then, for $h, h' \in H$, we have

$$g' = hgh' \iff g'_s = hg_sh' \text{ and } X'_u = Ad(h)X_u.$$

Hereafter, $g = (\exp X_u)g_s$ denotes the Jordan decomposition of $g \in G$ for the symmetric space $X = G/H$ as in Lemma 3.1 (1). We fix an element x_0 of $\bigcup_{l \in L} J_l$. Let

$$X_{\{x_0\}} = \bigsqcup_{i \in I} H_1 v_i H_1 \qquad (v_i \in G_1) \tag{3.1}$$

be the H_1-orbit decomposition of the symmetric space $X_{\{x_0\}} = G_1/H_1$. Since, for every $i \in I$, there exist some $h, h' \in H_1$ with $h(v_i)_s h' \in \bigcup_{x_0 \in J_l} J_l$, we may assume from the beginning that the family $\{v_i \mid i \in I\}$ of representatives of $H_1 \backslash G_1 / H_1$ has the property:

(3.2) For all $i \in I$, $\quad (v_i)_s H_1 \in \bigcup_{x_0 \in J_l} J_l$.

Let us write $x_0 = g_0 H_1$ with $g_0 \in G_1$. We can choose an open neighborhood $\tilde{U}$ of g_0 in G_1 satifying the following properties:

$$
(3.3) \quad
\begin{aligned}
&(a)\ \tilde{u} \in \tilde{U} \implies h\tilde{u}h' \in \tilde{U} \quad \text{for any } h, h' \in H_1 \\
&(b)\ Z_G(\tilde{\sigma}(gH))/Z_H(\tilde{\sigma}(gH)) \subset X_{\{x_0\}} \quad \text{for all } g \in \tilde{U} \\
&(c)\ \tilde{u} \in \tilde{U} \implies \tilde{u}_s \in \tilde{U} \\
&(d)\ \text{For } h \in H, \text{ and } \tilde{u}_i \in \tilde{U} \text{ satisfying } \tilde{u}_i H_1 \in \bigcup_{x_0 \in J_l} J_l \quad (i = 1, 2), \\
&\qquad\qquad h\tilde{u}_1 H = \tilde{u}_2 H \implies h \in H_1.
\end{aligned}
$$

Remark 3.1. For each $u \in \tilde{U}$, let us write $u = hv_i h'$ for some $h, h' \in H_1$ and some $i \in I$. Then, the property (a) in (3.3) implies $v_i \in \tilde{U}$.

Lemma 3.2. *Let S be a C^ω local cross section defined on an open neighborhood of the origin eH_1 of H/H_1. Then, for $x_0 \in \bigcup_{l \in L} J_l$, there exists an H_1-invariant open neighborhood U of x_0 in $X_{\{x_0\}}$ such that*

(1) *For every $H_i \in H$, $u_i \in U$,*

$$h_1 u_1 = h_2 u_2 \in X = G/H \implies h_1 H_1 = h_2 H_1.$$

(2) *For every $s_i \in S$, $u_i \in U$,*

$$s_1 u_1 = s_2 u_2 \in X = G/H \implies s_1 = s_1, \quad u_1 = u_2.$$

Proof. Fix $g_0 \in G_1$ such that $x_0 = g_0 H_1$. Fix an open neighborhood $\tilde{U}$ of $g_0 \in G_1$ with the property (3.3). Let π_1 be the natural projection $\pi_1 : G_1 \to X_{\{x_0\}} = G_1/H_1$. If we put $U = \pi_1(\tilde{U})$, then U is a H_1-invariant open neighborhood of x_0 in $X_{\{x_0\}} = G_1/H_1$. We shall verify that (1) (2) hold for U.

(1) Suppose that $h_1u_1 = h_2u_2$ ($h_1, h_2 \in H$, $u_1, u_2 \in U$) in $X = G/H$. When we write $u_i = \tilde{u}_i H_1 \in U$ ($\tilde{u}_i \in \tilde{U}$; $i = 1, 2$), this condition is rewritten as

$$(3.4) \qquad h_1\tilde{u}_1 h' = h_2\tilde{u}_2 \qquad \text{for some } h' \in H.$$

We set

$$(3.5) \qquad \tilde{u}_i = \xi_i v_{\alpha_i} \xi_i' \qquad (\xi_i,\ \xi_i' \in H_1,\ \alpha_i \in I;\ i = 1, 2),$$

according to (3.1). Note that $v_{\alpha_1}, v_{\alpha_2} \in \tilde{U}$ by Remark 3.1. Let us write

$$v_{\alpha_i} = (\exp X_{\alpha_i})_u\, (v_{\alpha_i})_s \qquad (i = 1, 2),$$

for the Jordan decomposition of v_{α_i}. It follows from (3.4) and (3.5) that we have

$$v_{\alpha_2} = (h_2\xi_2)^{-1} h_1 \xi_1 v_{\alpha_1} \xi_1' h' \xi_2'^{-1}.$$

In view of Lemma 3.1, we get

$$(v_{\alpha_2})_s = (h_2\xi_2)^{-1} h_1 \xi_1\, (v_{\alpha_1})_s\, \xi_1' h' \xi_2'^{-1}.$$

On the other hand, with the combination of the property (c) in (3.3) and the assumption (3.2) for v_i ($i \in I$), we have $(v_{\alpha_i})_s \in \tilde{U}$ and $(v_{\alpha_i})_s H_1 \in \bigcup_{x_0 \in J_l} J_l$ for $i = 1, 2$. Thus, it follows from the property (d) in (3.3) that $(h_2\xi_2)^{-1} h_1 \xi_1 \in H_1$, which asserts $h_1 H_1 = h_2 H_1$.

(2) Suppose that $s_1 u_1 = s_2 u_2$ in $X = G/H$ ($s_i \in S$, $u_i \in U$). Then, as above, we write $s_1 \tilde{u}_1 h' = s_2 \tilde{u}_2$ for $u_i = \tilde{u}_i H_1$ ($\tilde{u}_i \in \tilde{U}$; $i = 1, 2$) and some $h' \in H$. Since S is a subset of H, we have $s_1 H_1 = s_2 H_1$ by the assersion of (1), and whence $s_1 = s_2$. Thus we have $\tilde{u}_1 h = \tilde{u}_2$, and consequently we get $\tilde{u}_1 H_1 = \tilde{u}_2 H_1$ because $\tilde{u}_1^{-1}\, \tilde{u}_2 = h \in G_1 \cap H = H_1$. Q.E.D.

Now we are ready to prove Proposition 2.2. Let S and U be the ones as Lemma 3.2. For any function $g \in C_c^\infty(U)$, we define a function $f \in C_c^\infty(X)$ determined by g. We fix a function $\Psi \in C_c^\infty(S)$ such that

$$\int_{H/H_1} \Psi(s(\bar{h}))\, d\bar{h} = 1.$$

Here we write $s = s(\bar{h})$ for $sH_1 = \bar{h} \in H/H_1$. Put $U_S = \{s.u \in X \mid s \in S,\ u \in U\}$. Then Lemma 3.2 (2) tells us that

$$s_1u_1 = s_2u_2 \in U_S \quad (s_i \in S,\ u_i \in U) \implies s_1 = s_2 \text{ and } u_1 = u_2.$$

On the other hand, since we find by an easy observation that the dimension of U_S coincides with the dimension of G/H, U_S is an open subset in $X = G/H$. Thus, we can define $f \in C_c^\infty(X)$ by

$$f(x) = \begin{cases} \Psi(s)g(u) & (x = su \in U_s) \\ 0 & (x \notin U_s). \end{cases}$$

Obviously one has

$$\operatorname{supp} f \subset H.\operatorname{supp} g.$$

Lemma 3.3. *Keep the notations in Lemma 3.2. Fix a Cartan subspace J_l such that $x_0 \in J_l$. Let x be an element of $J_l' \cap U$. If $h \in H$ satisfies the condition $\int_{H_1/Z_{H_1}(J_l)} f(h\xi x)\,d\tilde{\xi} \neq 0$, then we have $\bar{h} = hH_1 \in \pi(S)$ and $\xi x \in U$. Here π is the natural projection of H to H/H_1.*

Proof. Let h be an element of H satisfying the above condition. Then there exists some $\xi \in H_1$ such that $h\xi x \in U_S = SU$. We write $h\xi x = h'x'$ $(h' \in S, x' \in U)$. Since U is H_1-invariant, $\xi x \in U$ follows. Hence, thanks to Lemma 3.2 (1), we have $hH_1 = h'H_1$, which implies $\bar{h} \in \pi(S)$. Q.E.D.

Let us consider the integration:

$$\begin{aligned} {}_XF_f(x) &= \int_{H/Z_H(J_l)} f(h.x)\,d\tilde{h} \\ &= \int_{H/H_1} \left(\int_{H_1/Z_{H_1}(J_l)} f(h\xi x)\,d\tilde{\xi} \right) d\bar{h} \qquad \text{for } x \in J_l' \cap U, \end{aligned}$$

with $J_l' \subset \bigsqcup_{x_0 \in J_l} J_l'$. (The H-invariant measure $d\bar{h}$ on H/H_1 is already normalized in §2 so that the above formula holds). In view of Lemma 3.3, the condition $\int_{H_1/Z_{H_1}(J_l)} f(h\xi x)\,d\tilde{\xi} \neq 0$ means $\bar{h} = hH_1 \in \pi(S)$ and $\xi x \in U$. Hence the above integration is written as follows:

$$\begin{aligned} \int_{H/H_1} \left(\int_{H_1/Z_{H_1}(J_l)} \Psi(s(\bar{h}))g(\xi x)\,d\tilde{\xi} \right) d\bar{h} &= \int_{H/H_1} \Psi(s(\bar{h}))\,d\bar{h} \int_{H_1/Z_{H_1}(J_l)} g(\xi x)\,d\tilde{\xi} \\ &= {}_{X_{\{x_0\}}}F_f(x). \end{aligned}$$

The proof of Proposition 1.2 is now complete.

§4. Invariant Eigendistribution

Let $X = G/H$ be a semisimple symmetric space and $\mathcal{O}$ an H-invariant open subset of X. Let $\boldsymbol{D}(X)$ be the ring of invariant differential operators on X, and χ a character of $\boldsymbol{D}(X)$: $\chi \in \mathrm{Hom}(\boldsymbol{D}(X), \boldsymbol{C})$.

Definition 4.1. A distribution $\Theta \in \mathcal{D}'(\mathcal{O})$ is said to be an *Invariant Eigendistribution (IED)* with χ on $\mathcal{O}$, if it satisfies the following two conditions:
i) Θ : H-invariant
ii) $D.\Theta = \chi(D)\Theta \quad (\forall D \in \boldsymbol{D}(X))$

Remarks. i) If $\mathcal{O}_1 \subset \mathcal{O}_2$, then, for any IED Θ with χ on $\mathcal{O}_2$, the restriction $\Theta|_{\mathcal{O}_1}$ is an IED with χ on $\mathcal{O}_1$.
ii) Let X' be the set of regular semisimple elements of X, which is open dense and H-invariant in X. Any IED on X' is necessarily a real analytic function.

In view of above remarks, for any IED Θ on X, the restriction $\Pi = \Theta|_{X'}$ is an IED on X', hence it is a real analytic function on $X' = \bigsqcup_{l \in L} H.J'_l$. In this way, putting $\Pi_l := \Theta|_{J'_l}$, we have a system of real analytic functions $\{\Pi_l\}_{l \in L}$. We study global matching conditions, i.e. compatibility conditions among Π_l's, to give an explicit form of IED's.

To this end, for every semiregular semisimple element $x_0 \in \bigcup_{l \in L} J_l$, we establish, by the procedure described in Diag. 1, local matching conditions, i.e. compatibility conditions on a neighborhood of x_0.

(a) Explicit form of radial parts of invariant differential operators
↓
(b) Explicit form of IED's on $H.J'_l$ $(J_l \ni x_0)$

(c) Behavior of invariant integral F_f around x_0

↘ ↙

↓← Integration by parts via Weyl's integral formula

Local matching conditions of IED around x_0

Diag. 1

§5. Case of $X = GL(4, \boldsymbol{R})/\,(GL(2, \boldsymbol{R}) \times GL(2, \boldsymbol{R}))$

Let $G = GL(4, \boldsymbol{R})$, and σ an involution defined by $\sigma(g) = I_{2,2} g I_{2,2}$, where $I_{2,2} = \mathrm{diag}(1, 1, -1, -1)$. Then $H = G^\sigma$ is isomorphic to $GL(2, \boldsymbol{R}) \times GL(2, \boldsymbol{R})$. In this section, we treat IED's on $X = G/H = GL(4, \boldsymbol{R})/\,(GL(2, \boldsymbol{R}) \times GL(2, \boldsymbol{R}))$.

Cartan Subspaces.

$$j_0(\theta_1, \theta_2; \epsilon_1, \epsilon_2) = \begin{pmatrix} \epsilon_1 \cosh\theta_1 & 0 & \epsilon_1 \sinh\theta_1 & 0 \\ 0 & \epsilon_2 \cosh\theta_2 & 0 & \epsilon_2 \sinh\theta_2 \\ \epsilon_1 \sinh\theta_1 & 0 & \epsilon_1 \cosh\theta_1 & 0 \\ 0 & \epsilon_2 \sinh\theta_2 & 0 & \epsilon_2 \cosh\theta_2 \end{pmatrix}$$

$$j_1(\theta_1, \theta_2; \epsilon) = \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & \epsilon \cosh\theta_2 & 0 & \epsilon \sinh\theta_2 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & \epsilon \sinh\theta_2 & 0 & \epsilon \cosh\theta_2 \end{pmatrix}$$

$$\tilde{j}_1(\theta_1, \theta_2) = \begin{pmatrix} \cos\theta_1 & 0 & 0 & \sin\theta_1 \\ 0 & \cos\theta_1 & -\sin\theta_1 & 0 \\ 0 & \sin\theta_1 & \cos\theta_1 & 0 \\ -\sin\theta_1 & 0 & 0 & \cos\theta_1 \end{pmatrix} j_0(\theta_2, \theta_2; +, +)$$

$$j_2(\theta_1, \theta_2) = \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$$

Table 1

Using the notations of Table 1, put

$$\begin{aligned}
J_0^{\epsilon_1,\epsilon_2} &= \{j_0(\theta_1,\theta_2;\epsilon_1,\epsilon_2)\,|\,\theta_1,\theta_2\in \boldsymbol{R}\} \qquad (\epsilon_1,\epsilon_2=\pm),\\
J_1^{\epsilon} &= \{j_1(\theta_1,\theta_2;\epsilon)\,|\,\theta_1,\theta_2\in \boldsymbol{R}\} \qquad (\epsilon=\pm),\\
J_2 &= \{j_2(\theta_1,\theta_2)\,|\,\theta_1,\theta_2\in \boldsymbol{R}\},\\
\tilde{J}_1 &= \{\tilde{j}_1(\theta_1,\theta_2)\,|\,\theta_1,\theta_2\in \boldsymbol{R}\},\\
J_0 &= J_0^{++}\sqcup J_0^{+-}\sqcup J_0^{-+}\sqcup J_0^{--}, \quad \text{and}\\
J_1 &= J_1^{+}\sqcup J_1^{-}.
\end{aligned}$$

Then we have a complete system $\{J_0, J_1, \tilde{J}_1, J_2\}$ of representatives of H-conjugacy classes of Cartan subspaces for $X = GL(4,\boldsymbol{R})/\left(GL(2,\boldsymbol{R})\times GL(2,\boldsymbol{R})\right)$. We note that these Cartan subspaces are now considered as the subspaces of $\tilde{\sigma}(X)$ which is identified with X.

For an element x in J_l $(l = 0,1,2)$, we denote by t_j its (j,j)-entry $(j = 1, 2)$. For example, we have $t_1 = \epsilon_1\cosh\theta_1$, $t_2 = \epsilon_2\cosh\theta_2$ for $x = j_0(\theta_1,\theta_2;\epsilon_1,\epsilon_2)\in J_0$. Furthermore, for an element $x = j_1(\theta_1,\theta_2;\epsilon)\in \tilde{J}_1$, we put $t_1 = \cosh(-\theta_2+i\theta_1)$ and $t_2 = \cosh(\theta_2+i\theta_1)$. By the correspondence of $x\in J_0\cup J_1\cup \tilde{J}_1\cup J_2$ to the "t-variables" (t_1, t_2), one can identify $J_0/(\boldsymbol{Z}_2)^2$ with $\left((-\infty,-1]\sqcup[1,\infty)\right)^2$, $J_1/(\boldsymbol{Z}_2)^2$ with $[-1,1]\times\left((-\infty,-1]\sqcup[1,\infty)\right)$, and $J_2/(\boldsymbol{Z}_2)^2$ with $\left([-1,1]\right)^2$. Then, for elements $(t_1,t_2),(t_1',t_2')\in J_l/(\boldsymbol{Z}_2)^2$ with $l=0,1,2$, (t_1,t_2) is mapped to (t_1',t_2') by the Weyl group $W(J_l)$ if and only if

$$\begin{cases} (t_1,t_2)=(t_1',t_2') \text{ or } (t_1,t_2)=(t_2',t_1') & (l=0,2)\\ (t_1,t_2)=(t_1',t_2') & (l=1).\end{cases}$$

For $\tilde{J}_1$, $\tilde{j}_1(\theta_1,\theta_2)$ is mapped to $\tilde{j}_1(\theta_1',\theta_2')$ by the Weyl group $W(\tilde{J}_l)$ if and only if

$$\theta_1 = \pm\theta_1' \pmod{2\pi} \text{ and } \theta_2=\pm\theta_2'.$$

Put

$$\begin{aligned}
\omega &= t_2 - t_1,\\
L &= 4(t^2-1)\frac{d^2}{dt^2} + 8t\frac{d}{dt},\\
L_j &= 4(t_j^2-1)\frac{\partial^2}{\partial t_j^2} + 8t_j\frac{\partial}{\partial t_j} \qquad (j=1,\,2), \text{ and}\\
\mathcal{S} &= \{\omega^{-1}S(L_1,L_2)\omega \mid S \text{ is a symmetric polynomial with 2-variables}\}.
\end{aligned}$$

Then we have the isomorphism Φ of $\boldsymbol{D}(X)$ to $\mathcal{S}$ satisfying

$$\Phi(D)|_{J'}(f|_{J'}) = (Df)|_{J'}$$

$$\text{for any } D \in \boldsymbol{D}(X),\ f \in C_H^\infty(X),\ J = J_0, J_1, \tilde{J}_1, J_2,$$

where $C_H^\infty(X)$ is the space of all H-invariant C^∞-functions on X. (cf. Hoogenboom [8]). In virtue of this fact, we denote by $\chi_{\lambda_1,\lambda_2}$ the unique character χ of $\boldsymbol{D}(X)$ satisfying the condition

$$\chi(\Phi^{-1}(\omega^{-1}S(L_1, L_2)\omega)) = S(\lambda_1, \lambda_2)$$

for any symmetric polynomial S with 2-variables. In case $\lambda_1 \neq \lambda_2$, $\chi = \chi_{\lambda_1,\lambda_2}$ is called *regular*. We call $\Phi(D)$ the *radial part* of D.

<u>**Symmetric Subspaces.**</u>

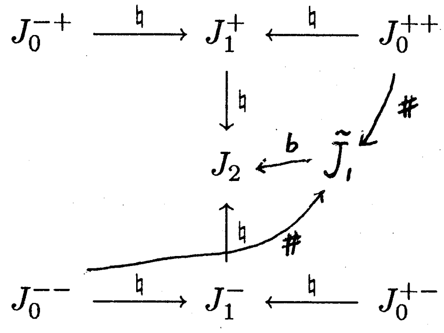

Fig. 1

In case of $X = GL(4,\boldsymbol{R})/(GL(2,\boldsymbol{R}) \times GL(2,\boldsymbol{R}))$, a semiregular semisimple element x_0 in $J_0 \cup J_1 \cup \tilde{J}_1 \cup J_2$ corresponds to one of three symbols ♮, ♯, ♭ in Fig. 1:

i) x_0 corresponds to ♮ if it belongs to $(J_0 \cap J_1) \cup (J_1 \cap J_2)$

ii) x_0 corresponds to ♯ if it belongs to $J_0 \cap \tilde{J}_1$

iii) x_0 corresponds to ♭ if it belongs to $(hJ_2h^{-1} \cap \tilde{J}_1) \cup (J_2 \cap h^{-1}\tilde{J}_1 h)$,

where $h = h^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. In each case, the symmetric subspace attached to x_0 is as follows:

i) $X_{\{x_0\}} \simeq GL(2,\boldsymbol{R})/(GL(1,\boldsymbol{R}) \times GL(1,\boldsymbol{R})) \times (\boldsymbol{R} \text{ or } \boldsymbol{T})$,

ii) $X_{\{x_0\}} \simeq (GL(2,\boldsymbol{R}) \times GL(2,\boldsymbol{R}))/\triangle GL(2,\boldsymbol{R}) \simeq SL(2,\boldsymbol{R}) \times \boldsymbol{R}$,

iii) $X_{\{x_0\}} \simeq GL(2,\boldsymbol{C})/GL(2,\boldsymbol{R}) \simeq SL(2,\boldsymbol{C})/SL(2,\boldsymbol{R}) \times \boldsymbol{T}$,

where we denote by $\simeq$ a local isomorphy.

Let x_0 be a semiregular semisimple element corresponding to $\natural$. For simplicity, we assume moreover $x_0 \in J_1^+ \cap J_0^{++}$. Note that $t_1 = 1$ for x_0. In the case of $X_1 = GL(2, \boldsymbol{R})/(GL(1, \boldsymbol{R}) \times GL(1, \boldsymbol{R}))$ (a symmetric space of rank 1), we have, on a small neighborhood of the origine ($t = 1$),

$$(*) \quad \{F_f(t) \mid f \in C_c^\infty(X_1)\} \\ = \{\phi(t) \mid \phi(t) = \phi_0(t) + \phi_1(t) \log|t-1| \quad (\phi_0, \phi_1 \in C^\infty)\}.$$

Since $X_{\{x_0\}}$ is (locally) isomorphic to $GL(2, \boldsymbol{R})/(GL(1, \boldsymbol{R}) \times GL(1, \boldsymbol{R})) \times \boldsymbol{R}$, we have, from Proposition 1.1, 1.2, and ($*$), on a small neighborhood of x_0,

$$(**) \quad \{F_f(t_1, t_2) \mid f \in C_c^\infty(X)\} \\ = \{\phi(t_1, t_2) \mid \phi(t_1, t_2) = \phi_0(t_1, t_2) + \phi_1(t_1, t_2) \log|t_1 - 1|;\ \phi_0, \phi_1 \in C^\infty\}.$$

From ($**$), by the procedure sketched in §4, we have the local matching conditon around x_0:

Let Θ be an IED defined on $X = GL(4, \boldsymbol{R})/(GL(2, \boldsymbol{R}) \times GL(2, \boldsymbol{R}))$, and Ψ_j's ($j = 0, 1$) real analytic functions defined on a neighborhood of x_0. Then we have:

$$\Theta|_{J_0^{++\prime}}(t_1, t_2) = \Psi_0(t_1, t_2) + \Psi_1(t_1, t_2) \log|t_1 - 1| \quad (t_1 > 1)$$
$$\Longleftrightarrow \Theta|_{J_1^{+\prime}}(t_1, t_2) = \Psi_0(t_1, t_2) + \Psi_1(t_1, t_2) \log|t_1 - 1| \quad (t_1 < 1)$$

In other words, $\Theta|_{J_0^{++\prime}}$ and $\Theta|_{J_1^{+\prime}}$ are determined mutually in a natural way.

For other semiregular semisimple elements of type $\natural$, we can prove similar local matching conditions.

We note that these local matching conditions are the same as those between two adjacent Cartan subspaces for IED's on $X = U(2,2)/GL(2, \boldsymbol{C})$.

For semiregular semisimple elements of type $\sharp$ or of type $\flat$, local matching conditions are essentially the same as those between two adjacent Cartan subspaces for IED's on $X = U(2,2)/(U(1,1) \times U(1,1))$.

Although the details are different and more complicated, we can also derive local matching conditions in these cases, along the lines mentioned above for elements of type $\natural$ (cf. Aoki-Kato [4]). For example, in the case of $x_0 \in J_0 \cap \tilde{J}_1$, the local matching conditions around x_0 is given as follows:

Let Θ be an IED defined on $X = GL(4, \boldsymbol{R})/(GL(2, \boldsymbol{R}) \times GL(2, \boldsymbol{R}))$. Then $\omega\Theta|_{J_0'}$ is extensible as a real analytic function to a neighborhood of x_0 in J_0, and we have

$$\frac{1}{i}\frac{d}{ds}\omega\Theta|_{\tilde{J}_1'}(\tilde{j}_1(s,0)x_0)\bigg|_{s=\pm 0} = \frac{d}{ds}\omega\Theta|_{J_0'}(j_0(-s,s;+,+)x_0)\bigg|_{s=\pm 0}.$$

In case of $x_0 \in J_2 \cap \tilde{J}_1$, we have a similar local matching conditions around x_0.

Gathering and reformulating all the local matching conditions mentioned above, we have a theorem which gives a rather explicit form for IED's on $J_0' \sqcup J_1' \sqcup \tilde{J}_1' \sqcup J_2'$ (cf. Aoki-Kato [5]).

Explicit Form of IED's.

Let λ be a complex number and put $L = 4(t^2-1)\dfrac{d^2}{dt^2} + 8t\dfrac{d}{dt}$. Any solution of the equation

$$(\star) \qquad L\phi = \lambda\phi$$

is real analytic on $(-\infty,-1) \sqcup (-1,1) \sqcup (1,\infty)$, and a solution ϕ defined on $(-1,1)$ is of the form:

$$\begin{aligned}\phi(t) &= \phi_1(t) + \log|\,1-t\,|\phi_2(t)\\ &= \phi_3(t) + \log|\,1+t\,|\phi_4(t),\end{aligned}$$

where $\phi_1(t)$, $\phi_2(t)$ are real analytic on $(-1,\infty)$, and $\phi_3(t)$, $\phi_4(t)$ are real analytic on $(-\infty,1)$. So we can extend naturally to $(-\infty,-1) \sqcup (-1,1) \sqcup (1,\infty)$ the solution ϕ initially defined only on $(-1,1)$ as follows:

$$\phi(t) := \begin{cases} \phi_1(t) + \log|1-t|\phi_2(t) & (-1 < t) \\ \phi_3(t) + \log|1+t|\phi_4(t) & (t < 1). \end{cases}$$

Let α, β be complex numbers satisfying the conditions $\alpha+\beta = 1$, $\alpha\beta = -\lambda$. We consider the following two solutions of the equation $(\star)$ on $(-1,1)$:

$$\begin{aligned}\phi_+(t,\lambda) &:= F\big(\alpha,\beta,1;\frac{1-t}{2}\big),\\ \phi_-(t,\lambda) &:= F\big(\alpha,\beta,1;\frac{1-t}{2}\big)\log\Big|\frac{1-t}{2}\Big| + F^*\big(\frac{1-t}{2}\big),\end{aligned}$$

where $F^*(z)$ is analytic on $|z| < 1$,

and extend these solutions naturally to $(-\infty,-1) \sqcup (-1,1) \sqcup (1,\infty)$ in the manner described above. We denote by the same symbols the extended solutions.

Let $A(\lambda)$ be the 2×2 matrix determined by the equation :

$$(\phi_+(t,\lambda), \phi_-(t,\lambda)) = (\phi_+(-t,\lambda), \phi_-(-t,\lambda))A(\lambda).$$

Using these notations we can state:

Theorem 5.1. *Let $\chi = \chi_{\lambda_1,\lambda_2}$ be a generic character of $\boldsymbol{D}(X)$. Then, for an IED Θ with χ defined on X, there exsist constants C_{ν_1,ν_2}'s ($\nu_1, \nu_2 = \pm$) such that we have*

(i)
$$\Theta|_{J_0'\sqcup J_1'\sqcup J_2'}(t_1,t_2) = \omega^{-1} \sum_{\nu_1,\nu_2=\pm} C_{\nu_1,\nu_2} \begin{vmatrix} \phi_{\nu_1}(t_1,\lambda_1) & \phi_{\nu_2}(t_1,\lambda_2) \\ \phi_{\nu_1}(t_2,\lambda_1) & \phi_{\nu_2}(t_2,\lambda_2) \end{vmatrix}$$

(ii)
$$\Theta|_{\tilde{J}_1'}(t_1,t_2) = \omega^{-1} \left\{ \sum_{\nu_1,\nu_2=\pm} C_{\nu_1,\nu_2} \begin{vmatrix} \phi_{\nu_1}(t_1,\lambda_1) & \phi_{\nu_2}(t_1,\lambda_2) \\ \phi_{\nu_1}(t_2,\lambda_1) & \phi_{\nu_2}(t_2,\lambda_2) \end{vmatrix} \right.$$
$$\left. + \operatorname{sgn}(\theta_1\theta_2) \sum_{\nu_1,\nu_2=\pm} C'_{\nu_1,\nu_2} \{\phi_{\nu_1}(t_1,\lambda_1)\phi_{\nu_2}(t_2,\lambda_2) + \phi_{\nu_2}(t_1,\lambda_2)\phi_{\nu_1}(t_2,\lambda_1)\} \right\},$$

where
$$C'_{--} = 0, \qquad \begin{pmatrix} C'_{++} \\ C'_{+-} \\ C'_{-+} \end{pmatrix} = B^{-1} \begin{pmatrix} \frac{\pi}{i} C_{--} \\ 0 \\ \frac{\pi}{i} C_{--} \end{pmatrix}.$$

Here B is a non-singular 3×3 matrix explicitly determined by $A(\lambda_1)$ and $A(\lambda_2)$.

Estimate of the Dimension.

As a corollary of the above theorem, we have

Corollary 5.1. *If χ is generic, then*

$$\dim \{\Theta|_{X'};\ \Theta \text{ is an IED on } X \text{ with } \chi\} \leq 4.$$

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