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Evolution Semigroups and Harmonic Analysis of Bounded Solutions of Evolution Equations:

Spectral Decomposition Technique and Criteria for Almost Periodic Solutions

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We consider in this lecture the following linear inhomogeneous integral equation

$$x(t)=U(t,s)x(s)+\int_{s}^{t}U(t,\xi)g(\xi)d\xi,\forall t\geq s;t,s\in\mathbf{R}, \tag{1}$$

where f is continuous, $x(t)\in\mathbf{X}$, $\mathbf{X}$ is a Banach space, $(U(t,s))_{t\geq s}$ is assumed to be a 1-periodic evolutionary process on $\mathbf{X}$. This notion of evolutionary processes arises naturally from the well-posed evolution equations

$$\frac{dx}{dt}=A(t)x+f(t),t\in\mathbf{R},x\in\mathbf{X}, \tag{2}$$

where $A(t)$ is a (in general, unbounded) linear operator for every fixed t and is 1-periodic in t.

A central problem to be studied in the qualitative theory of solutions to Eq.(1) is to find conditions for the existence of (almost) periodic solutions to Eq.(1). In this direction, it is known (see e.g. [Pr], [V-S], [N-M], [M-N-M]) that if the following *nonresonnant condition* holds

$$(\sigma(P)\cap S^{1})\cap\overline{e^{isp(f)}}=\oslash, \tag{3}$$

where $P:=U(1,0)$, S^{1} denotes the unit circle of the complex plane, and f is almost periodic, then there exists an almost periodic solution x_{f} to Eq.(1) which is unique if one requires

$$\overline{e^{isp(x_{f})}}\subset\overline{e^{isp(f)}}.$$

We may ask a question as what happens in the resonnant case where condition (3) fails. In fact, in the particular case where *the forcing term f is 1-periodic and the monodromy*

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operator P is compact this question has been answered with an additional assumption that there exists a bounded uniformly continuous solution to Eq.(1). Historically, this question goes back to a classical result by Massera saying that for Eq.(2) in the finite dimensional case to have a 1-periodic solution it is necessary and sufficient that it has a bounded solution. This famous result serves as the starting point for many papers extending it to various classes of equations among which we would like to mention [C-H], [D-M, Thrm 11.20], [S-N], [N-M-M-S] for extensions to the infinite dimensional case.

It is the purpose of our work to give an answer to the general problem as mentioned above (*Massera-typed problem*): *Let Eq.(1) have a bounded (uniformly continuous) solution x_f with given almost periodic forcing term f. Then, when does Eq.(1) have an almost periodic solution w (which may be different from x_f) such that*

$$\overline{e^{isp(w)}} \subset \overline{e^{isp(f)}} \quad ?$$

Our method is to employ the *evolution semigroup* associated with $(U(t,s))_{t\geq s}$ to study the harmonic analysis of bounded solutions to Eq.(1). As a result we will prove a spectral decomposition theorem for bounded solutions which seems to be useful in dealing with the above problem. In fact, using the notation $\sigma_\Gamma(P) := \sigma(P)\cap S^1$ we have the following:

Theorem 1 *Let f be almost periodic, $\sigma_\Gamma(P)\backslash\overline{e^{isp(f)}}$ be closed. Moreover, let $\overline{e^{isp(f)}}$ be countable and $\mathbf{X}$ not contain any subspace which is isomorphic to c_0 [2]. Then if there exists a bounded uniformly continuous solution u to Eq.(1), there exists an almost periodic solution w to Eq.(1) such that $\overline{e^{isp(w)}} = \overline{e^{isp(f)}}$.*

Our method provides not only the information on the existence of such an almost periodic solution, but also the information on its spectrum. Hence, in case $\sigma_\Gamma(P)$ is countable, since the bounded uniformly continuous solution u is almost periodic we have

Theorem 2 *Let all assumptions of Theorem 1 be satisfied. Moreover, let $\sigma_\Gamma(P)$ be countable. Then if there exists a bounded uniformly continuous solution u to Eq.(1), it is almost periodic. Moreover, the following part of the Fourier series of u*

$$\Sigma b_\lambda e^{i\lambda x}, \quad b_\lambda = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{-i\lambda\xi}u(\xi)d\xi, \tag{4}$$

where $e^{i\lambda} \in \overline{e^{isp(f)}}$, is again the Fourier series of another almost periodic solution to Eq.(1).

In the case where the process $(U(t,s))_{t\geq s}$ is generated by an autonomous equation instead of the spectrum $\sigma_\Gamma(P)$ one can use the part $\sigma_i(A) := \sigma(A)\cap i\mathbf{R}$ of the generator A, i.e. Eq.(1) now takes the form:

$$x(t) = T(t-s)x(s) + \int_s^t T(t-\xi)f(\xi)d\xi, \forall t \geq s, \tag{5}$$

[2] c_0 is defined to be the space of all numerical sequences converging to 0

where $(T(t))_{t\geq 0}$ is a C_0-semigroup of linear operators on $\mathbf{X}$ with the infinitesimal generator A. This will improve a little the statement of Theorem 1 in view of the failure of the Spectral Mapping Theorem. Moreover, we have

Theorem 3 *Let the above assumptions be satisfied. Moreover, let $\sigma_i(A)$ be bounded and $\sigma_i(A)\backslash isp(f)$ be closed. Then if Eq.(4) has a bounded uniformly continuous solution u, it has a bounded uniformly continuous solution w such that $sp(w) = sp(f)$.*

Theorem 3 is useful in dealing with the case where f is quasi-periodic. In fact we see that if $\sigma_i(A)$ is countable and $\mathbf{X}$ does not contain c_0 and $sp(f)$ has "an integer and finite basis", then w is quasi-periodic.

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