<table>
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<th>Title</th>
<th>Radial symmetry of self-similar solutions for semilinear heat equations (Methods and Applications for Functional Equations)</th>
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<tr>
<td>Author(s)</td>
<td>Naito, Yuki; Suzuki, Takashi</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1999), 1083: 181-186</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1999-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62758">http://hdl.handle.net/2433/62758</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Radial symmetry of self-similar solutions for semilinear heat equations

We consider the symmetry properties of positive solutions of the equation

$$\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u + u^p = 0 \quad \text{in } \mathbb{R}^n, \quad (1.1)$$

where $n \geq 2$ and $p > 1$. This equation arises in the study of (forward) self-similar solutions of the semilinear heat equation

$$w_t = \Delta w + w^p \quad \text{in } \mathbb{R}^n \times (0, \infty). \quad (1.2)$$

It is well known that if $w(x, t)$ satisfies (1.2), then, for $\mu > 0$ the rescaled functions

$$w_\mu(x, t) = \mu^{2/(p-1)} w(\mu x, t)$$

define a one parameter family of solutions to (1.2). A solution $w$ is said to be self-similar, when $w_\mu(x, t) = w(x, t)$ for all $\mu > 0$. It can be easily checked that $w$ is a self-similar solution to (1.2) if and only if $w$ has the form

$$w(x, t) = t^{-1/(p-1)} u \left( \frac{x}{\sqrt{t}} \right), \quad (1.3)$$

where $u$ satisfies the elliptic Eq. (1.1). Moreover, if $u$ has spherical symmetry, that is if $u = u(r)$, $r = |x|$, then $u$ satisfies the ordinary differential equation

$$u'' + \left( \frac{n-1}{r} + \frac{r}{2} \right) u' + \frac{1}{p-1} u + u^p = 0, \quad r > 0. \quad (1.4)$$

Such self-similar solutions are often used to describe the large time behavior of global solutions to the Cauchy problem, see, e.g., [11, 13, 3, 14, 5, and 15], and to show nonuniqueness of solution to (1.2) with zero initial data in a certain functional space, see [12].
First we state the result concerning the symmetry properties of the solution of (1.1).

**Theorem 1.1.** Let \( u \in C^2(\mathbb{R}^n) \) be a positive solution of (1.1) such that
\[
    u(x) = o(|x|^{-2/(p-1)}) \quad \text{as} \quad |x| \to \infty.
\]

Then \( u \) must be radially symmetric about the origin.

The proof of Theorem 1.1 is based on the moving planes argument. This technique was developed by Serrin [18] in PDE theory, and extended and generalized by Gidas, Ni, and Nirenberg [9, 10]. We remark that with a change of variables we are still able to prove a radial symmetry result for Eq. (1.1).

Let us consider the problem
\[
    \begin{cases}
    u'' + \left( \frac{n-1}{r} + \frac{r}{2} \right) u' + \frac{1}{p-1} u + |u|^{p-1} u = 0, & r > 0, \\
    u'(0) = 0 \quad \text{and} \quad u(0) = \alpha \in \mathbb{R}.
    \end{cases}
\]

The problem (1.6) has been investigated extensively in [12, 16, 20, and 2]. We denote by \( u(r; \alpha) \) the unique solution of (1.6). We recall that \( u(r; \alpha) \) has the following properties:

(i) \( \lim_{r \to \infty} r^{2/(p-1)} u(r; \alpha) = L(\alpha) \) exists and is finite for every \( \alpha \in \mathbb{R} \) (see [12, Theorem 5]);

(ii) if \( L(\alpha) = 0 \), then there exists a constant \( A \neq 0 \) such that
\[
    u(r; \alpha) = A e^{-r^2/4 r^{2/(p-1)-n} \{1 + O(r^{-2})\}} \quad \text{as} \quad r \to \infty
\]
(see [16, Theorem 1]);

(iii) if \( p \geq (n+2)/(n-2) \), then \( u(r; \alpha) \) is positive on \([0, \infty)\) and \( L(\alpha) > 0 \) for every \( \alpha > 0 \) (see [12, Theorem 5]);

(iv) if \( (n+2)/n < p < (n+2)/(n-2) \), then there exists a unique \( \alpha > 0 \) such that \( u(r; \alpha) \) is positive on \([0, \infty)\) and \( L(\alpha) = 0 \) (see [20, Theorem 1] and [2, Theorem 1.2 and Corollary 1.3]).

By virtue of Theorem 1.1 we obtain the following:

**Corollary 1.1.** (i) Assume that \( p \geq (n+2)/(n-2) \). Then there exists no positive solution \( u \) of (1.1) satisfying (1.5).

(ii) Assume that \( (n+2)/n < p < (n+2)/(n-2) \). Then there exists a unique positive solution \( u(x) \) satisfying (1.5). Moreover, the solution \( u \) is radially symmetric about the origin.
Remark. The result (i) is differently proven by [3, Proposition 4.3] based on the Pohozaev identity.

Following the notations in [3] and [14], we define
\[ L^2(K) = \{ u : \mathbb{R}^n \to \mathbb{R}; \int_{\mathbb{R}^n} |u|^2 K(x) dx < \infty \} \quad \text{and} \]
\[ H^1(K) = \{ u : \mathbb{R}^n \to \mathbb{R}; \int_{\mathbb{R}^n} (|u|^2 + |\nabla u|^2) K(x) dx < \infty \}, \]
where \( K(x) = \exp(|x|^2/4) \). Escobedo and Kavian have shown in [3, Proposition 3.5] that if \( 1 < p < (n+2)/(n-2) \) and if \( u \in H^1(K) \) is a solution of (1.1), then \( u \in C^2(\mathbb{R}^n) \) and satisfies \( u(x) = O(\exp(-|x|^2/8)) \) as \( |x| \to \infty \). As a consequence of Corollary 1.1, we obtain the following:

**COROLLARY 1.2.** Assume that \( (n+2)/n < p < (n+2)/(n-2) \). Then the problem
\[
\begin{cases}
\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u + u^p = 0 & \text{in } \mathbb{R}^n, \\
u \in H^1(K) \text{ and } u > 0 & \text{in } \mathbb{R}^n,
\end{cases}
\tag{1.7}
\]
has a unique solution.

Let us consider the Cauchy problem
\[
\begin{cases}
w_t = \Delta w + w^p & \text{in } \mathbb{R}^n \times (0, \infty), \\
w(x, 0) = \tau w_0 & \text{in } \mathbb{R}^n,
\end{cases}
\tag{1.8}
\]
where \( w_0 \in L^2(K) \cap L^\infty(\mathbb{R}^n) \), \( w_0 \geq 0 \), and \( \tau > 0 \) is a parameter. We denote by \( w(x, t; \tau) \) the unique solutions of (1.8) (see [15]). Combining the result by Kawanago [15, Theorem 1] and Corollary 1.2, we obtain the following, where the asymptotic behavior of \( w(\cdot, t; \tau) \) as \( t \to \infty \) becomes clearer.

**COROLLARY 1.3.** Assume that \( (n+2)/n < p < (n+2)/(n-2) \). Then there exists a unique \( \tau_0 > 0 \) such that the solution \( w(x, t; \tau) \) is a global solution if \( \tau \in (0, \tau_0] \), and \( w(x, t; \tau) \) blows up in finite time if \( \tau \in (\tau_0, \infty) \). Moreover, \( w(x, t; \tau_0) \) satisfies
\[
\lim_{t \to \infty} \left\| t^{1/(p-1)} w(\cdot, t; \tau_0) - u_0 \left( \cdot \sqrt{t} \right) \right\|_{L^\infty(\mathbb{R}^n)} = 0,
\]
where \( u_0 \) is a unique solution of the problem (1.7).
Next we consider the existence of nonradial solutions of (1.1). Let $p > (n+2)/n$ and let $U(r)$ be a positive solution of (1.4) satisfying
\begin{equation}
U'(0) = 0 \quad \text{and} \quad \lim_{r \to \infty} r^{2/(p-1)}U(r) > 0.
\end{equation}
(1.9)
The existence of such $U$ is obtained by [12, Theorem 5]. Define $\ell = \ell(U) > 0$ as
\begin{equation}
\ell = \lim_{r \to \infty} r^{2/(p-1)}U(r).
\end{equation}
(1.10)
We investigate the Cauchy problem for Eq. (1.2) with
\begin{equation}
w(x, 0) = w_0 \in L^{1}_{\text{loc}}(\mathbb{R}^n),
\end{equation}
(1.11)
where
\begin{equation}
0 \leq w_0(x) \leq \ell|x|^{-2/(p-1)}, \quad w_0 \neq 0; \quad x \in \mathbb{R}^n \setminus \{0\}.
\end{equation}
(1.12)
Relation (1.11) is taken in the sense of $L^{1}_{\text{loc}}(\mathbb{R}^n)$, that is,
\begin{equation}
\int_{K} |w(x, t) - w_0(x)| \, dx \to 0 \quad \text{as} \quad t \to 0
\end{equation}
for any compact subset $K$ of $\mathbb{R}^n$. We note that $w_0 \in L^{1}_{\text{loc}}(\mathbb{R}^n)$ if (1.12) holds with $p > (n+2)/n$.

**THEOREM 1.2.** Let $p > (n+2)/n$. Assume that (1.12) holds, where $\ell$ is the constant in (1.10). Then there exists a positive solution $w \in C^{2,1}(\mathbb{R}^n \times (0, \infty))$ of (1.2) and (1.11). Assume, furthermore, that $w_0 \in C(\mathbb{R}^n \setminus \{0\})$, then $w$ satisfies
\begin{equation}
w(x, t) \to w_0(x) \quad \text{as} \quad t \to 0 \quad \text{uniformly in} \quad |x| \geq r \quad \text{for every} \quad r > 0.
\end{equation}
(1.13)
Moreover, $w$ is self-similar if $\mu^{2/(p-1)}w_0(\mu x) = w_0(x)$ for every $\mu > 0$.

**COROLLARY 1.4.** Let $p > (n+2)/n$. Assume that $A : S^{n-1} \to \mathbb{R}$ is continuous and satisfies
\begin{equation}
0 \leq A(\sigma) \leq \ell, \quad A \neq 0, \quad \sigma \in S^{n-1}.
\end{equation}
(1.14)
Then there exists a positive self-similar solution $w \in C^{2,1}(\mathbb{R}^n \times (0, \infty))$ of (1.2) satisfying (1.11) and (1.13) with $w_0(x) = A(x/|x|)|x|^{-2/(p-1)}$.

Recall that self-similar solutions $w$ to (1.2) have the form (1.3) with $u$ satisfying (1.1). Therefore, $w(\sigma, t) = r^{2/(p-1)}u(r\sigma)$ for $\sigma \in S^{n-1}$, where $r = 1/\sqrt{\ell}$. Then we obtain the following corollary, which shows that condition (1.5) in Theorem 1.1 is crucial.
**Corollary 1.5.** Let $p > (n + 2)/n$. Assume that $A : S^{n-1} \to \mathbf{R}$ is continuous and satisfies (1.14). Then there exists a positive non-radial solution $u$ of (1.1) satisfying

$$r^{2/(p-1)}u(r\sigma) \to A(\sigma) \quad \text{as} \quad r \to \infty \quad \text{uniformly in} \quad \sigma \in S^{n-1}.$$

**Remark.** (i) If $1 < p \leq (n + 2)/n$, no time global, non-negative, and nontrivial solution exists in (1.2) (see, e.g., [7], [19]). Therefore, (1.1) admits a positive solution only if $p > (n + 2)/n$.

(ii) We find that the solution $w$ of (1.2) and (1.11) obtained in Theorem 1.2 is a minimal solution of the integral equation

$$w(x, t) = \int_{\mathbf{R}^n} \Gamma(x - y : t)w_0(y)dy + \int_0^t \int_{\mathbf{R}^n} \Gamma(x - y : t - s)[w(y, s)]^pdyds,$$

where $\Gamma(x : t) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$. See the proof of Theorem 1.2 below.

(iii) Galaktionov and Vazquez [8] studied the Cauchy problem (1.2) and (1.11) with singular initial values for the case $p > n/(n - 2)$.

**References**


[16] L. A. Peletier, D. Terman, and F. B. Weissler, On the equation $\Delta u + \frac{1}{2} x \cdot \nabla u + f(u) = 0$, Arch. Rational Mech. Anal. 94 (1986), 83-99.


