

The asymptotic stability of $x_{n+1} - x_n + Ax_{n-k} = 0$

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1. Introduction and main results

The asymptotic stability of delay difference equations has been investigated by many authors. In scalar case, many results can be found in several books and papers [1-5]. Levin *et.al* [5] and Kuruklis [4] have shown the nice result as follows:

The delay difference equation

$$x_{n+1} - x_n + qx_{n-k} = 0, \quad n = 0, 1, \dots,$$

where q is a real number and k is a nonnegative integer, is asymptotically stable if and only if

$$0 < q < 2 \cos \frac{k\pi}{2k+1}.$$

In this paper we give some new necessary and sufficient conditions for the asymptotic stability of a 2-dimensional linear delay difference system

$$x_{n+1} - x_n + Ax_{n-k} = 0, \quad n = 0, 1, \dots, \quad (1)$$

where k is a nonnegative integer and A is a 2×2 constant matrix.

By the transformation $x_n = Py_n$ with an appropriate regular matrix P , we can rewrite (1) as

$$y_{n+1} - y_n + P^{-1}APy_{n-k} = 0, \quad n = 0, 1, \dots.$$

Thus, we only have to consider (1) where the matrix A is either of the following two matrices :

$$(I) \quad A = qR(\theta) \equiv q \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (II) \quad A = \begin{pmatrix} q_1 & b \\ 0 & q_2 \end{pmatrix},$$

where q, q_1, q_2, b and θ ($|\theta| \leq \frac{\pi}{2}$) are real numbers.

For the case (I), we have

Theorem 1. *The system (1) is asymptotically stable if and only if*

$$0 < q < 2 \cos \frac{k\pi + |\theta|}{2k+1}. \quad (2)$$

For the case (II), we have

Theorem 2. *The system (1) is asymptotically stable if and only if*

$$0 < q_1 < 2 \cos \frac{k\pi}{2k+1} \quad \text{and} \quad 0 < q_2 < 2 \cos \frac{k\pi}{2k+1}. \quad (3)$$

In this paper we only give the proof of the Theorem 1 since both theorems are proved in very similar way.

2. Proof of Theorem 1

Theorem 1 is proved by using the fact that the system (1) is asymptotically stable if and only if all the roots of its characteristic equation

$$F(\lambda) \equiv \det(\lambda^{k+1}I - \lambda^k I + qR(\theta)) = 0 \quad (4)$$

are inside the unit disk. Hence, we investigate the characteristic roots of (4) to prove Theorem 1.

Let

$$f^+(\lambda) \equiv \lambda^{k+1} - \lambda^k + qe^{i\theta}, \quad f^-(\lambda) \equiv \lambda^{k+1} - \lambda^k + qe^{-i\theta}.$$

Then we have

$$\begin{aligned} F(\lambda) &= \det \begin{pmatrix} \lambda^{k+1} - \lambda^k + q \cos \theta & -q \sin \theta \\ q \sin \theta & \lambda^{k+1} - \lambda^k + q \cos \theta \end{pmatrix} \\ &= (\lambda^{k+1} - \lambda^k + q \cos \theta)^2 + q^2 \sin^2 \theta \\ &= (\lambda^{k+1} - \lambda^k + q \cos \theta + iq \sin \theta)(\lambda^{k+1} - \lambda^k + q \cos \theta - iq \sin \theta) \\ &= f^+(\lambda)f^-(\lambda). \end{aligned}$$

Note that $f^-(\lambda) = 0$ implies $f^+(\bar{\lambda}) = 0$, where $\bar{\lambda}$ is the complex conjugate of any complex λ . Also, when $-\frac{\pi}{2} \leq \theta \leq 0$, substituting $\tilde{\theta} = -\theta$ in $f^+(\lambda) = 0$ and $f^-(\lambda) = 0$ implies $0 \leq \tilde{\theta} \leq \frac{\pi}{2}$. Therefore, we only have to consider the case $f^+(\lambda) = 0$ under the condition $0 \leq \theta \leq \frac{\pi}{2}$ to investigate the characteristic roots of (4). We also notice that (4) has no real roots if $q \neq 0$ and $\theta \neq 0$.

When $k = 0$, it follows from (4) that $\lambda = 1 - q \cos \theta \pm iq \sin \theta$. Then

$$|\lambda|^2 = (1 - q \cos \theta)^2 + (q \sin \theta)^2 = q^2 - 2q \cos \theta + 1.$$

It is easy to see that $|\lambda| < 1$ if and only if $0 < q < 2 \cos \theta$, and so, Theorem 1 stands if $k = 0$. Hereafter, let k be a positive integer.

As a beginning, we shall examine the existence region of the arguments of complex roots of $f^+(\lambda) = 0$.

Lemma 1. *Assume that $q > 0$ and $0 \leq \theta \leq \frac{\pi}{2}$. Let $re^{i\omega}$, with $r > 0$ and $0 < |\omega| < \pi$, be a complex root of $f^+(\lambda) = 0$. Then*

$$\frac{\theta + 2p\pi}{k} < \omega < \frac{\theta + (2p+1)\pi}{k+1}, \quad p = 0, 1, \dots, \left[\frac{k-1}{2} \right], \quad \text{if } \omega > 0;$$

$$\frac{\theta - \pi}{k+1} < \omega < 0, \quad \frac{\theta - (2p+1)\pi}{k+1} < \omega < \frac{\theta - 2p\pi}{k}, \quad p = 1, 2, \dots, \left[\frac{k}{2} \right], \quad \text{if } \omega < 0,$$

where $[a]$ represents the integer part of a .

Proof. From $f^+(re^{i\omega}) = 0$, we have $r^{k+1}e^{i((k+1)\omega-\theta)} - r^k e^{i(k\omega-\theta)} + q = 0$, namely

$$r^k \{ \cos(k\omega - \theta) - r \cos((k+1)\omega - \theta) \} = q, \quad (5)$$

and

$$r^k \{ \sin(k\omega - \theta) - r \sin((k+1)\omega - \theta) \} = 0. \quad (6)$$

It is obvious that $\sin((k+1)\omega - \theta) \neq 0$, so (6) implies

$$r = \frac{\sin(k\omega - \theta)}{\sin((k+1)\omega - \theta)}. \quad (7)$$

(5) and (7) yield

$$\begin{aligned} q &= r^k \frac{\sin((k+1)\omega - \theta) \cos(k\omega - \theta) - \sin(k\omega - \theta) \cos((k+1)\omega - \theta)}{\sin((k+1)\omega - \theta)} \\ &= r^k \frac{\sin \omega}{\sin((k+1)\omega - \theta)}. \end{aligned} \quad (8)$$

We consider the case $0 < \omega < \pi$. (In case $-\pi < \omega < 0$, the proof is similar.) From (7) and (8), we must have $\sin(k\omega - \theta) > 0$ and $\sin((k+1)\omega - \theta) > 0$ because of $q > 0, r > 0$ and $\sin \omega > 0$. Hence,

$$\frac{\theta + 2m\pi}{k} < \omega < \frac{\theta + (2m+1)\pi}{k}, \quad m = 0, 1, \dots, \left[\frac{k-1}{2} \right],$$

and

$$\frac{\theta + 2n\pi}{k+1} < \omega < \frac{\theta + (2n+1)\pi}{k+1}, \quad n = 0, 1, \dots, \left[\frac{k}{2} \right],$$

which imply

$$\frac{\theta + 2p\pi}{k} < \omega < \frac{\theta + (2p+1)\pi}{k+1}, \quad p = 0, 1, \dots, \left[\frac{k-1}{2} \right].$$

The proof is complete.

When $q < 0$, we have the following analogous result.

Lemma 2. Assume that $q < 0$ and $0 \leq \theta \leq \frac{\pi}{2}$. Let $re^{i\omega}$, with $r > 0$ and $0 < |\omega| < \pi$, be a complex root of $f^+(\lambda) = 0$. Then

$$0 < \omega < \frac{\theta}{k+1}, \quad \frac{\theta + (2p-1)\pi}{k} < \omega < \frac{\theta + 2p\pi}{k+1}, \quad p = 1, 2, \dots, \left[\frac{k}{2}\right], \text{ if } \omega > 0;$$

$$\frac{\theta - (2p+2)\pi}{k+1} < \omega < \frac{\theta - (2p+1)\pi}{k}, \quad p = 0, 1, \dots, \left[\frac{k-1}{2}\right], \text{ if } \omega < 0.$$

The next lemma determines the value of q and the root's argument ω on the unit circle.

Lemma 3. Assume that $q > 0$ and $0 \leq \theta \leq \frac{\pi}{2}$. Then the arguments of complex roots of $f^+(\lambda) = 0$ on the unit circle are given by ω_p^+ or ω_p^- , where

$$\omega_p^+ \equiv \frac{2\theta + (4p+1)\pi}{2k+1} > 0, \quad p = 0, 1, \dots, \left[\frac{k-1}{2}\right],$$

$$\omega_p^- \equiv \frac{2\theta - (4p+1)\pi}{2k+1} < 0, \quad p = 0, 1, \dots, \left[\frac{k}{2}\right].$$

Moreover, the following relation stands :

$$q = \sqrt{2 - 2 \cos \omega}. \quad (9)$$

Proof. Substituting $r = 1$ into (5) and (6), we get

$$\cos(k\omega - \theta) - \cos((k+1)\omega - \theta) = q, \quad (10)$$

and

$$\sin(k\omega - \theta) - \sin((k+1)\omega - \theta) = 0. \quad (11)$$

(11) implies that $2 \cos \frac{(2k+1)\omega - 2\theta}{2} \sin \frac{\omega}{2} = 0$. Since $\sin \frac{\omega}{2} \neq 0$, we have

$$\cos \frac{(2k+1)\omega - 2\theta}{2} = 0. \quad (12)$$

We consider the case $0 < \omega < \pi$. (In case $-\pi < \omega < 0$, the proof is similar.) Then (12) yields

$$\omega = \frac{2\theta + (2n+1)\pi}{2k+1}, \quad n = 0, 1, \dots, k-1. \quad (13)$$

By Lemma 1, (13) is suitable when n is only even, and therefore, we obtain

$$\omega = \omega_p^+ \equiv \frac{2\theta + (4p+1)\pi}{2k+1}, \quad p = 0, 1, \dots, \left[\frac{k-1}{2}\right].$$

Next, by squaring both sides of (10) and (11), and adding them together, we have

$$\begin{aligned} q^2 &= \{\cos(k\omega - \theta) - \cos((k+1)\omega - \theta)\}^2 + \{\sin(k\omega - \theta) - \sin((k+1)\omega - \theta)\}^2 \\ &= 2 - 2\{\cos((k+1)\omega - \theta)\cos(k\omega - \theta) + \sin((k+1)\omega - \theta)\sin(k\omega - \theta)\} \\ &= 2 - 2\cos\omega. \end{aligned}$$

Hence, (9) stands. The proof is complete.

Remark 1. In view of the definitions of ω_p^+ and ω_p^- , the value of $q(\omega)$ given by (9) is minimum when $\omega = \omega_0^-$. Taking account of $\sqrt{2 - 2\cos\omega} = 2|\sin\frac{\omega}{2}|$, we obtain that

$$q(\omega_0^-) = 2\sin\left|\frac{2\theta - \pi}{2(2k+1)}\right| = 2\sin\left(\frac{\pi}{2} - \frac{k\pi + \theta}{2k+1}\right) = 2\cos\frac{k\pi + \theta}{2k+1}.$$

When $q < 0$, we have the following result which is analogous to Lemma 3.

Lemma 4. Assume that $q < 0$ and $0 \leq \theta \leq \frac{\pi}{2}$. Then the arguments of complex roots of $f^+(\lambda) = 0$ on the unit circle are given by α_p^+ or α_p^- , where

$$\begin{aligned} \alpha_p^+ &\equiv \frac{2\theta + (4p-1)\pi}{2k+1} > 0, & p = 1, 2, \dots, \left[\frac{k}{2}\right], \\ \alpha_p^- &\equiv \frac{2\theta - (4p+3)\pi}{2k+1} < 0, & p = 0, 1, \dots, \left[\frac{k-1}{2}\right]. \end{aligned}$$

Furthermore, we shall observe the crossing of the unit circle by the roots of $f^+(\lambda) = 0$ when the value of q varies.

Lemma 5. Assume that $0 \leq \theta < \frac{\pi}{2}$. Then the simple root $\lambda = 1$ of $f^+(\lambda) = 0$ with $q = 0$ moves inside the unit disk (resp. outside the unit disk) as q increases from 0 (resp. decreases from 0).

Proof. It suffices to show that $(dr/dq)|_{r=1} < 0$. If $\theta = 0$, let $\lambda = r$ be a positive root of $f^+(\lambda) = 0$, then

$$r^{k+1} - r^k + q = 0. \quad (14)$$

Note that $r = 1$ implies $q = 0$. Taking the derivative of r with q on (14), we have

$$(k+1)r^k \frac{dr}{dq} - kr^{k-1} \frac{dr}{dq} + 1 = 0,$$

or

$$\frac{dr}{dq} = \frac{1}{kr^{k-1} - (k+1)r^k}.$$

Hence, we arrive at

$$\left. \frac{dr}{dq} \right|_{r=1} = \frac{1}{k - (k+1)} = -1 < 0.$$

If $0 < \theta < \frac{\pi}{2}$, it follows from (8) that

$$\begin{aligned} \frac{dr}{dq} &= \frac{dr}{d\omega} \left(\frac{dq}{d\omega} \right)^{-1} \\ &= \frac{dr}{d\omega} \left(kr^{k-1} \frac{dr}{d\omega} \frac{\sin \omega}{\sin((k+1)\omega - \theta)} + r^k \frac{d}{d\omega} \left(\frac{\sin \omega}{\sin((k+1)\omega - \theta)} \right) \right)^{-1} \\ &= \frac{dr}{d\omega} \left(\frac{qk}{r} \frac{dr}{d\omega} + r^k \frac{d}{d\omega} \left(\frac{\sin \omega}{\sin((k+1)\omega - \theta)} \right) \right)^{-1}. \end{aligned} \quad (15)$$

Also, by (7), we have

$$\begin{aligned} \frac{dr}{d\omega} &= \frac{k \cos(k\omega - \theta) \sin((k+1)\omega - \theta) - (k+1) \sin(k\omega - \theta) \cos((k+1)\omega - \theta)}{\sin^2((k+1)\omega - \theta)} \\ &= \frac{k \sin \omega - \sin(k\omega - \theta) \cos((k+1)\omega - \theta)}{\sin^2((k+1)\omega - \theta)}, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{d\omega} \left(\frac{\sin \omega}{\sin((k+1)\omega - \theta)} \right) &= \frac{\cos \omega \sin((k+1)\omega - \theta) - (k+1) \sin \omega \cos((k+1)\omega - \theta)}{\sin^2((k+1)\omega - \theta)} \\ &= \frac{\sin(k\omega - \theta) - k \sin \omega \cos((k+1)\omega - \theta)}{\sin^2((k+1)\omega - \theta)}. \end{aligned}$$

Denote

$$G(\omega) \equiv k \sin \omega - \sin(k\omega - \theta) \cos((k+1)\omega - \theta), \quad (16)$$

and

$$H(\omega) \equiv \sin(k\omega - \theta) - k \sin \omega \cos((k+1)\omega - \theta) \quad (17)$$

then (15) yields

$$\frac{dr}{dq} = \frac{G(\omega)}{(qk/r)G(\omega) + r^k H(\omega)}. \quad (18)$$

Noticing that $q = 0$ is equivalent to $\omega = 0$, we obtain that

$$\left. \frac{dr}{dq} \right|_{\substack{r=1 \\ q=0}} = \frac{G(0)}{H(0)} = \frac{-\sin(-\theta) \cos(-\theta)}{\sin(-\theta)} = -\cos \theta < 0.$$

The proof is complete.

Lemma 6. Assume that $q > 0$ and $0 \leq \theta \leq \frac{\pi}{2}$. Then all the roots of $f^+(\lambda) = 0$ on the unit circle move outside as q increases.

Proof. By Lemma 3, it suffices to show that

$$\left. \frac{dr}{dq} \right|_{\substack{r=1 \\ q=q(\omega_p^+)}} > 0, \quad p = 0, 1, \dots, \left[\frac{k-1}{2} \right], \quad (19)$$

and

$$\left. \frac{dr}{dq} \right|_{\substack{r=1 \\ q=q(\omega_p^-)}} > 0, \quad p = 0, 1, \dots, \left[\frac{k}{2} \right]. \quad (20)$$

From (18), we have

$$\left. \frac{dr}{dq} \right|_{\substack{r=1 \\ q=q(\omega_p^+)}} = \frac{G(\omega_p^+)}{qkG(\omega_p^+) + H(\omega_p^+)},$$

where $G(\omega)$ and $H(\omega)$ are defined by (16) and (17) respectively. Note that $\sin \omega_p^+ > 0$ and $\sin(k\omega_p^+ - \theta) > 0$ because of Lemma 1. Also, since

$$\frac{2\theta + (4p+1)\pi}{2(k+1)} < \omega_p^+ < \frac{\theta + (2p+1)\pi}{k+1},$$

it is easy to see that $\cos((k+1)\omega_p^+ - \theta) < 0$. Thus, we obtain that $G(\omega_p^+) > 0$ and $H(\omega_p^+) > 0$, and so (19) holds. Similarly, we can show (20). The proof is complete.

When $q < 0$, using Lemmas 2 and 4 instead of Lemmas 1 and 3, we have the following result which is analogous to Lemma 6.

Lemma 7. Assume that $q < 0$ and $0 \leq \theta \leq \frac{\pi}{2}$. Then all the roots of $f^+(\lambda) = 0$ on the unit circle move outside as $|q|$ increases.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let λ be a characteristic root of (4). Here, recalling the argument above, we only have to consider the value of λ satisfying $f^+(\lambda) = 0$ with $0 \leq \theta \leq \frac{\pi}{2}$.

(*Sufficiency*) From (2), we must have $0 \leq \theta < \frac{\pi}{2}$. Thus, by virtue of Lemma 5 and continuity of λ with respect to q , we notice that, if $q > 0$ is sufficient small, then $|\lambda| < 1$ holds for any λ and the system (1) is asymptotically stable.

If the increasing of q leads the system (1) to instability, there exists a root λ^* of $f^+(\lambda) = 0$ such that $|\lambda^*| = 1$. By Remark 1, we find that $2 \cos \frac{k\pi + \theta}{2k+1}$ is the minimum value of q when $|\lambda^*| = 1$. This fact indicates that, if (2) is true, then $|\lambda| < 1$ holds for any λ and the system (1) is asymptotically stable.

(Necessity) Suppose that the system (1) is asymptotically stable, that is, for any λ ,

$$|\lambda| < 1. \quad (21)$$

For the sake of contradiction, (2) is false. Then we consider two cases.

Case 1: $q \geq 2 \cos \frac{k\pi+\theta}{2k+1}$. By Lemma 6, there exists a root λ^* of $f^+(\lambda) = 0$ such that $|\lambda^*| \geq 1$ as q increases from $2 \cos \frac{k\pi+\theta}{2k+1}$, which contradicts (21).

Case 2: $q \leq 0$. By Lemma 5, the simple root $\lambda = 1$ with $q = 0$ moves outside the unit disk as $|q|$ increases from 0. Hence, in view of the fact above and Lemma 7, there exists a root λ^* of $f^+(\lambda) = 0$ such that $|\lambda^*| \geq 1$, which also contradicts (21).

References

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