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On the Complex WKB Analysis for Products of the Airy Functions

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§1. Introduction.

1.1. We consider a third order linear ordinary differential equation containing a small parameter ε:

$$\varepsilon^3 y''' + 3\varepsilon^2 p_2(x)\, y'' + 3\varepsilon p_1(x)\, y' + p_0(x)\, y = 0, \quad 0 < \varepsilon \le \varepsilon_0, \quad |x| \le x_0, \tag{1.1}$$

where x is complex.

We suppose that the coefficients of (1.1) are linear functions of x such as

$$p_2(x) = ax + b,\ p_1(x) = cx + d,\ p_0(x) = ex + f,$$

where $a, b, \cdots, f$ are complex constants.

The characteristic equation of (1.1) is given by

$$k(\lambda, x) := \lambda^3 + 3p_2(x)\lambda^2 + 3p_1(x)\lambda + p_0(x) = 0, \tag{1.2}$$

whose roots are called the characteristic roots for (1.1) and they are denoted by $\lambda_j(x)$ $(j = 1, 2, 3)$. We denote by $\boldsymbol{D}$ a discriminant of the algebraic equation (1.2).

1.2. We give a brief summary on the complex WKB analysis about (1.1).

Definition 1. *Zeros of the discriminant $\boldsymbol{D}$ are called turning points of the equation* (1.1), *or a point $x = a$ satisfying $\lambda_j(a) = \lambda_l(a)$ $(j \ne l)$ is a turning point of the equation* (1.1).

The turning point x satisfying $\dfrac{\partial^2 k(\lambda(x), x)}{\partial \lambda^2} \ne 0$ and $\dfrac{\partial k(\lambda(x), x)}{\partial x} \ne 0$ are called of simple order.

Definition 2. *Curves on the x-plane determined by*

$$\Re\,\xi_{jl}(a, x) = 0, \quad \xi_{jl}(a, x) := \int_a^x \Big\{\lambda_j(x) - \lambda_l(x)\Big\}\, dx, \quad \lambda_j(a) = \lambda_l(a) \quad (j \ne l) \tag{1.3}$$

are called Stokes curves of the equation (1.1). *Curves determined by $\Im\,\xi_{jl}(a, x) = 0$ are called anti-Stokes curves of the equation* (1.1). *They emerge from the turning point $x = a$.*

Curves $\Re\,\xi_{jl}(a, x) =$const. and curves $\Im\,\xi_{jl}(a, x) =$const. are called level curves.

Both of Stokes and anti-Stokes curves are level curves of level zero. It is known that Stokes curves of the equation (1.1) emerging from one turning point do not intersect each other except for this turning point and the point at infinity, and a Stokes curve of the equation (1.1) does not make a loop (Kelly [12]).

We remark that someone calls curves defined by $\Re \xi_{jl}(a, x) = 0$ anti-Stokes curves, and calls curves defined by $\Im \xi_{jl}(a, x) = 0$ Stokes curves. Terminology is sometimes used conversely.

Definition 3. *The main term $\tilde{y}_j(x, \varepsilon)$ of a formal series solution of* (1.1) *is called a formal WKB solution of the equation* (1.1) :

$$(1.4)\quad \tilde{y}_j(x,\varepsilon) := \frac{1}{\sqrt{(\lambda_j - \lambda_{j+1})(\lambda_j - \lambda_{j+2})}} \left(\frac{1}{\varepsilon}\int_a^x \lambda_j(x)dx\right) \ (j = 1, 2, 3;\ \lambda_4 := \lambda_1, \lambda_5 := \lambda_2).$$

This is derived from the formulae given in Fedoryuk [7], [8] or Nakano et. al. [15].

Lemma 1. *There exists an x-region D_j such that the formal WKB solutions $\tilde{y}_j(x, \varepsilon)$ possess double asymptotic property:*

$$y_j(x, \varepsilon) \sim \tilde{y}_j(x, \varepsilon)$$

(1.5) *as $x \to \infty$ in D_j for ε,*

(1.6) *as $\varepsilon \to 0$ for $x \in D_j$,*

where $y_j(x, \varepsilon)$ is a true solution of (1.1).

This lemma can be proved by the similar method used for second order differential equations (Evgrafov-Fedoryuk [5] or Nakano et. al. [15]).

W-K-B are originated from **W**entzel [20], **K**ramers [13] and **B**rillouin [4].

Definition 4. *The maximal region D_j of the x-plane, in which a formal WKB solution $\tilde{y}_j(x, \varepsilon)$ is an asymptotic expansion of the true solution $y_j(x, \varepsilon)$, is called a λ_j-admissible region of the equation* (1.1).

An intersection of three λ_j-admissible regions $D_1 \cap D_2 \cap D_3$ is called a canonical region of the equation (1.1).

The canonical region is the maximal region in which three linearly independent solutions $y_j(x, \varepsilon)$'s of (1.1) possess formal WKB solutions $\tilde{y}_j(x, \varepsilon)$'s as asymptotic solutions. There are several canonical regions of (1.1) (see §6).

In §2 it is shown that the equation (1.1) is classified into nine classes and they are shown on the table. From §3 we study mainly about the equation type Ib on the table.

In §4 we study location of turning points and local Stokes curves for the equation type Ib, in §5 global Stokes curves are considered and they are shown in several figures, in §6 the canonical region, eistence region of three independent solutions with some asymptotic property, are gained, in §7 we show that the solution can be represented by the Laplace integral, in §8 we give a brief skecth of the Airy functions and in §9, the last section, we study relation between solutions of the equation type Ib and products of the Airy functions.

This article is a revised edition of Nakano [14].

§2. Classification of 3rd order equations.

2.1. We can classify the differential equation (1.1) in a six-dimensional space with respect to the order and numbers of the turning points of (1.1) by using the discriminant D of (1.2).

The characteristic equation (1.2) can be reduced to

$$(2.1)_1 \qquad \eta^3 + 3P\eta + Q = 0,$$

where

$$(2.1)_2. \qquad \eta := \lambda + p_2(x), \quad P := p_1(x) - p_2(x)^2, \quad Q := 2p_2(x)^3 - 3p_2(x)p_1(x) + p_0(x).$$

The solutions ξ of $(2.1)_1$ are given by the Cardano's formula as follows:

$$(2.2)_1 \qquad \eta_1 := \sqrt[3]{\alpha} + \sqrt[3]{\beta}, \quad \eta_2 := \omega\sqrt[3]{\alpha} + \omega^2\sqrt[3]{\beta}, \quad \eta_3 := \omega^2\sqrt[3]{\alpha} + \omega\sqrt[3]{\beta},$$

where

$$(2.2)_2 \qquad \alpha := \frac{-Q+\sqrt{D}}{2}, \quad \beta := \frac{-Q-\sqrt{D}}{2}, \quad D := 4P^3 + Q^2.$$

'D' is a discriminant of the characteristic equation of (1.2) (and (2.1)) since roots of (1.2) (and (2.1)) coincide at zeros of D, and these zeros are the turning points of (1.1) (see Def. 1).

2.2. The discriminant D is expressed by a polynomial of x as follows:

$$\begin{aligned}
D &:= 4\Big\{p_1(x) - p_2(x)^2\Big\}^3 + \Big\{(2p_2(x)^3 - 3p_2(x)p_1(x) + p_0(x)\Big\}^2 \\
&= -3p_2(x)^2\, p_1(x)^2 + 4p_1(x)^3 + 4p_2(x)^3\, p_0(x) - 6p_2(x)\, p_1(x)\, p_0(x) + p_0(x)^2 \\
&= \ (-3b^2d^2 + 4d^3 + 4b^3f - 6bdf + f^2) \\
&\qquad + (-6b^2cd - 6abd^2 + 12cd^2 + 4b^3e - 6bde + 12ab^2f - 6bcf - 6adf + 2ef)\, x \\
&\qquad + (-3b^2c^2 - 12abcd + 12c^2d - 3a^2d^2 + 12ab^2e - 6bce - 6ade + e^2 + 12a^2bf - 6acf)\, x^2 \\
&\qquad + (-6abc^2 + 4c^3 - 6a^2cd + 12a^2be - 6ace + 4a^3f)\, x^3 \\
&\qquad + a^2(-3c^2 + 4ae)\, x^4.
\end{aligned}$$

Since D is of degree 4, there are at most four roots. By defining constants $a, b, \cdots, f$ appropriately, we get the following typical examples of the characteristic equations.

type	1ple	2ble	3ple	characteristic equation	discriminant D	turning points
Ia	oo			$\lambda^3 - 3\lambda - 2x = 0$	$4(x^2-1)$	$-1, 1$
Ib	o o o			$\lambda^3 - 4x\lambda - 2 = 0$	$4\{1-(4x/3)^3\}$	$3/4, 3\omega/4, 3\omega^2/4$
Ib′	o o o			$\lambda^3 - 3x\lambda^2 + 4 = 0$	$16(1-x^3)$	$1, \omega, \omega^2$
Ic	o o oo			$\lambda^3+3ax\lambda^2-3a\lambda+x=0$	$4a^3(x^4-1), a{:=}\sqrt{1+2/\sqrt{3}}$	$-1, 1, -i, i$
IIa		o		$\lambda^3 - x = 0$	x^2	0,0
IIb	o	o		$\lambda^3 - \lambda^2 - x\lambda + x = 0$	$-4x(x-1)^2/27$	0,1,1
IIc	oo	o		$\lambda^3 + 3x\lambda^2 - 4x = 0$	$16x^2(1-x^2)$	$-1, 1, 0, 0$
IId		oo		$\lambda^3 - x\lambda^2 - \lambda + x = 0$	$-4(x^2-1)^2/27$	$-1, -1, 1, 1$
IIIa			o	$\lambda^3 - x\lambda = 0$	$-4x^3/27$	0,0,0
IIIb	o		o	$\lambda^3 + x\lambda^2 + x\lambda + 1 = 0$	$-(x-3)^3(x+1)/27$	$-1, 3, 3, 3$

REMARK: The mark 'o' represents a number of n-ple zeros. There exists no case where D has only one simple zero, and there exists also no case where D has a 4-ple zero.

§3. The equation Ib.

3.1. From now on we are mainly considering the third order linear ordinary differential equation of type Ib on the table in §2.2:

$$(3.1) \qquad \varepsilon^3 y''' - 4\varepsilon x y' - 2y = 0.$$

The equation (3.1) has three simple turning points at $x = 3/4,\ 3\omega/4,\ 3\omega^2/4$ ($\omega^3 = 1$, $\omega \neq 1$) as shown on the table (see Def. 1), and the point at infinity is an irregular singular point.

When $\varepsilon = 1$ the solutions of (3.1) are $\mathrm{Ai}(x)^2$, $\mathrm{Ai}(x)\cdot\mathrm{Bi}(x)$ and $\mathrm{Bi}(x)^2$, where $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the Airy functions. The Airy functions are linearly independent solutions of the Airy equation

$$Y'' - xY = 0. \tag{3.2}$$

The equation Ib$'$ has same property as the equation Ib, but we prefer the equation Ib because it relates directly to the Airy equation.

The characteristic roots $\lambda := \lambda_j(x)$ $(j = 1, 2, 3)$ for (3.1) are given by

$$\begin{cases} \lambda_1(x) := \alpha^{1/3} + \dfrac{4x}{3} \cdot \alpha^{-1/3}, \\ \lambda_2(x) := \omega^2\alpha^{1/3} + \dfrac{4x}{3} \cdot \omega\alpha^{-1/3}, \\ \lambda_3(x) := \omega\alpha^{1/3} + \dfrac{4x}{3} \cdot \omega^2\alpha^{-1/3}, \end{cases} \qquad \left(\alpha := 1 + \sqrt{D}, \quad D := 1 - \left(\tfrac{4x}{3}\right)^3\right) \tag{3.3}$$

The formal WKB solutions of (3.1) are got from (1.4)

$$\tilde{y}_1(x,\varepsilon) = x^{(1-3\varepsilon)/4\varepsilon} e^{\frac{1}{\varepsilon}\frac{4}{3}x^{3/2}}, \quad \tilde{y}_2(x,\varepsilon) = x^{-1/2\varepsilon}, \quad \tilde{y}_3(x,\varepsilon) = x^{(1-3\varepsilon)/4\varepsilon} e^{-\frac{1}{\varepsilon}\frac{4}{3}x^{3/2}}. \tag{3.4}$$

3.2. In the case of second order linear differential equations there are only two characteristic values. Then, there is only one difference of them if we take no account of signature. Turning points and Stokes curves are determined by this difference (see Def. 1, 2).

However, in the case of higher order differential equations there are many differences of characteristic roots, and Stokes curves are determined by these differences. Therefore, Stokes curves may cross each other. Indeed, the crossing of Stokes curves happens for the equation (3.1) (see Fig. 1). In this sense the equation (3.1) is a typical example with general property which the general higher order differential equations possess, nevertheless the equation (3.1) looks very simple.

Third order equations are studied by, for instance, Aoki et. al. [2], Berk et. al. [3] and Nakano et. al. [15]. Berk et. al. studied the equation of type Ia introducing a new Stokes curve and showing that a Stokes phenomenon happens on the new Stokes curve, and computed a Stokes multiplier. However we need no new Stokes curves and we can get canonical regions without new Stokes curves. We use 'old' Stokes curves only (see Theorem 4).

§4. Turning points and local Stokes curves.

4.1. We can see that every characteristic root $\lambda_j(x)$ is obtained by other characteristic roots by changing arguments and we get the following relations which we call *the rotation rules.*

Theorem 1. *Between characteristic roots the following equations are valid:*

$$\lambda_1(x\omega) = \omega^2\lambda_3(x), \quad \lambda_2(x\omega) = \omega^2\lambda_1(x), \quad \lambda_3(x\omega) = \omega^2\lambda_2(x); \tag{4.1}$$

$$\lambda_{12}(x) = \omega\lambda_{23}(x\omega), \quad \lambda_{23}(x) = \omega\lambda_{31}(x\omega), \quad \lambda_{31}(x) = \omega\lambda_{12}(x\omega), \tag{4.2}$$

where $\omega := e^{2\pi i/3}$, $\lambda_{jk}(x) := \lambda_j(x) - \lambda_k(x)$.

PROOF. These rotation rules are easily derived from the definition of $\lambda_j(x)$ and by inserting $x\omega$ into (3.3). Q.E.D.

From the rotation rules we get the relations:

$$\lambda_j(xe^{2\pi i}) = \lambda_j(x), \quad \lambda_{jk}(xe^{2\pi i}) = \lambda_{jk}(x) \quad (j \neq k; j, k = 1, 2, 3). \tag{4.3}$$

Therefore $\lambda_j(x)$ *and* $\lambda_{jk}(x)$ *are single-valued.*

As stated already, turning points of (3.1) are $x = 3/4, 3\omega/4$ and $3\omega^2/4$. But, in order to construct precisely canonical regions, we must know which turning point is derived from which two characteristic roots.

Theorem 2. *The turning points are determined as follows.*
The turning point $x = \frac{3}{4}e^{4\pi i/3} \left(= \frac{3}{4}\omega^2\right)$ *is induced by the equation* $\lambda_1(x) = \lambda_2(x)$.
The turning point $x = \frac{3}{4}$ *is induced by the equation* $\lambda_2(x) = \lambda_3(x)$.
The turning point $x = \frac{3}{4}e^{2\pi i/3} \left(= \frac{3}{4}\omega\right)$ *is induced by the equation* $\lambda_3(x) = \lambda_1(x)$

PROOF. We show how to get the turning point $x = \frac{3}{4}\omega^2$ induced by two characteristic roots $\lambda_1(x)$ and $\lambda_2(x)$.

From (3.3) we get $\lambda_{12}(x) := \lambda_1(x) - \lambda_2(x) = (1-\omega)\cdot\left(e^{\pi i/3}\alpha^{1/3} + \frac{4x}{3}\alpha^{-1/3}\right)$.

Since zeros of the discriminant $D := 1 - \left(\frac{4x}{3}\right)^3$ are $x = \frac{3}{4}\cdot 1,\ \frac{3}{4}\cdot\omega,\ \frac{3}{4}\cdot\omega^2$, all turning points can be represented in the form of $x = \frac{3}{4}\cdot e^{2k\pi i/3}$ (k =integer).

Thus, by inserting $x = \frac{3}{4}e^{2k\pi i/3}$ into $\lambda_{12}(x)$, we get

$$\lambda_{12}\left(\frac{3}{4}e^{2k\pi i/3}\right) = (1-\omega)\left(e^{\pi i/3} + e^{2k\pi i/3}\right) = 0,$$

from which we obtain $e^{(2k-1)\pi i/3} = 1$, then $k = \cdots, -1, 2, 5, \cdots$. Thus we get

$$\begin{aligned} x &= \cdots, \frac{3}{4}e^{-2\pi i/3}, \frac{3}{4}e^{4\pi i/3}, \frac{3}{4}e^{10\pi i/3}, \cdots \\ &= \frac{3}{4}\omega^2. \end{aligned}$$

We can show others similarly. Q.E.D.

We notice that all the three characteristic roots do not coincide at one point. Only any two of them can coincide at only one point.

4.2. At the turning point $x = \frac{3}{4}$ the equality $\lambda_2(x) = \lambda_3(x)$ or $\lambda_{23}(x) = 0$ is valid, and near $x = \frac{3}{4}$ we get $\alpha = 1 + 2\,i\,t^{1/2} + \cdots$ $(x := t + 3/4)$. Then

$$\begin{aligned} \lambda_{23}(x) &:= \lambda_2(x) - \lambda_3(x) \\ &= \frac{4}{\sqrt{3}}\,t^{1/2} + (\text{higher order terms}), \end{aligned}$$

and

$$\begin{aligned} \xi_{23}\left(\tfrac{3}{4}, x\right) &:= \int_{3/4}^{x} \lambda_{23}(x)dx \\ &= \frac{4}{\sqrt{3}}\cdot\frac{2}{3}\,t^{3/2} + (\text{higher order terms}). \end{aligned}$$

Therefore we can get the relation

$$\Re\,\xi_{23} = 0 \Longleftrightarrow \cos\frac{3\theta}{2} = 0 \quad (\theta := \arg t).$$

Thus, we get angles $\theta = \pm\frac{\pi}{3}, \pi$ near the turning point $x = \frac{3}{4}$, and we can see that there exist three Stokes curves emerging from the turning point $x = \frac{3}{4}$ defined by $\lambda_{23}(x) = 0$.

4.3. The point at infinity is an irregular singular point of the equation (3.1), and so we can say that Stokes curves have to emerge from (or enter to) the point at infinity due to the local theory about the point at infinity (Wasow [18]).

When $|x| \gg 1$ we get

$$\alpha := 1 + \left(1 - \left(\frac{4x}{3}\right)^3\right)^{1/2} \sim \left(\frac{4}{3}\right)^{3/2} e^{\pi i/2} x^{3/2} \quad (x \to \infty).$$

Then

$$\lambda_{23}(x) \sim -i\sqrt{3}\,(e^{\pi i/6} - e^{-\pi i/6})x^{1/2} \sim \sqrt{3}\,x^{1/2} \quad (x \to \infty),$$

and we have

$$\begin{aligned} \xi_{23} &:= \int^x \lambda_{23}(x)dx \\ &\sim \frac{2}{\sqrt{3}} x^{3/2} \quad (x \to \infty). \end{aligned}$$

Therefore, from the equality $\Re\xi_{23} = 0$ we can get arguments of x near the point at infinity: $\arg x = \pm\pi/3,\ \pi$.

§5. Global Stokes curves.

5.1. Since we got local behavior of Stokes curves near the particular points, we are getting global Stokes curves on the whole plane.

Firstly, we determine the global Stokes curves derived from two characteristic values $\lambda_2(x)$ and $\lambda_3(x)$. From (3.3) we have

$$\begin{aligned} \lambda_{23}(x) &:= \lambda_2(x) - \lambda_3(x) \\ &= (\omega^2 - \omega)\left(\alpha^{1/3} - \frac{4x}{3}\alpha^{-1/3}\right), \end{aligned}$$

where $\alpha := 1 + \{1 - (4x/3)^3\}^{1/2}$.

Now, we see $\omega^3 - \omega = -\sqrt{3}i$ and $1 - 4x/3 \geq 0$ $(x \leq 3/4)$, then $\alpha \geq 0$, and so we get $\lambda_{23}(x) = -\sqrt{3}i \cdot C$ $(C \geq 0)$. Thus, we can see a part of the real axis, i.e., the semi-infinite interval $x \leq 3/4$ is a Stokes curve l_0 (Fig.1), because

$$\Re \int_{3/4}^x \lambda_{23}(x)\,dx = 0 \quad \text{for} \quad x \leq \frac{3}{4}.$$

By the same way, we can see a part of the real axis $x \geq 3/4$ is an anti-Stokes curve L_0. Other two Stokes curves (l_1 and l_2) emerging from the turning point $x = 3/4$ are shown in Fig.1.

The curve l_1 tends to the point at infinity of a direction with $\arg x = -\pi/3$ $\ (|x| \gg 1)$. Indeed, l_1 can not cross l_0, because two Stokes curves can not cross except for turning points and the point at infinity.

Also, l_1 does not cross L_0. Because the Stokes curve l_1 and the anti-Stokes curve L_0 emerge from the same turning point $x = 3/4$ and so they cannot cross each other at other points by the general theory (Kelly [12]).

Similarly we get a Stokes curve l_2 as shown in Fig.1.

5.2. Stokes and anti-Stokes curves defined by $\lambda_1(x)$ and $\lambda_3(x)$ emerging from the turning points $x = 3\omega/4$ are shown in Fig.1, too.

The Stokes curve l_0' is a straight line passing through the origin. Indeed, we can get l_0' as follows: For $x \in l_0'$, i.e., for x satisfying $-\infty \cdot \omega < x \leq 3\omega/4$ we have by the rotation rule (Theorem 1)

$$\begin{aligned} \xi_{31} &:= \int_{3\omega/4}^{x\omega} \lambda_{31}(x)dx \\ &= \int_{3/4}^{x} \lambda_{23}(x)\,dx =: \ \xi_{23}. \end{aligned}$$

Thus we get

$$\Re\,\xi_{31}=0 \quad \left(-\infty\cdot\omega<x\le\frac{3\omega}{4}\right) \quad \Longleftrightarrow \quad \Re\,\xi_{23}=0 \quad \left(-\infty<x\le\frac{3}{4}\right).$$

Therefore, the line l_0' defined by $\Re\xi_{31}=0$ is a Stokes line rotated l_0 by an angle $2\pi/3$ around the origin.

In Fig.1, the line l_2' emerging from $x=\frac{3\omega}{4}$ does not cross the negative real axis, which is a part of Stokes curve l_0 emerging from the turning point $x=3/4$. Indeed, when we choose three integral paths: a diameter from $x=\frac{3\omega}{4}$ to the origin, an interval on the negative real axis from the origin to $x=\frac{3}{4}re^{\pi i}$ and a curve from $x=\frac{3\omega}{4}$ to $x=\frac{3}{4}re^{\pi i}$ $(r\ge 0)$, we get the equation

$$\xi_{31}\left(\frac{3\omega}{4},0\right)+\xi_{31}\left(0,\frac{3}{4}re^{\pi i}\right)=\xi_{31}\left(\frac{3\omega}{4},\frac{3}{4}re^{\pi i}\right)$$

by the Cauchy's integral theorem, because there are no singularities of the integrand in the interior region bounded by three integral paths. Since the point on the diameter is $x=\frac{3}{4}r\omega$ $(0\le r\le)$, we get $\alpha=1+\sqrt{1-r^3}$ on the diameter, and so α is real. Then we get $\xi_{31}(\frac{3\omega}{4},0)=i\frac{3\sqrt{3}}{4}\int_0^1(\alpha^{1/3}-r\alpha^{-1/3})dr$, which is purely imaginary.

Since the point on the negative real axis is $x=\frac{3}{4}re^{\pi i}$ $(r\ge 0)$, α takes values $\alpha=1+\sqrt{1+r^3}$ (≥ 2). Then we get

$$\xi_{31}\left(0,\frac{3}{4}re^{\pi i}\right)=-\frac{3\sqrt{3}}{8}\int_{-3r/4}^{0}\left\{\sqrt{3}(\alpha^{1/3}-r\alpha^{-1/3})-i(\alpha^{1/3}-r\alpha^{-1/3})\right\}dr,$$

whose real part is negative. Thus we see $\Re\,\xi(\frac{3\omega}{4},\frac{3}{4}re^{\pi i})<0$.

If the Stokes curve l_2' crosses the negative real axis, the following property must be true:

$$\Re\,\xi\left(\frac{3\omega}{4},\frac{3}{4}re^{\pi i}\right)=0 \quad \text{for some } r\ (\ge 0).$$

Therefore the Stokes curve l_2' cannot cross the Stokes curve l_0.

Similarly, we can get the Stokes curves derived from the characteristic values $\lambda_1(x)$ and $\lambda_2(x)$. Summing up above results, we get the Stokes curve configuration as shown in Fig.1.

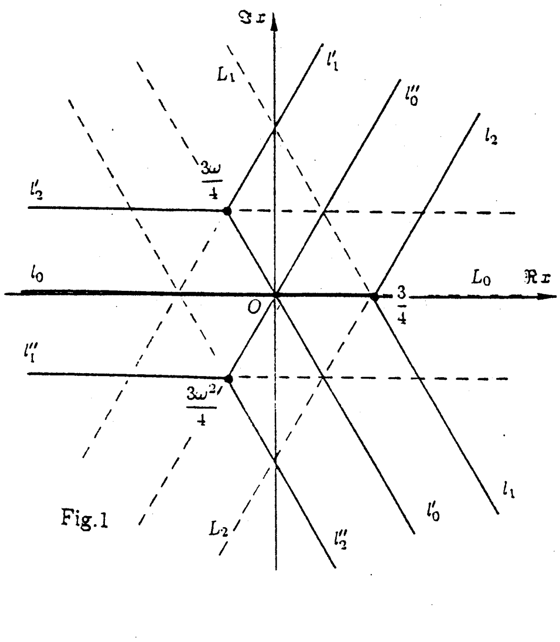

Fig.1

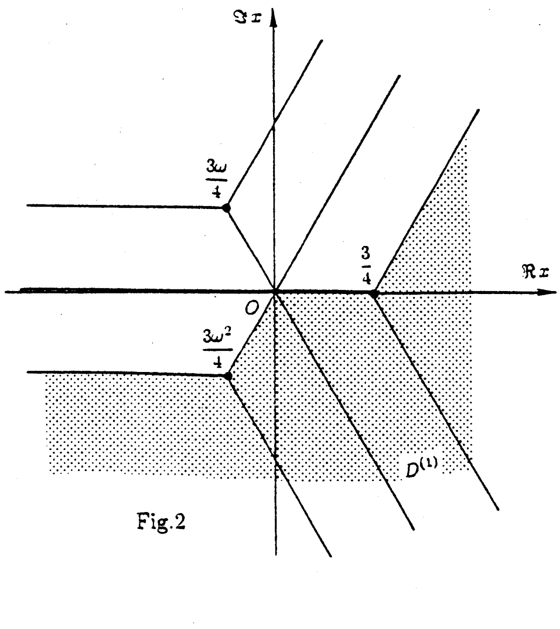

Fig.2

Theorem 3. *The Stokes curve configuration is shown in Fig.1. The real lines show the Stokes curves and the broken lines show the anti-Stokes curves.*

All Stokes curves do not cross each other except for the origin, where three Stokes curves l_0, l_0' and l_0'' only cross.

The origin is neither a turning point nor a irregular singular point of (1.1).

Here we notice that three Stokes curves cross at the origin which is an ordinary point. In the case of second order differential equations any two Stokes curves do not cross at a point except for the turning points and irregular singularities.

§6. Canonical regions.

6.1. A λ_j-admissible region D_j $(j = 1, 2, 3)$ is the maximal region in which a formal WKB solution $\tilde{y}_j(x, \varepsilon)$ has the double asymptotic property (1.5) and (1.6). To determine the λ_j-admissible regions, we need the following

Lemma 2. *In the λ_j-admissible region D_j the inequality*

$$\text{(6.1)} \qquad \Re\, \xi_{lj}(a, x) \leq 0, \quad \xi_{lj}(a, x) := \int_a^x \Big\{\lambda_l(x) - \lambda_j(x)\Big\} dx, \quad (l = j+1, j+2)$$

must be valid along any integral path in the region D_j from the turning point a to x.

The proof is given in Nakano et. al. [15] and so we omit it here.

To find points x satisfying (6.1), it suffices to draw level curves on the x-plane defined by $\Re\, \xi_{lj}(a, x) = \text{const.}$ and $\Im\, \xi_{lj}(a, x) = \text{const.}$ Since the λ_j-admissible region is maximal in the x-plane, the image of D_j in the ξ-plane under the conformal mapping $\xi = \xi(x)\, (:= \xi_{lj}(a, x))$ must be also maximal in the ξ-plane.

Since the λ_j-admissible region is maximal, Stokes phenomenon must occur if we continue the solution $y_j(x, \varepsilon)$ analytically beyond any boundary of the λ_j-admissible region.

In the intersection $D^{(\cdot)}$ of three λ_j-admissible regions, three formal WKB solutions $\tilde{y}_j(x, \varepsilon)$ are asymptotic solutions of (1.1). This intersection $D^{(\cdot)}$ is the maximal region in which three independent solutions $y_j(x, \varepsilon)$'s exist, and this is called a canonical region of (1.1) (Def. 4).

If we try to continue analytically the solution $y_j(x, \varepsilon)$ (whose asymptotic property is represented by a linear combination of some formal WKB solutions $\tilde{y}_j(x, \varepsilon)$'s) beyond the boundary of the canonical region, the solution $y_j(x, \varepsilon)$ must have another asymptotic representation, that is to say, the Stokes phenomenon must occur.

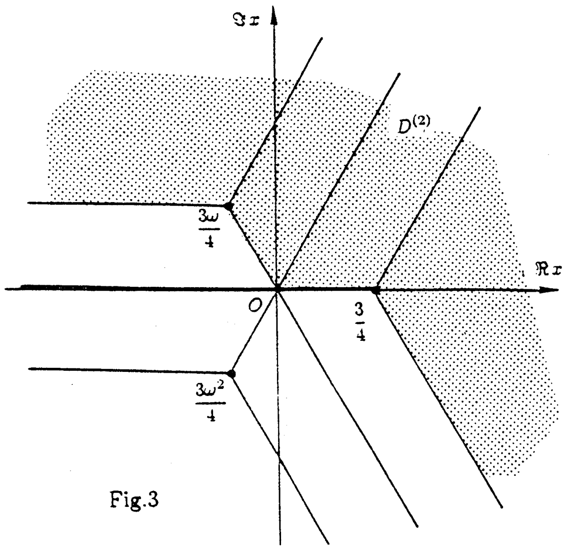

Fig.3

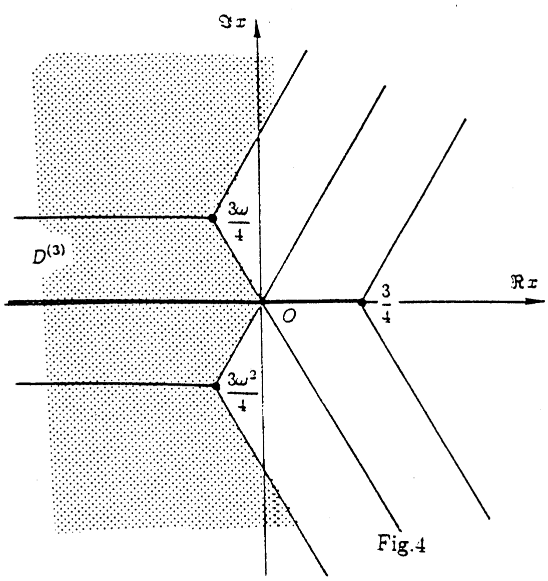

Fig.4

Thus we get

Theorem 4. *There exist three canonical regions $D^{(1)}, D^{(2)}$ and $D^{(3)}$ of the equation* (1.1) *as shown in Fig.2~4. They are situated symmetrically around the origin, especially $D^{(2)} = \bar{D}^{(1)} := \{\bar{x} : x \in D^{(1)}\}$.*

6.2. Berk et. al. [3] study the equation Ia with two simple turning points and they assert that the Stokes phenomenon occurs on the new Stokes curve, which emerges from the intersection point (which is called a new turning point by Aoki et. al. [2]) of the 'old' Stokes curves, in order to continue solutions by using Furry's rule which was obtained for second order equations with simple turning points.

But the equation Ib with three simple turning points needs no new Stokes curves to construct canonical regions (Theorem 4).

By the way, Berk et. al. state that there exist six directions of Stokes curves to ∞ emerging from two simple turning points, although the local theory at an irregular singular point asserts that there exist eight Stokes curves emerging from ∞ (Wasow [17]).

The equation Ib has nine Stokes curves emerging from three simple turning points and they tend to ∞ in three different directions, and the local theory at an irregular singular point asserts that three Stokes curves emerge from ∞ (cf. §4.3).

Now, we propese

Conjecture. *Let N_t be a number of directions of Stokes curves tending to ∞ emerging from all the turning points and let N_∞ be a number of directions of Stokes curves emerging from ∞. If $N_t = N_\infty$, then there exist no new Stokes curves.*

§7. Laplace transforms.

7.1. As known well, a solution of a linear ordinary differential equation with linear coefficients, which is called a Laplace equation, can be represented by the Laplace transform or the Laplace integral. The Laplace transform of (3.1) is

$$(7.1)\qquad \begin{aligned} y(x,\varepsilon) &= \int_\gamma \frac{1}{-4s} \exp\left\{\frac{1}{\varepsilon}\left(xs - \frac{s^3}{12} + \log s^{1/2}\right)\right\} ds \\ &= -\frac{1}{4}\int_\gamma \frac{1}{s^{1-1/2\varepsilon}} \exp\left\{\frac{1}{\varepsilon}\left(xs - \frac{s^3}{12}\right)\right\} ds, \end{aligned}$$

if we suppose that the integral converges.

When we put $S(s,x) := xs - s^3/12 + \log s^{1/2}$, then $\partial S/\partial s = x - s^2/4 + 1/2s = -(s^3 - 4xs - 2)/(4s)$.

Zeros of $\partial S/\partial s$ are called saddle points of the integral (7.1). The numerator $s^3 - 4xs - 2$ of $\partial S/\partial s$ coincides with the characteristic polynomial of the equation (3.1), and its zeros are the characteristic values of (3.1). Thus we get

LEMMA 3. *The characteristic values of the equation* (3.1) *are saddle points of* (7.1).

When s is sufficiently large, the integral (7.1) must converge. The convergence regions in the s-plane are derived from $Re(-s^3/12) < 0$. The origin of the s-plane is a singular point of the integrand, but the integral converges at the origin, because the exponent $1 - 1/2\varepsilon$ ($\varepsilon > 0$) is smaller than 1.

7.2. If we choose the integral path γ such that it passes through the saddle point and comes from and goes to ∞ or from 0 to ∞ in the convergence regions, we can get asymptotic representations of the Laplace integral by the saddle point method or the method of the steepest descent as follows:

$$(7.2)\qquad y(x,\varepsilon) \sim C(\varepsilon)\frac{\lambda^{1/2\varepsilon}}{\sqrt{\lambda^3+1}} e^{\frac{1}{\varepsilon}\frac{2}{3}x\lambda} \quad (\varepsilon \to 0,\ x \to \infty;\quad \textit{for any } \lambda);$$

$$(7.3) \quad y_j(x,\varepsilon) \sim \frac{x^{1/4\varepsilon}}{\sqrt{\lambda_j^3+1}} e^{(-1)^{(j+1)/2}\frac{1}{\varepsilon}\frac{4}{3}x^{3/2}} \ (j=1,3),\ y_2(x,\varepsilon) \sim \frac{x^{-1/2\varepsilon}}{\sqrt{\lambda_2^3+1}} \ (\varepsilon \to 0,\ x \to \infty).$$

Right hand sides of (7.3) are same as WKB solutions (1.4) if we calculate a root of (1.4) much more. After a short calculation we get more precise form from (7.3):

$$(7.4) \qquad y_1(x,\varepsilon) \sim x^{(1-3\varepsilon)/4\varepsilon} e^{\frac{1}{\varepsilon}\frac{4}{3}x^{3/2}},\ y_2(x,\varepsilon) \sim x^{-1/2\varepsilon},\ y_3(x,\varepsilon) \sim x^{(1-3\varepsilon)/4\varepsilon} e^{-\frac{1}{\varepsilon}\frac{4}{3}x^{3/2}}.$$

Thus we get

LEMMA 4. *The Laplace transform has the formal WKB solutions as asymptotic expansions.*

§8. The Airy functions.

8.1. From now on, we assume $\varepsilon = 1$. Then the equation (3.1) becomes

$$(8.1) \qquad y''' - 4xy' - 2y = 0.$$

The Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are linearly independent solutions of the Airy equation

$$(3.2) \qquad Y'' - xY = 0.$$

If we put $y := Y^2$, then y satisfies (8.1), i.e., Y^2 is a solution of (8.1). More precisely speaking, Ai^2, $\mathrm{Ai}\cdot\mathrm{Bi}$ and Bi^2 are linearly independent solutions of (8.1). The wronskian of Ai^2, $\mathrm{Ai}\cdot\mathrm{Bi}$ and Bi^2 is $2\pi^{-3}$.

Asymptotic properties of the Airy functions are: for $x \to \infty$

$$(8.2) \quad \mathrm{Bi}(x) \sim \frac{1}{\sqrt{\pi}} x^{-1/4} e^{\frac{2}{3}x^{3/2}} \quad \Big(|\arg x| < \frac{\pi}{3}\Big), \quad \mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} \quad \Big(|\arg x| < \pi\Big),$$

then by making simply products we get

$$(8.3) \qquad \begin{cases} \mathrm{Bi}(x)^2 \sim \dfrac{1}{\pi} x^{-1/2} e^{\frac{4}{3}x^{3/2}} & \Big(x \to \infty,\ |\arg x| < \dfrac{\pi}{3}\Big), \\ \mathrm{Ai}(x)\cdot\mathrm{Bi}(x) \sim \dfrac{1}{2\pi} x^{-1/2} & \Big(x \to \infty,\ |\arg x| < \dfrac{\pi}{3}\Big), \\ \mathrm{Ai}(x)^2 \sim \dfrac{1}{4\pi} x^{-1/2} e^{-\frac{4}{3}x^{3/2}} & \Big(x \to \infty,\ |\arg x| < \pi\Big). \end{cases}$$

Right hand sides of (8.3) are formal WKB solutions of (8.1) corresponding to the characteristic values $\lambda_1(x)$, $\lambda_2(x)$ and $\lambda_3(x)$ in order. Thus, (8.3) coincides with (7.4) when $\varepsilon = 1$ if we take no account of constants.

8.2. Between two Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ there is a linear relation (Abranowitz-Stegun [1])

$$(8.4) \qquad 2\mathrm{Ai}(xe^{\pm 2\pi i/3}) = e^{\pm \pi i/3}\{\mathrm{Ai}(x) \mp i\,\mathrm{Bi}(x)\}.$$

By squaring this we get the relation

$$(8.5) \qquad 4\mathrm{Ai}(xe^{\pm 2\pi i/3})^2 = e^{\pm 2\pi i/3}\{\mathrm{Ai}(x)^2 \mp 2i\mathrm{Ai}(x)\mathrm{Bi}(x) - \mathrm{Bi}(x)^2\}.$$

This equation contains four functions which are solutions of (8.1). Therefore, (8.5) represents a linear relation between four solutions of (8.1) and it is a connection formula. The asymptotic property (8.2) and the relation (8.4) are gained from the Laplace integral for the Airy equation:

$$(8.6) \qquad Y = \frac{1}{2\pi i}\int_\gamma e^{tx - t^3/3}\,dt.$$

The Laplace integral for (8.1) is got from (7.1) by putting $\varepsilon = 1$ and it is

(8.7) $$y = \int_\gamma \frac{1}{\sqrt{s}} e^{xs - s^3/12} ds.$$

We must notice that (8.5) is not got from (8.7) but simply got by making a product of (8.4).

§9. Products of the Airy functions.

9.1. By formal calculation, we see that a product of (8.6) becomes (8.7):

(9.1) $$\left(\int e^{tx - t^3/3} dt\right)^2 = \text{const.} \int \frac{1}{\sqrt{s}} e^{xs - s^3/12} ds.$$

The Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are composed of three parts, $I_1(x)$, $I_2(x)$ and $I_3(x)$, as follows (Jeffreys-Jeffreys [11]):

(9.2) $$\mathrm{Ai}(x) = I_2(x) - I_3(x), \quad \mathrm{Bi}(x) = i\{2I_1(x) - I_2(x) - I_3(x)\},$$

where $I_j(x)$'s are defined by the Laplace integral of the Airy equation:

(9.3) $$\begin{cases} I_1(x) := \dfrac{1}{2\pi i} \displaystyle\int_0^{+\infty} e^{tx - t^3/3} dt, \\ I_2(x) := \dfrac{1}{2\pi i} \displaystyle\int_0^{\infty\omega} e^{tx - t^3/3} dt, \\ I_3(x) := \dfrac{1}{2\pi i} \displaystyle\int_0^{\infty\omega^2} e^{tx - t^3/3} dt. \end{cases}$$

Here we must notice that $I_2(x) - I_3(x)$ and $2I_1(x) - I_2(x) - I_3(x)$ are solutions of the Airy equation (3.2), but each of $I_j(x)$'s is not a solution of the Airy equation (8.6).
Then, by squaring (9.2) or making a product of them, we get

(9.4) $$\begin{cases} \mathrm{Ai}^2 = I_2^2 - 2I_2I_3 + I_3^2, \\ \mathrm{Ai}\cdot\mathrm{Bi} = i(2I_1I_2 - I_2^2 - 2I_1I_3 + I_3^2), \\ \mathrm{Bi}^2 = -4I_1^2 - I_2^2 - I_3^2 + 4I_1I_2 - 2I_2I_3. \end{cases}$$

From (9.1) and (9.3), we want to expect the following relations

(9.5) $$I_1^2 = \int_0^{+\infty} \frac{1}{\sqrt{s}} e^{xs - s^3/12} ds, \quad I_2^2 = \int_0^{\infty\omega} \frac{1}{\sqrt{s}} e^{xs - s^3/12} ds, \quad I_3^2 = \int_0^{\infty\omega^2} \frac{1}{\sqrt{s}} e^{xs - s^3/12} ds.$$

However, the relations (9.5) are not valid.

9.2. The right hand sides of (9.4) are too complicated to define solutions of (8.1). $\mathrm{Ai}(x)$, $\mathrm{Ai}(\omega x)$, $\mathrm{Ai}(\omega^2 x)$, $\mathrm{Bi}(x)$, $\mathrm{Bi}(\omega x)$ and $\mathrm{Bi}(\omega^2 x)$ are solutions of the Airy equation, and $\mathrm{Ai}(x)^2$, $\mathrm{Ai}(\omega x)^2$, $\mathrm{Ai}(\omega^2 x)^2$ and other products of them are solutions of (8.1), but we adopt more simply the right hand sides of (9.5) as the standard solutions of (8.1) and denote them by

(9.6) $$\begin{cases} \mathrm{Ap}(x) := \displaystyle\int_0^{+\infty} \frac{1}{\sqrt{s}} e^{xs - s^3/12} ds, \\ \mathrm{Bp}(x) := \displaystyle\int_0^{\infty\omega} \frac{1}{\sqrt{s}} e^{xs - s^3/12} ds, \\ \mathrm{Cp}(x) := \displaystyle\int_0^{\infty\omega^2} \frac{1}{\sqrt{s}} e^{xs - s^3/12} ds. \end{cases}$$

From (9.6), we see that

$$\mathrm{Ap}(xe^{2\pi i}) = \mathrm{Ap}(x), \quad \mathrm{Bp}(xe^{2\pi i}) = \mathrm{Bp}(x), \quad \mathrm{Cp}(xe^{2\pi i}) = \mathrm{Cp}(x) \tag{9.7}$$

are valid. Therefore $\mathrm{Ap}(x)$, $\mathrm{Bp}(x)$ and $\mathrm{Cp}(x)$ are single-valued and entire functions.

Three functions Ap, Bp and Cp are defined by independent integral paths, then they are linearly independent solutions of (8.1) and all other solutions of (8.1) can be represented by a linear combination of Ap, Bp and Cp.

Summing up we get

Thorem 6. *Three functions* $\mathrm{Ap}(x)$, $\mathrm{Bp}(x)$ *and* $\mathrm{Cp}(x)$ *defined by* (9.6) *are not created from parts* $I_j(x)$*'s of the Airy functions (see* (9.2)*). They are linearly independent solutions of* (8.1) *and single-valued entire functions. The right hand sides of* (7.4) *becomes the formal WKB solutions of* (8.1) *by putting* $\varepsilon = 1$.

In Zwillinger [21] the equation (8.1) is cited but it has no name, and so we propose here to name the equation (8.1) the **Pairy equation** and three functions Ap, Bp and Cp **Pairy functions**. The name 'Pairy' is originated from ***Pairy***=(**P**roducts+***Airy***)/2.

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