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Necessary and sufficient condition for global stability of a Lotka-Volterra system with two delays

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1. Introduction

We consider the following symmetrical Lotka-Volterra type predator-prey system with two delays $\tau_{1}$ and $\tau_{2}$

\[
\begin{align*}
  x'(t) &= x(t)[r_{1} + ax(t) + \alpha x(t - \tau_{1}) - \beta y(t - \tau_{2})] \\
  y'(t) &= y(t)[r_{2} + ay(t) + \beta x(t - \tau_{1}) + \alpha y(t - \tau_{2})].
\end{align*}
\]

(1)

The initial condition of (1) is given as

\[
\begin{align*}
  x(s) &= \phi(s) \geq 0, \quad -\tau_{1} \leq s \leq 0; \quad \phi(0) > 0 \\
  y(s) &= \psi(s) \geq 0, \quad -\tau_{2} \leq s \leq 0; \quad \psi(0) > 0.
\end{align*}
\]

(2)

Here $a$, $\alpha$, $\beta$, $r_{1}$, $r_{2}$, $\tau_{1}$ and $\tau_{2}$ are constants with $a < 0$, $\tau_{1} \geq 0$ and $\tau_{2} \geq 0$, and $\phi$, $\psi$ are continuous functions. Obviously, we can take $\beta \geq 0$ without loss of generality. We assume that (1) has a positive equilibrium $(x^{*}, y^{*})$, that is

\[
x^{*} = \frac{-(a + \alpha)r_{1} - \beta r_{2}}{(a + \alpha)^{2} + \beta^{2}} > 0, \quad y^{*} = \frac{\beta r_{1} - (a + \alpha)r_{2}}{(a + \alpha)^{2} + \beta^{2}} > 0.
\]

The positive equilibrium $(x^{*}, y^{*})$ is said to be globally asymptotically stable if $(x^{*}, y^{*})$ is stable and attracts any solution of (1) with (2). Our purpose is to seek a sharp condition for the global asymptotic stability of $(x^{*}, y^{*})$ for all $\tau_{1}$ and $\tau_{2}$, making the best use of the symmetry of (1). In this paper we give the following necessary and sufficient condition for the global asymptotic stability of $(x^{*}, y^{*})$ for all $\tau_{1} \geq 0$ and $\tau_{2} \geq 0$.

**Theorem.** The positive equilibrium $(x^{*}, y^{*})$ of (1) is globally asymptotically stable for all $\tau_{1} \geq 0$ and $\tau_{2} \geq 0$ if and only if

\[
\sqrt{\alpha^{2} + \beta^{2}} \leq -a
\]

holds.
Gopalsamy [2] showed that if $|\alpha| + |\beta| < -a$ holds, then the positive equilibrium $(x^*, y^*)$ is globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$. It is clear that Theorem improves the Gopalsamy’s condition for (1). Recently, Lu and Wang [7] also considered the global asymptotic stability of $(x^*, y^*)$ for (1) with $\alpha = 0$.

When the system (1) has no delay, that is $\tau_1 = \tau_2 = 0$, it is easy to see that $(x^*, y^*)$ is globally asymptotically stable if and only if $a + \alpha < 0$ [cf. Appendix].

In the proof of the sufficiency of Theorem, we use an extended LaSalle’s invariance principle (also see [8] and [9] for ODE), by which our proof is more complete than that in [7].

2. Proof of Theorem

In order to consider the global asymptotic stability of the positive equilibrium $(x^*, y^*)$ of (1), we first introduce an extension of the LaSalle’s invariance principle.

For some constant $\Delta > 0$, let $C^n = C([-\Delta, 0], \mathbb{R}^n)$. Consider the delay differential equations

$$ z'(t) = f(z_t) \tag{3} $$

where $z_t \in C^n$ is defined as $z_t(\theta) = z(t+\theta)$ for $-\Delta \leq \theta \leq 0$, $f : C^n \rightarrow \mathbb{R}^n$ is completely continuous, and solutions of (3) are continuously dependent on the initial data in $C^n$. The following lemma is actually a corollary of LaSalle invariance principle and the proof is omitted. (see, for example, [4, 5]).

**Lemma.** Assume that for a subset $G$ of $C^n$ and $V : G \rightarrow \mathbb{R}$,

(i) $V$ is continuous on $G$.

(ii) For any $\phi \in \partial G$ (the boundary of $G$), the limit $l(\phi)$

$$ l(\phi) = \lim_{\psi \rightarrow \phi} V(\psi) $$

exists or is $+\infty$.

(iii) $\dot{V}(\phi) \leq 0$ on $G$, where $\dot{V}(\phi)$ is the upper right-hand derivative of $V$ along the solution of (3).

Let $E = \{ \phi \in \bar{G} | l(\phi) < \infty \text{ and } \dot{V}(\phi) = 0 \}$ and $M$ denote the largest subset in $E$ that is invariant with respect to (3). Then every bounded solution of (3) that remains in $G$ approaches $M$ as $t \rightarrow +\infty$. 
Proof of Theorem.

(Sufficiency.) By using the transformation
\[ x = x - x^*, \quad y = y - y^*, \]
the system (1) is reduced to
\[ \begin{cases} \dot{x}(t) = (x^* + x(t)) [ax(t) + \alpha x(t - \tau_1) - \beta y(t - \mathcal{T}_2)] \\ \dot{y}(t) = (y^* + y(t)) [ay(t) + \beta x(t - \tau_1) + \alpha y(t - \mathcal{T}_2)] \end{cases} \] (4)
where we used \( x(t) \) and \( y(t) \) again instead of \( \overline{x}(t) \) and \( \overline{y}(t) \) respectively. Define
\[ G = \{ \phi = (\phi_1, \phi_2) \in C^2 \mid \phi_i(s) \geq 0, \phi_i(0) + x^*_i > 0, i = 1, 2 \} \]
where \( C^2 = C([-\Delta, 0], \mathbb{R}^2), \Delta = \max\{\tau_1, \tau_2\} \) and \( (x^*_1, x^*_2) = (x^*, y^*) \). We consider the functional \( V \) defined on \( G \),
\[ V(\phi) = -2a \sum_{i=1}^{2} \left\{ \phi_i(0) - x^*_i \log \frac{\phi_i(0) + x^*_i}{x^*_i} \right\} + (\alpha^2 + \beta^2) \sum_{i=1}^{2} \int_{-\tau_i}^{0} \phi_i^2(\theta) d\theta. \] (5)
It is clear that \( V \) is continuous on \( G \) and that
\[ \lim_{\psi \to \phi \in \partial G} V(\psi) = +\infty. \]
Furthermore,
\[ \dot{V}(\phi) = -2a \left[ a \phi_1(0) + \alpha \phi_1(-\tau_1) - \beta \phi_2(-\tau_2) \right] \phi_1(0) \]
\[ -2a \left[ a \phi_2(0) + \beta \phi_1(-\tau_1) + \alpha \phi_2(-\tau_2) \right] \phi_2(0) \]
\[ + (\alpha^2 + \beta^2) \left\{ \frac{\phi_1^2(0) - \phi_1^2(-\tau_1)}{x^*_1} + \frac{\phi_2^2(0) - \phi_2^2(-\tau_2)}{x^*_2} \right\} \]
\[ = - [a \phi_1(0) + \alpha \phi_1(-\tau_1) - \beta \phi_2(-\tau_2)]^2 \]
\[ - [a \phi_2(0) + \beta \phi_1(-\tau_1) + \alpha \phi_2(-\tau_2)]^2 \]
\[ - \left[ a^2 - (\alpha^2 + \beta^2) \right] \frac{\phi_1^2(0) + \phi_2^2(0)}{x^*_1 + x^*_2} \leq 0 \] (6)
on \( G \). From (5) and (6), we see that the trivial solution of (4) is stable and that every solution is bounded.

Let
\[ E = \{ \phi \in \bar{G} \mid l(\phi) < \infty \text{ and } \dot{V}(\phi) = 0 \}, \]
\[ M \text{ : the largest subset in } E \text{ that is invariant with respect to (4)}. \]
For \( \phi \in M \), the solution \( z_t(\phi) = (x(t + \theta), y(t + \theta)) \) \((-\Delta \leq \theta \leq 0) \) of (4) through \((0, \phi)\) remains in \( M \) for \( t \geq 0 \) and satisfies for \( t \geq 0 \),
\[ \dot{V}(z_t(\phi)) = 0. \]
Hence, for $t \geq 0$,
\[
\begin{align*}
  ax(t) + \alpha x(t - \tau_1) - \beta y(t - \tau_2) &= 0 \\
  ay(t) + \beta x(t - \tau_1) + \alpha y(t - \tau_2) &= 0,
\end{align*}
\]
which implies that for $t \geq 0$,
\[
x'(t) = y'(t) = 0.
\]
Thus, for $t \geq 0$,
\[
x(t) = c_1, \quad y(t) = c_2
\]
for some constants $c_1$ and $c_2$. From (7) and (8), we have
\[
\begin{bmatrix}
  a + \alpha & -\beta \\
  \beta & a + \alpha
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0
\end{bmatrix},
\]
which implies that $c_1 = c_2 = 0$ by our assumptions and thus we have
\[
x(t) = y(t) = 0 \quad \text{for} \quad t \geq 0.
\]
Therefore, for any $\phi \in M$, we have
\[
\phi(0) = (x(0), y(0)) = 0.
\]
By Lemma, any solution $z_t = (x(t + \theta), y(t + \theta))$ tends to $M$. Thus
\[
\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} y(t) = 0.
\]
Hence, $(x^*, y^*)$ is globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$.

(*Necessity.*) The proof is by contradiction. Assume the assertion were false. That is, let $(x^*, y^*)$ be globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$ and $\sqrt{\alpha^2 + \beta^2} > -a$.

Linearizing (4), we have
\[
\begin{align*}
x'(t) &= x^* \left[ ax(t) + \alpha x(t - \tau_1) - \beta y(t - \tau_2) \right] \\
y'(t) &= y^* \left[ ay(t) + \beta x(t - \tau_1) + \alpha y(t - \tau_2) \right].
\end{align*}
\]
Now, we will show that there exists a characteristic root $\lambda_0$ of (9) such that
\[
Re(\lambda_0) > 0
\]
for some $\tau_1$ and $\tau_2$, which implies that the trivial solution of (4) is not stable (see [1, p.160, 161]).

When $\alpha \geq -a$, $(x^*, y^*)$ is not globally asymptotically stable in case $\tau_1 = \tau_2 = 0$ [cf. Appendix]. Therefore, we have only to consider the case $\alpha < -a$. 
(1) The case $0 < |\alpha| < -a$.

Let $\tau_1 = \tau_2 = \tau$, then the characteristic equation of (9) takes the form

$$\lambda^2 + p\lambda + q + (r + s\lambda)e^{-\lambda \tau} + ve^{-2\lambda \tau} = 0$$

(11)

where $p = -a(x^* + y^*)$, $q = a^2x^*y^*$, $r = 2ax^*y^*$, $s = -\alpha(x^* + y^*)$ and $v = (\alpha^2 + \beta^2)x^*y^*$.

When $x^* = y^*$, (11) can be factorized as

$$\left[\lambda - x^*\{a + (\alpha + i\beta)e^{-\lambda \tau}\}\right]\left[\lambda - x^*\{a + (\alpha - i\beta)e^{-\lambda \tau}\}\right] = 0.$$  

(12)

Let us consider the equation

$$\lambda - x^*\{a + (\alpha + i\beta)e^{-\lambda \tau}\} = 0.$$  

(13)

Set $\alpha = b\cos \theta$ and $\beta = b\sin \theta$, where $b$ and $\theta$ are constants with $b \geq 0$. Then, we note that $b > 0$ because of $a < 0$ and $\sqrt{\alpha^2 + \beta^2} > -a$. Substituting $\lambda = iy$ into (13), we have

$$iy - x^*[a + b\{\cos(y\tau - \theta) - i\sin(y\tau - \theta)\}] = 0.$$  

(14)

By separating the real and imaginary parts of (14), we obtain

$$\begin{align*}
 bx^*\cos(y\tau - \theta) &= -ax^* \\
 bx^*\sin(y\tau - \theta) &= -y.
\end{align*}$$  

(15)

From (15), we have

$$(bx^*)^2 = (ax^*)^2 + y^2.$$  

In order to solve $y$ in (15), define the following function

$$f_1(Y) = Y + (ax^*)^2 - (bx^*)^2$$  

(16)

where $Y = y^2$. Then $f_1$ is an increasing linear function and

$$f_1(0) = x^{*2}\{a^2 - (\alpha^2 + \beta^2)\} < 0.$$  

Thus, it follows that there exists a positive root $Y_0$ of $f_1(Y) = 0$. Substituting $y_0$, which satisfies $Y_0 = y_0^2$, into (15), we can get $\tau_0$ such that (13) has a characteristic root $iy_0$ when $\tau = \tau_0$.

Furthermore, taking the derivative of $\lambda$ with $\tau$ on (13), we have

$$\frac{d\lambda}{d\tau} = \frac{-x^*be^{i\theta}\lambda e^{-\lambda \tau}}{1 + x^*be^{i\theta}\lambda e^{-\lambda \tau}}.$$  

Using (13), we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{1}{-\lambda(\lambda - x^*a)} - \frac{\tau}{\lambda}.$$
Hence,
\[
\text{sign} \left[ \Re \left( \frac{d\lambda}{d\tau} \bigg|_{\lambda=i\gamma_0, \tau=\tau_0} \right) \right] = \text{sign} \left[ \Re \left( \left( \frac{d\lambda}{d\tau} \right)^{-1} \bigg|_{\lambda=i\gamma_0, \tau=\tau_0} \right) \right] = \text{sign} \left[ \Re \left( \frac{1}{-i\gamma_0(i\gamma_0 - x^*a) - \tau_0 / i\gamma_0} \right) \right] = \text{sign} \left[ \Re \left( \frac{1}{\gamma_0^2 + i\gamma_0 x^*a} \right) \right] > 0,
\]
which implies that (10) holds. Therefore, the trivial solution of (4) is not stable, that is, \((x^*, y^*)\) is not stable near \(\tau_0\), which is a contradiction.

When \(x^* \neq y^*\), (11) cannot be factorized as (12). Substituting \(\lambda = iy\) into (11), we have
\[
(-y^2 + piy + q)e^{iy\tau} + r + siy + ve^{-iy\tau} = 0. \tag{17}
\]
By separating the real and imaginary parts of (17), we have
\[
\begin{align*}
\left[(-y^2 + q)^2 - v^2 + p^2y^2\right] \cos(y\tau) &= (r - sp)y^2 - r(q - v) \\
\left[(-y^2 + q)^2 - v^2 + p^2y^2\right] \sin(y\tau) &= sy^3 + [rp - s(q + v)]y,
\end{align*}
\tag{18}
\]
and thus
\[
\left[(-y^2 + q)^2 - v^2 + p^2y^2\right]^2 = \left[(r - sp)y^2 - r(q - v)\right]^2 + \left[ sy^3 + [rp - s(q + v)]y\right]^2.
\]
Define the following function
\[
f_2(Y) = \left[(-Y + q)^2 - v^2 + p^2Y\right]^2 - \left[(r - sp)Y - r(q - v)\right]^2
\]
\[
= -Y[sY + rp - s(q + v)]^2, \tag{19}
\]
where \(Y = y^2\), then \(f_2\) is a quartic function such that \(f_2 \to +\infty\) as \(|Y| \to +\infty\). Since
\[
f_2(0) = [a^2 - (\alpha^2 + \beta^2)][(a + \alpha)^2 + \beta^2][(a - \alpha)^2 + \beta^2](x^*y^*)^4 > 0,
\]
we cannot immediately find positive zeros of (19) and so we have to investigate \(f_2\) in more detail. Define
\[
F(Y) = \left[(-Y + q)^2 - v^2 + p^2Y\right]^2
\]
\[
G(y) = -[(r - sp)Y - r(q - v)]^2
\]
\[
H(y) = -Y[sY + rp - s(q + v)]^2,
\]
then \(f_2 = F + G + H\). It is easy to see that positive zeroes of \(F, G\) and \(H\) are mutually different as long as \(x^* \neq y^*\). Hence, the value of \(f_2\) at the positive zero of \(F\) is negative, which, together with \(f_2(0) > 0\), implies that there exists a positive root of \(f_2(Y) = 0\). It is also clear that there exists another positive root of \(f_2(Y) = 0\) because \(f_2 \to +\infty\) as \(Y \to +\infty\). Thus, one of the two positive roots is a simple root at least.
Let $Y_0$ be such a simple root. Substituting $y_0$, which satisfies $Y_0 = y_0^2$, into (18), we can get some $\tau$ such that (11) has a characteristic root $iy_0$ at $\tau$. We note that $iy_0$ is a simple root of (11) because $Y_0$ is a simple root of $f_2(Y) = 0$.

Furthermore, taking the derivative of $\lambda$ with $\tau$ on (11), we have
\[
\frac{d\lambda}{d\tau} = \frac{-2\lambda(\lambda^2 + p\lambda + q) - \lambda(r + s\lambda)e^{-\lambda\tau}}{2\lambda + p + 2\tau(\lambda^2 + p\lambda + q) + e^{-\lambda\tau}[s + \tau(r + s\lambda)]},
\]
and
\[
\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + p + se^{-\lambda\tau}}{-2\lambda(\lambda^2 + p\lambda + q) - \lambda(r + s\lambda)e^{-\lambda\tau}} - \frac{\tau}{\lambda}.
\]

Hence, we have
\[
\operatorname{sign}\left[\operatorname{Re}\left(\left|\frac{d\lambda}{d\tau}\right|_{\lambda=iy_0} \right)\right] = \operatorname{sign}\left[\operatorname{Re}\left(\left|\frac{d\lambda}{d\tau}\right|^{-1} \right|_{\lambda=iy_0} \right)
\]
\[
= \operatorname{sign}\left[\operatorname{Re}\left(\frac{2iy_0 + p + se^{-iy_0\tau}}{-2iy_0(-y_0^2 + piy_0 + q) - iy_0(r + siy_0)e^{-iy_0\tau}} - \frac{\tau}{iy_0} \right)\right]
\]
\[
= \operatorname{sign}\left[\operatorname{Re}\left(\left(\frac{2iy_0 + p + se^{-iy_0\tau}}{-2iy_0(-y_0^2 + piy_0 + q) - iy_0(r + siy_0)e^{-iy_0\tau}}\right)^{-1}\right)\right]
\]
\[
= \operatorname{sign}\left[1 + \frac{(a^2 + a\alpha \cos(y_0\tau))(x^* - y^*)^2}{(p + s \cos(y_0\tau))^2 + (2y_0 - s \sin(y_0\tau))^2}\right].
\]

Since
\[
(a^2 + a\alpha \cos(y_0\tau))(x^* - y^*)^2 \geq a(a + |\alpha|)(x^* - y^*)^2 > 0,
\]
the last expression in (20) is positive. This implies that (10) holds, which is a contradiction.

(II) The case $\alpha = 0$.

Let $\tau_1 = \tau_2 = \tau$, then the characteristic equation of (9) takes the form
\[
\lambda^2 + p\lambda + q + ve^{-2\lambda\tau} = 0.
\]

Substituting $\lambda = iy$ into (21), we have
\[
-y^2 + piy + q + ve^{-2iy\tau} = 0.
\]

By separating the real and imaginary parts of (22), we have
\[
\begin{cases}
    v \cos(2y\tau) = y^2 - q \
    v \sin(2y\tau) = py
\end{cases}
\]

and
\[
v^2 = (y^2 - q)^2 + (py)^2.
\]
Define the following function

$$f_3(Y) = (Y - q)^2 + p^2 Y - v^2$$

(24)

where $Y = y^2$, then $f_3$ is a downwards convex quadratic function and

$$f_3(0) = (a^4 - \beta^4)x^*y^* < 0.$$ 

Thus, it follows that there exists a positive simple root $Y_0$ of $f_3(Y) = 0$. Substituting $y_0$, which satisfies $Y_0 = y_0^2$, into (23), we can get some $\tau$ such that (21) has a characteristic root $iy_0$ at $\tau$. Here $iy_0$ is a simple root of (21) by the same reason as above.

Taking the derivative of $\lambda$ with $\tau$ on (21), we have

$$\frac{d\lambda}{d\tau} = \frac{2v\lambda e^{-2\lambda\tau}}{2\lambda + p - 2v\tau e^{-2\lambda\tau}},$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + p}{2\lambda(-\lambda^2 - p\lambda - q)} - \frac{\tau}{\lambda}.$$ 

Hence,

$$\text{sign} \left[ \text{Re} \left( \frac{d\lambda}{d\tau} \bigg|_{\lambda=iy_0} \right) \right] = \text{sign} \left[ \text{Re} \left( \left(\frac{d\lambda}{d\tau}\right)^{-1} \bigg|_{\lambda=iy_0} \right) \right]$$

$$= \text{sign} \left[ \text{Re} \left( \frac{2iy_0 + p}{2iy_0(y_0^2 - p^2y_0 - q)} - \frac{\tau}{iy_0} \right) \right]$$

$$= \text{sign} \left[ \text{Re} \left( \frac{2iy_0 + p}{2y_0[p + iy_0 + i(y_0^2 - q)]} \right) \right]$$

$$= \text{sign} \left[ 2y_0^2 + a^2(x^* + y^*)^2 \right] > 0.$$ 

This implies that (10) holds, which is a contradiction.

(III) The case $\alpha \leq a$.

Let $\tau_1 = \tau$ and $\tau_2 = 0$, then the characteristic equation of (9) takes the form

$$\lambda^2 + \tilde{p}\lambda + \tilde{q} + (\tilde{r} + \tilde{s}\lambda)e^{-\lambda\tau} = 0$$

(25)

where $\tilde{p} = -ax^* - (a + \alpha)y^*$, $\tilde{q} = a(a + \alpha)x^*y^*$, $\tilde{r} = [\alpha(a + \alpha) + \beta^2]x^*y^*$, $\tilde{s} = -a x^*$. Let us use $p$, $q$, $r$ and $s$ again instead of $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ and $\tilde{s}$ respectively. Substituting $\lambda = iy$ into (25), we have

$$-y^2 + piy + q + (r + siy)e^{-iy\tau} = 0.$$ 

(26)

By separating the real and imaginary parts of (26), we have

$$\begin{align*}
(r^2 + s^2y^2)\cos(y\tau) &= r(y^2 - q) - spy^2 \\
(r^2 + s^2y^2)\sin(y\tau) &= sy(y^2 - q) + pry
\end{align*}$$

(27)
and

\[
[r^2 + s^2y^2]^2 = [r(y^2 - q) - spy^2]^2 + [sy(y^2 - q) + pry]^2.
\]

Define the following function

\[
f_4(Y) = Y [s(Y - q) + pr]^2 + [r(Y - q) - sp]^2 - [r^2 + s^2Y]^2
\]  

where \(Y = y^2\), then \(f_4\) is an upwards cubic function to the right and

\[
f_4(0) = [\alpha(a + \alpha) + \beta^2][a^2 - (\alpha^2 + \beta^2)](x^*y^*) < 0.
\]

Thus, there can exist some positive roots of \(f_4(Y) = 0\). Now, let us show that there exists a simple root in such positive roots. We see that

\[
f_4'(Y) = 3s^2Y^2 + 2 [s^2(p^2 - 2q - s^2) + r^2] Y + s^2(q^2 - 2r^2) + r^2(p^2 - 2q)
\]

and

\[
f_4''(Y) = 6s^2Y + 2 [s^2(p^2 - 2q - s^2) + r^2].
\]

Let \(f_4''(Y) = 0\), then

\[
3s^2Y + [s^2(p^2 - 2q - s^2) + r^2] = 0,
\]

and thus we have

\[
-3s^2f_4'(Y) = [s^2(p^2 - 2q - s^2) + r^2]^2 - 3s^2 [s^2(q^2 - 2r^2) + r^2(p^2 - 2q)]
\]

\[
= x^*y^2 \left[ x^2 + \{\alpha(a + \alpha) + \beta^2\}^2 y^2 \right]
\]

\[
\times \left[ \{\alpha(a + \alpha) + \beta^2\}^2 - \alpha^2(a + \alpha)^2 \right]
\]

\[
+ \alpha^4x^* \left[ (a^2 - \alpha^2)x^2 - (a + \alpha)^2y^2 \right]^2.
\]

Since \(\alpha \leq a < 0\), (29) is positive. This prove that there exists no triple root of \(f_4(Y) = 0\), which implies that there exists at least a positive simple root \(Y_0\) of \(f_4(Y) = 0\).

Substituting \(y_0\), which satisfies \(Y_0 = y_0^2\), into (27), we can get some \(\tau\) such that (25) has a characteristic root \(iy_0\) at \(\tau\). Here again \(iy_0\) is a simple root of (25).

Taking the derivative of \(\lambda\) with \(\tau\) on (25), we have

\[
\frac{d\lambda}{d\tau} = \frac{\lambda(r + s\lambda)e^{-\lambda\tau}}{2\lambda + p + ce^{-\lambda\tau}[s - \tau(r + s\lambda)]},
\]

\[
\left( \frac{d\lambda}{d\tau} \right)^{-1} = \frac{2\lambda + p + ce^{-\lambda\tau} - \tau}{\lambda(r + s\lambda)e^{-\lambda\tau} - \lambda}
\]

\[
= \frac{2\lambda + p}{-\lambda(\lambda^2 + p\lambda + q)} + \frac{s}{\lambda(r + s\lambda)} - \frac{\tau}{\lambda}.
\]
Hence, we have

\[
\text{sign} \left[ \text{Re} \left( \frac{d\lambda}{d\tau} \bigg|_{\lambda=iy_0} \right) \right] = \text{sign} \left[ \text{Re} \left( \left( \frac{d\lambda}{d\tau} \right)^{-1} \bigg|_{\lambda=iy_0} \right) \right] \\
= \text{sign} \left[ \text{Re} \left( \frac{2iy_0 + p}{-iy_0(-y_0 + piy_0 + q)} + \frac{s}{iy_0(r + siy_0)} - \frac{\tau}{iy_0} \right) \right] \\
= \text{sign} \left[ \frac{s^2y_0^4 + 2r^2y_0^2 - s^2q^2 - 2r^2q + p^2r^2}{[(py_0)^2 + (y_0 - q)^2]^2[r^2 + (sy_0)^2]} \right].
\]

(30)

Since

\[-s^2q^2 - 2r^2q + p^2r^2 \geq [a^2+(\alpha^2+y^2)][a(a+\alpha) + \beta^2]^2x^2y^2 - a^2\alpha^2(a+\alpha)^2x^4y^2 \]

the last expression in (30) is positive. This implies that (10) holds, which is a contradiction. This completes the proof.

Here, we give the following three portraits of the trajectory of (1) with (2), drawn by a computer using the Runge-Kutta method, to illustrate Theorem ( \( r_1 = 10, r_2 = -10 \)).

**Fig.1** \( a = -5, \alpha = 3, \beta = 3.99 \) \( (\sqrt{\alpha^2 + \beta^2} < -a) \)

\( \tau_1 = 1, \tau_2 = 2, (\phi, \psi) = (3 + 0.8t, 3.5 + \sin(8t)) \)
When $\tau_1 = \tau_2 = 0$, the system (1) become

\[
\begin{align*}
&x'(t) = x(t)[r_1 + (a + \alpha)x(t) - \beta y(t)] \\
y'(t) = y(t)[r_2 + \beta x(t) + (a + \alpha)y(t)].
\end{align*}
\]
By using the transformation

$\overline{x} = x - x^*$,  \hspace{1em} \overline{y} = y - y^*$,

(31) is reduced to

$$
\begin{align*}
    x'(t) &= (x^* + x(t))[(a + \alpha)x(t) - \beta y(t)] \\
    y'(t) &= (y^* + y(t))[(\beta x(t) + (a + \alpha)y(t)]
\end{align*}
$$

(32)

where we used $x(t)$ and $y(t)$ again instead of $\overline{x}(t)$ and $\overline{y}(t)$, respectively. Consider the following Liapunov function

$$
V(x, y) = \left( x - x^* \log \frac{x + x^*}{x} \right) + \left( y - y^* \log \frac{y + y^*}{y} \right)
$$

(33)

for $x > -x^*$ and $y > -y^*$, then $V$ is positive definite. Calculating the derivative of $V$ along the solution of (32), we have

$$
\dot{V}_{(32)}(x, y) = (a + \alpha)(x^2 + y^2).
$$

Clearly, $\dot{V}_{(32)}$ is negative definite if and only if $a + \alpha < 0$ holds. The well-known Liapunov theorem shows that the origin $(0,0)$ is globally asymptotically stable if and only if $a + \alpha < 0$ holds.

References


