Title
Star-shaped periodic solutions for $\mathbf{x}(t)=-\alpha||\mathbf{x}(t)||^2R(\theta)\mathbf{x}(\tau)$ (Methods and Applications for Functional Equations)

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Star-shaped periodic solutions for
\[ \dot{x}(t) = -\alpha \{1 - \|x(t)\|^2\} R(\theta) x([t]) \]

1. Introduction

Recently in [1], Hara considered a 2-dimensional delay differential system
\[ \dot{x}(t) = -\alpha \{1 - \|x(t)\|^2\} R(\theta) x(t - 1), \tag{1.1} \]
where \( \alpha > 0, R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, |\theta| < \frac{\pi}{2}, x = \begin{pmatrix} x \\ y \end{pmatrix} \) and \( \|x\|^2 = x^2 + y^2 \). He gave a conjecture:

**Conjecture.** There exists a constant \( \alpha_0 > \pi/2 - |\theta| \) such that \( \alpha > \alpha_0 \) implies the following:

(a) If \( \theta/\pi \) is rational, then (1.1) has a star-shaped periodic solution.
(b) If \( \theta/\pi \) is irrational, then each solution orbit densely fills out an annular region centered at the origin.

Our purpose is to give an answer in some sense to this conjecture for an approximate system to (1.1)
\[ \dot{x}(t) = -\alpha \{1 - \|x(t)\|^2\} R(\theta) x([t]), \tag{1.2} \]
where \([\cdot]\) means the greatest-integer function.

We shall consider the system (1.2) together with the initial condition
\[ x(t_0 + s) = \phi(s) \quad \text{for} \quad s \in [-1, 0], \tag{1.3} \]
where \( \phi \in C \), the family of all continuous functions from \([-1, 0]\) into \( \mathbb{R}^2 \). In what follows, \( N \) denotes the minimal integer not less than the initial time \( t_0 \). Then \( N = t_0 \) if \( t_0 \in \mathbb{Z} \), the set of all integers, and \( N = [t_0] + 1 \) if \( t_0 \notin \mathbb{Z} \). Furthermore, \( \mathbb{Q} \) means the set of all rational numbers.
Our results in this paper are similar to ones ([2]) for a linear system

\[ \dot{x}(t) = -\alpha R(\theta)x(t) \]  

(1.4)
which is the first approximate system for (1.2).

2. Preliminary propositions

In this section, we give preliminary propositions to prove our theorems.
For each solution \( x(t) \) of (1.2) and each integer \( n \geq N \), there exists one and only one \( \varphi \in [0, 2\pi) \) such that

\[ x(n) = R(\varphi) \left( \begin{array}{l} ||x(n)|| \\ 0 \end{array} \right). \]  

(2.1)
Changing variables

\[ u(t) = R(- (\theta + \varphi))x(t) \]  

(2.2)
or

\[ x(t) = R(\theta + \varphi)u(t), \]
we obtain the following proposition.

**Proposition 2.1.** Let \( x(t) \) be a solution of (1.2). Then \( u(t) \), determined by (2.1) and (2.2), satisfies for any integer \( n \geq N \):

- (a) \( ||u(t)|| = ||x(t)|| \) for \( t \geq n \).
- (b) \( u(n) = ||x(n)|| \cdot \left( \begin{array}{l} \cos \theta \\ -\sin \theta \end{array} \right) \).
- (c) \( \dot{u}(t) = -\alpha \{1 - ||u(t)||^2\} \left( \begin{array}{l} ||x(n)|| \\ 0 \end{array} \right) \) for \( t \in [n, n+1) \).

This proposition follows by elementary calculation and also shows:

**Proposition 2.2.** Let \( x(t) \) be a solution of (1.2). Then the following are valid:

- (a) \( x(N) = 0 \) implies \( x(t) = 0 \) for \( t \geq N \).
- (b) \( ||x(t_0)|| = 1 \) implies \( x(t) = x(t_0) \) for \( t \geq t_0 \).
- (c) \( ||x(t_0)|| < 1 \) implies \( ||x(t)|| < 1 \) for \( t \geq t_0 \).
- (d) \( ||x(t_0)|| > 1 \) implies \( ||x(t)|| > 1 \), whenever \( x(t) \) exists.

**Proof.** We prove only (c) and omit the proof of others. First, suppose \( ||x(t_1)|| = 1 \) for some \( t_1 \leq N \) and \( ||x(t)|| < 1 \) on \( [t_0, t_1) \). Using change of variables

\[ u(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = R(-(\theta + \varphi_0))x(t) \]
with
\[ x([t_0]) = R(\varphi_0) \left( \|x([t_0])\| \right), \]
we have \( v(t) = v(t_0) \) and
\[ \dot{u}(t) = -\alpha \|x([t_0])\| \{1 - u(t)^2 - v(t_0)^2\} \] (2.3)
for \( t \in [t_0, t_1) \), where \( u(t_0)^2 < 1 - v(t_0)^2 \). Since \( u = \pm \sqrt{1 - v(t_0)^2} \) are critical points for (2.3), uniqueness of solutions for (2.3) guarantees
\[ -\sqrt{1 - v(t_0)^2} < u(t) < \sqrt{1 - v(t_0)^2} \]
on \([t_0, t_1]\),
which implies
\[ \|x(t_1)\| = u(t_1)^2 + v(t_1)^2 < 1. \]
This contradicts the supposition \( \|x(t_1)\| = 1 \). Therefore \( x(t) \) satisfies \( \|x(t)\| < 1 \) on \([t_0, N]\). Next, suppose \( \|x(t_1)\| = 1 \) for some \( t_1 > N \) and \( \|x(t)\| < 1 \) on \([t_0, t_1]\). Then there is an integer \( n \geq N \) fulfilling \( n < t_1 \leq n + 1 \). For convenience sake, put
\[ \rho = \|x(n)\|, \quad \beta = \sqrt{1 - \rho^2 \sin^2 \theta}. \]
It follows from Proposition 2.1 that
\[ \dot{u}(t) = -\alpha \beta^2 - u(t)^2 \] (2.4)
and
\[ \dot{v}(t) = 0 \quad \text{or} \quad v(t) = -\rho \sin \theta \] (2.5)
for \( t \in [n, n+1) \), where \( u(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \). Therefore we can easily show that the inequality \( \|x(t_1)\| < 1 \) holds, a contradiction. Thus we conclude that \( \|x(t)\| < 1 \) for \( t \in [t_0, \infty) \). This completes the proof. \( \square \)

Remark 2.1. Propositions 2.1 and 2.2 show that every solution \( x(t) \) of (1.2) with \( x(N) \neq 0 \) moves straightly from \( x(n) \) to \( x(n+1) \) as \( t \) does from \( n \) to \( n+1 \). Therefore, if \( \|x(n)\| \to 0 \) as \( n \to \infty \), then the solution \( x(t) \) approaches the origin as \( t \to \infty \). Furthermore, if \( x(N+m) = x(N) \) for some integer \( m \), then \( x(t) \) runs on a star-shaped periodic orbit for all time.

Now, we prepare several lemmas for proving our theorems in the next section. Let \( 0 < \rho < 1 \) and put \( \beta = \sqrt{1 - \rho^2 \sin^2 \theta} \). Then it is easy to see \( 0 < \rho \cos \theta < \beta \leq 1 \). So, defining the function \( f \) on \((0,1)\) by
\[ f(\rho) = \frac{1}{\rho \beta} \log \frac{\beta + \rho \cos \theta}{\beta - \rho \cos \theta}, \]
we obtain the following lemma.
Lemma 2.1. The function $f$ is continuous and strictly increasing in $\rho$, and satisfies
\[ \lim_{\rho \to 0^+} f(\rho) = 2 \cos \theta, \quad \lim_{\rho \to 0^-} f(\rho) = \infty. \]

Proof. It is convenient to put
\[ g(\rho) = \log \frac{\beta + \rho \cos \theta}{\beta - \rho \cos \theta}. \]
Then it is obvious that $g$ is positive and continuous, and hence $f$ is also. Since $\beta$ tends to 1 as $\rho \to +0$, it follows that $g(\rho)$ tends to 0 as $\rho \to +0$, and so L'Hospital's theorem asserts
\[ \lim_{\rho \to 0^+} f(\rho) = \lim_{\rho \to 0^+} \frac{g(\rho)}{\rho} = \lim_{\rho \to 0^+} g'(\rho). \]
Here, elementary calculation shows
\[ g'(\rho) = \frac{2 \cos \theta}{(1 - \rho^2)\beta}. \]
This implies that $f(\rho)$ tends to $2 \cos \theta$ as $\rho \to +0$. On the other hand, since $\beta$ tends to $\cos \theta$ as $\rho \to 1 - 0$, the equalities
\[ \lim_{\rho \to 1^{-}} f(\rho) = \frac{1}{\cos \theta} \lim_{\rho \to 1^{-}} g(\rho) = \infty \]
hold. Differentiating $f(\rho)$, we have
\[ f'(\rho) = \frac{(2 \rho \cos \theta)/(1 - \rho^2) - g(\rho) \beta + (g(\rho) \rho \sin^2 \theta)}/(\rho \beta)^2 \]
and then
\[ f'(\rho) \geq \frac{h(\rho) - g(\rho)}{(\rho \beta)^2}, \quad (2.6) \]
where $h(\rho) = (2 \rho \cos \theta)/(1 - \rho^2)$. It is easy to see that $h(\rho)$ tends to 0 as $\rho \to +0$ and
\[ h'(\rho) = \frac{2 \cos \theta (1 + \rho^2)}{(1 - \rho^2)^2}. \]
Since $1 - \rho^2 < \beta^2 \leq \beta < \beta(1 + \rho^2)$, it follows that
\[ g'(\rho) < \frac{2 \cos \theta(1 + \rho^2)}{(1 - \rho^2)^2} = h'(\rho). \]
This, together with the fact
\[ \lim_{\rho \to 0^+} g(\rho) = \lim_{\rho \to 0^+} h(\rho) = 0, \]
implies that
\[ g(\rho) < h(\rho) \quad \text{for} \quad 0 < \rho < 1. \]
Hence we can conclude from (2.6) that $f(\rho)$ is strictly increasing in $\rho$. Thus the proof is now completed. \[ \square \]
Proposition 2.3. Let \( x(t) \) be a solution of (1.2) satisfying \( 0 < ||x(N)|| < 1 \). Then, for any integer \( n \geq N \), the following are valid:

(a) \( \alpha = f(||x(n)||) \) implies \( ||x(n+1)|| = ||x(n)|| \).
(b) \( \alpha < f(||x(n)||) \) implies \( ||x(n+1)|| < ||x(n)|| \).
(c) \( \alpha > f(||x(n)||) \) implies \( ||x(n+1)|| > ||x(n)|| \).

Proof. In the same manner as the proof of Proposition 2.2, we get (2.4), (2.5),

\[
u(n) = \rho \cos \theta
\]

and also \(-\beta < u(t) < \beta \) for \( t \in [n, n+1] \). Applying the quadrature to (2.4), we have

\[
\frac{\beta + u(n+1)}{\beta - u(n+1)} = \frac{\beta + u(n)}{\beta - u(n)} e^{-2\alpha \rho \beta}.
\]

On the other hand, \( u(n+1) = -u(n) \) if and only if

\[
\frac{\beta - u(n+1)}{\beta + u(n+1)} = \frac{\beta + u(n)}{\beta - u(n)}.
\]

Here, if \( \alpha = f(\rho) \), then (2.7) asserts

\[
\alpha = \frac{1}{\rho \beta} \log \frac{\beta + u(n)}{\beta - u(n)},
\]

and so (2.9) follows from (2.8). Hence we can conclude from Proposition 2.1 (a) and (2.5) that

\[
\alpha = f(||x(n)||) \implies ||x(n+1)|| = ||x(n)||.
\]

In the same way, we arrive at the conclusion that (b) and (c) of this lemma are valid. \( \square \)

The following lemma is an immediate consequence of Lemma 2.2 in [2].

Lemma 2.2. There exists a positive integer \( m \) such that \( R(m(2\theta - \pi)) = I \) if and only if the ratio \( \theta / \pi \) is rational.

3. Theorems

Let \( \phi \) be an initial function with \( ||\phi(0)|| < 1 \). Then Proposition 2.2 asserts that the solution \( x(t) \) of (1.2) and (1.3), satisfies \( ||x(t)|| < 1 \) on \([t_0, \infty) \). First of all, we give a sufficient condition for such a solution to approach the origin as \( t \to \infty \).

Theorem 3.1. Assume \( \alpha \leq 2 \cos \theta \). Then each solution \( x(t) \) of (1.2) with \( ||x(t_0)|| < 1 \) approaches the origin as \( t \to \infty \), and also the zero solution is stable.
Proof. We may assume that \(0 < \|x(N)\| < 1\). Then Lemma 2.1 asserts \(\alpha < f(\|x(N)\|)\). It follows from Proposition 2.3 that
\[
\|x(N+1)\| < \|x(N)\| < 1.
\]
Repeating this argument, we have
\[
\|x(n+1)\| < \|x(n)\| < 1
\]
for any integer \(n \geq N\). So, suppose the sequence \(\{\|x(n)\|\}\) converges to a positive \(\rho_0\) as \(n \to \infty\). Then it is clear that
\[
\|x(n)\| \geq \rho_0
\]
for any \(n\). Now, consider a system
\[
\dot{y}(t) = -\alpha\{1 - \|y(t)\|^2\}R(\theta)\xi, \quad y(0) = \xi,
\]
where \(\|\xi\| = \rho_0\). Proposition 2.3 asserts that the solution \(y(t;0,\xi)\) of (3.2) satisfies
\[
\|y(1;0,\xi)\| < \|\xi\| = \rho_0,
\]
because \(\alpha < f(\|\xi\|)\). Since the set \(S = \{\xi \in \mathbb{R}^2 : \|\xi\| = \rho_0\}\) is compact, continuous dependence of solutions on their initial values shows
\[
\sup\{\|y(1;0,\xi)\| : \xi \in S\} < \rho_0.
\]
Hence there exist a positive \(\varepsilon\) and an integer \(K\) such that \(n \geq K\) implies
\[
\|x(n+1)\| < \rho_0 - \varepsilon,
\]
because \(\|x(n)\| \to \rho_0\) as \(n \to \infty\). This contradicts (3.1). Therefore we arrive at \(\rho_0 = 0\), and so \(\|x(n)\|\) tends to 0 as \(n \to \infty\). Thus we conclude from Remark 2.1 that \(x(t)\) approaches the origin as \(t \to \infty\). Next, we choose \(\phi\) so that
\[
(1 + \alpha)\|\phi\| < 1,
\]
where \(\|\phi\| = \sup\{\|\phi(s)\| : -1 \leq s \leq 0\}\). Then it follows from Proposition 2.2 that
\[
\|x(t)\| < 1
\]
on \([t_0, \infty)\). Hence (1.2) implies that
\[
\|x(t)\| \leq \|x(t_0)\| + \alpha(t - t_0)\{1 - \|x(t)\|^2\}\|x([t_0])\| < (1 + \alpha)\|\phi\|
\]
for \(t \in [t_0, N]\). In particular,
\[
\|x(N)\| < (1 + \alpha)\|\phi\|.
\]
Since the sequence \(\{\|x(n)\|\}\) is strictly decreasing in \(n\), we have from Remark 2.1 that
\[
\|x(t)\| < \|x(N)\| < (1 + \alpha)\|\phi\|
for $t \geq N$, and hence
\[ \|\mathbf{x}(t)\| < (1 + \alpha)\|\phi\| \]
for $t \geq t_0$. This shows that the zero solution is stable. Thus the proof is completed. \(\square\)

Next, we shall give a sufficient condition for (1.2) to possess star-shaped periodic solutions. This result is a consequence of the following proposition.

**Proposition 3.1.** Assume that $\alpha > 2\cos\theta$ and $\theta/\pi \in \mathbb{Q}$, and let $\mathbf{x}(t)$ be a solution of (1.2) satisfying $f(\|\mathbf{x}(N)\|) = \alpha$. Then there exists a positive integer $m$ such that
\[ \mathbf{x}(t + m) = \mathbf{x}(t) \]
(3.3)
for $t \geq N$.

**Proof.** Since domain of $f$ is the interval $(0,1)$, it follows that $0 < \|\mathbf{x}(N)\| < 1$. Then Proposition 2.3 and its proof show
\[ u(N + 1) = \|\mathbf{x}(N)\| \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = R(2\theta - \pi)u(N) \]
or
\[ \mathbf{x}(N + 1) = R(2\theta - \pi)\mathbf{x}(N), \]
and of course $f(\|\mathbf{x}(N + 1)\|) = \alpha$. Repeating this argument, we have
\[ \mathbf{x}(N + n) = R(n(2\theta - \pi))\mathbf{x}(N) \]
for any positive integer $n$. Hence Lemma 2.2 ensures the existence of a positive integer $m$ such that
\[ \mathbf{x}(N + m) = \mathbf{x}(N). \]
(3.4)
Since the system (1.2) is autonomous, we then arrive at the conclusion that
\[ \mathbf{x}(t + m) = \mathbf{x}(t) \]
for $t \geq N$. This completes the proof. \(\square\)

**Theorem 3.2.** Assume that $\alpha > 2\cos\theta$ and $\theta/\pi \in \mathbb{Q}$. Then there exist star-shaped periodic solutions of (1.2).

**Proof.** It follows from Lemma 2.1 that there exists one and only one $\rho \in (0,1)$ satisfying $\alpha = f(\rho)$. Put $\sigma = t_0 - \lfloor t_0 \rfloor$, and choose $\phi \in C$ so that
\[ \phi(s) = \begin{pmatrix} \beta + \rho \cos\theta - (\beta - \rho \cos\theta)e^{2\rho(s+1)} \\ \beta + \rho \cos\theta + (\beta - \rho \cos\theta)e^{2\rho(s+1)} \\ \rho \sin\theta \end{pmatrix} \]
or
\[
\frac{\beta - \phi_u(s)}{\beta + \phi_u(s)} = \frac{\beta - \rho \cos \theta}{\beta + \rho \cos \theta} e^{2\alpha \rho \beta(s+1)}, \quad \phi_v(s) = -\rho \sin \theta
\]
for \(s \in [-1, 0]\), where \(\phi(s) = \begin{pmatrix} \phi_u(s) \\ \phi_v(s) \end{pmatrix}\) and \(\beta = \sqrt{1 - \rho^2 \sin^2 \theta}\). Then it is easy to see that \(\phi_u(-1) = \rho \cos \theta, \phi_u(0) = -\rho \cos \theta\) and
\[
\dot{\phi}_u(s) = -\alpha \rho \{\beta^2 - \phi_u(s)^2\},
\]
which implies
\[
\dot{\phi}(s) = -\alpha \{1 - ||\phi(s)||^2\} \begin{pmatrix} \rho \\ 0 \end{pmatrix}
\]
for \(s \in [-1, 0]\). So, define \(\psi \in C\) by
\[
\psi(s) = \begin{cases} 
\phi(s + \sigma - 1) & , -\sigma \leq s \leq 0 \\
R(\pi - 2\theta)\phi(s + \sigma) & , -1 \leq s \leq -\sigma.
\end{cases}
\]
Then the function \(\psi\) fulfills
\[
\dot{\psi}(s) = -\alpha \{1 - ||\psi(s)||^2\} \begin{pmatrix} \rho \\ 0 \end{pmatrix} \tag{3.5}
\]
for \(s \in [-\sigma, 0]\), and
\[
\dot{\psi}(s) = -\alpha \{1 - ||\psi(s)||^2\} R(\pi - 2\theta) \begin{pmatrix} \rho \\ 0 \end{pmatrix} \tag{3.6}
\]
for \(s \in [-1, -\sigma]\). Now, let \(x(t)\) be the solution of (1.2) with the initial condition
\[
x(t_0 + s) = R(\theta)\psi(s) \text{ on } [-1, 0]. \tag{3.7}
\]
And, consider the case of \(t_0 \notin \mathbb{Z}\). Then \([t_0] = N - 1 < t_0 < N\) and it follows from (3.4) that
\[
R(\pi - 2\theta)x(N + m) = R(\pi - 2\theta)x(N)
\]
and so
\[
x([t_0] + m) = x([t_0]).
\]
Thus \(x(t)\) fulfills
\[
\dot{x}(t) = -\alpha \{1 - ||x(t)||^2\} R(\theta)x([t_0]) \tag{3.8}
\]
for \([t_0] + m \leq t < N + m\). Furthermore (3.7) implies
\[
x([t_0]) = R(\theta)\psi(-\sigma) = R(\theta)\phi(-1) = \begin{pmatrix} \rho \\ 0 \end{pmatrix}.
\]
On the other hand, by (3.7), the equality (3.5) becomes (3.8) for \([t_0] \leq t < t_0\). Hence \(x(t)\) fulfills (3.8) on \([|t_0|, N]\). By uniqueness of solutions for (3.8), we can conclude that (3.3) holds on \([|t_0|, N]\). Similarly, it follows from (3.6) and the equality
\[
x([t_0] - 1 + m) = R(\pi - 2\theta)x([t_0])
\]
that (3.3) holds on \([t_0 - 1, [t_0]]\). Therefore, by Proposition 3.1, we arrive at the conclusion that (3.3) holds for all \(t \geq t_0 - 1\). Next, consider the case of \(t_0 \in \mathbb{Z}\). Since \(t_0 = N\), (3.4) implies
\[
x(t_0 - 1 + m) = R(\pi - 2\theta)x(t_0 + m) = R(\pi - 2\theta)x(t_0).
\]
On the other hand, (3.6) becomes
\[
x(t) = -\alpha\{1 - ||x(t)||^2\}R(\theta)R(\pi - 2\theta)x(t_0).
\]
By uniqueness of solutions, we arrive again at the conclusion that (3.3) holds for all \(t \geq t_0 - 1\). This shows that \(x(t)\) is a periodic solution, more precisely a star-shaped periodic solution. Moreover, for any \(\varphi \in (0, 2\pi)\), the solution of (1.2) with the initial condition
\[
x(t_0 + s) = R(\theta + \varphi)\psi(s) \quad \text{on } [-1, 0]
\]
is also periodic. Thus the proof is now completed.

In the case that \(\alpha > 2\cos \theta\) and \(\theta/\pi\) is irrational, the system (1.2) does not possess nontrivial periodic solutions. But we obtain a similar result to Theorem 3.4 in [2].

**Theorem 3.3.** Assume that \(\alpha > 2\cos \theta\) and \(\theta/\pi \notin \mathbb{Q}\), and let \(x(t)\) be a solution of (1.2) with \(f(||x(N)||) = \alpha\). Then the trajectory of \(x(t)\) for \(t \geq N\) is everywhere dense on the closed annular region \(\{\xi \in \mathbb{R}^2 : ||x(N)|| \cdot |\sin \theta| \leq ||\xi|| \leq ||x(N)||\}\).

The proof of this theorem is analogous to one of Theorem 3.4 in [2], and so it is omitted.

Finally we describe a result which is more precise than Proposition 2.2 (d).

**Theorem 3.4.** Any solution \(x(t)\) of (1.2) with \(||x(t_0)|| > 1\) possesses a finite escape time \(T\), that is, \(||x(t)|| \to \infty\) as \(t \to T - 0\).

**Proof.** Suppose \(x(t)\) exists in the future. Then it follows from Proposition 2.2 (d) that \(||u(t)|| > 1\) and so
\[
\dot{u}(t) > 0 \quad \text{on } [n, n + 1)
\]
for each $n \geq N$, where $u(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ is the function determined by (2.1) and (2.2).

Since $u(n) > 0$, $\|u(t)\|$ is strictly increasing in $t$ and hence

$$\|u(t)\| \geq \rho_N \text{ on } [n, n+1),$$

where $\rho_N = \|\mathbf{x}(N)\|$. This implies

$$\dot{u}(t) \geq \alpha \rho_n (\rho_n^2 - 1),$$

so that

$$u(t) \geq \alpha \rho_n (\rho_n^2 - 1)(n - N)$$

on $[n, n+1]$ for each $n \geq N$. Thus we conclude that

$$\|\mathbf{x}(t)\| \to \infty \text{ as } t \to \infty. \tag{3.9}$$

Now, consider the case of $\theta \neq 0$. Then there exists a positive $\rho^*$ such that $\rho > \rho^*$ implies

$$\alpha \rho (\rho^2 \sin^2 \theta - 1) > \pi. \tag{3.10}$$

On the other hand, according to the quadrature, we have from Proposition 2.1 (c) that

$$\tan^{-1} \frac{u(t)}{\delta_n} = \tan^{-1} \frac{\rho_n \cos \theta}{\delta_n} + \alpha \rho_n \delta_n (t - n) > \alpha \rho_n \delta_n (t - n)$$

for $t \in [n, n+1)$, where $\rho_n = \|\mathbf{x}(n)\|$ and $\delta_n = \rho_n^2 \sin^2 \theta - 1$. But (3.9) implies that $\rho_n > \rho^*$ for $n$ large enough. Hence (3.10) shows that for such an integer $n$, the inequality

$$\tan^{-1} \frac{u(n + \frac{1}{2})}{\delta_n} > \frac{\pi}{2}$$

holds, which is a contradiction. Therefore our supposition is false in the case of $\theta \neq 0$.

Next, consider the case of $\theta = 0$. In this case, (c) in Proposition 2.1 becomes

$$\dot{u}(t) = \alpha \rho_n \{u(t)^2 - 1\}, \quad \dot{v}(t) = 0.$$

According to the quadrature again, we have

$$\frac{u(t) - 1}{u(t) + 1} = \frac{\rho_n - 1}{\rho_n + 1} e^{2 \alpha \rho_n (t - n)} \tag{3.11}$$

on each interval $[n, n+1)$, because $u(t) > 1$ on $[n, n+1)$. But (3.9) implies that the inequality

$$\frac{\rho_n - 1}{\rho_n + 1} e^{\alpha \rho_n} > 1$$
holds for $n$ large enough. Hence it follows from (3.11) that

$$u(n + \frac{1}{2}) - 1 > u(n + \frac{1}{2}) + 1$$

for $n$ above, which is a contradiction. Therefore the solution possesses a finite escape time. This completes the proof. \hspace{1cm} \square

4. Numerical examples

The following figures are some orbits of (2.1) which illustrate Theorems 3.1–3.3.

![Fig. 1. $\alpha = 1.800 < 2 \cos \theta$](image1)

$\theta = \frac{\pi}{7}$, $t_0 = 0$, $\phi(t) = \begin{pmatrix} 0.2 \\ 0 \end{pmatrix}$

![Fig. 2A. $\alpha = 1.805 > 2 \cos \theta$](image2)

$\theta = \frac{\pi}{7}$, $t_0 = 0$, $\phi(t) = \begin{pmatrix} 0.2 \\ 0 \end{pmatrix}$; $t \geq 600$

![Fig. 2B. $\alpha = 1.805 > 2 \cos \theta$](image3)
Fig. 3. \( \alpha = 8.211 \)
\[ \theta = \frac{\pi}{7}, \ t_0 = 0, \ \phi(t) = \begin{pmatrix} 0.999 \\ 0 \end{pmatrix} \]

Fig. 4. \( \alpha = 8.193 \)
\[ \theta = \frac{\pi}{7.1}, \ t_0 = 0, \ \phi(t) = \begin{pmatrix} 0.999 \\ 0 \end{pmatrix} \]

Fig. 5. \( \alpha = 8.198 \)
\[ \theta = \frac{\pi}{\sqrt{50}}, \ t_0 = 0, \ \phi(t) = \begin{pmatrix} 0.999 \\ 0 \end{pmatrix} \]

References
