THE UNIQUE EXISTENCE OF A CLOSED ORBIT OF FITZHUGH–NAGUMO SYSTEM

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1. Introduction

Our purpose is to give a parameter range for which the ordinary differential equations governing the FitzHugh–Nagumo system have a unique non-trivial closed orbit. It is wider than those already known.

To explicate the ion mechanism for the excitation and the conduction of nerve Hodgkin and Huxley ([H–H]) developed the system of four nonlinear ordinary differential equations as a model of nerve conduction in the squid giant axon (Loligo). R. FitzHugh ([Fi]) and J. Nagumo et al. ([Na]) simplified the system by introducing the following two dimensional autonomous system of ordinary differential equations:

\[
\begin{align*}
\dot{w} &= v - \frac{1}{3} w^3 + w + I \\
\dot{v} &= \rho(a - w - bv),
\end{align*}
\]

(FHN)

where the dot (\(\cdot\)) denotes differentiation and \(a, \rho, b\) are real constants such that

\[a \in \mathbb{R}, \quad \rho > 0, \quad 0 < b < 1.\]

The variable \(w\) corresponds to the potential difference through the axon membrane and \(v\) represents the potassium activation. The quantity \(I\) is the current through the membrane. The system (FHN) for special values of \(I\) has been investigated by using numerical methods and phase space analysis in [Fi] or [Na].

The system (FHN) has a unique equilibrium point \((x_I, y_I)\) for each \(I \in \mathbb{R}\), where

\[
x_I = \sqrt[3]{\frac{3(I + a/b) + \sqrt{9(I + a/b)^2 + 4(1/b - 1)^3}}{2}} + \frac{\sqrt[3]{3(I + a/b) - \sqrt{9(I + a/b)^2 + 4(1/b - 1)^3}}}{2}
\]
and
\[ y_I = (a - x_I)/b. \]
Instead of the parameter $I$ we introduce a new parameter $\eta$. By the transformation $\eta = x_I$, $x = w - \eta$ and $y = v - a/b + \eta/b + \rho b (w - \eta)$, the system (FHN) is transformed to the following system:
\[
\begin{cases}
\dot{x} = y - \left( \frac{1}{3} x^3 + \eta x^2 + (\eta^2 + \rho b - 1) x \right) \\
\dot{y} = -\frac{\rho b}{3} \left( x^3 + 3 \eta x^2 + 3(\eta^2 + \frac{1}{b} - 1) x \right)
\end{cases}
\] (FNS)

The system (FNS) is called the FitzHugh nerve system and has a unique equilibrium point $E(0, 0)$. We shall give some results for the system (FNS) is equivalent to the system (FHN).

Note that, if $\rho b \geq 1$, then the system (FNS) has no non-trivial closed orbits. Thus instead of [C1], we can assume the condition
\[ [C2] \quad a \in \mathbb{R}, \quad 0 < b < 1, \quad 0 < \rho < 1/b. \]

It was studied in such papers as [H1], [K-S] and [Su] for the system (FNS) with the condition [C2]. Let $\eta_0 = \sqrt{1 - \rho b}$. The following is our main result.

**Theorem.** The system (FNS) has a unique non-trivial closed orbit if $\eta^2 < \eta_0^2$.

This result improves those given in [K-S] and [H1]. In fact, the result that 'If either $\eta^2 \leq 4^{-1} \eta_0^2$ or $\{\rho b^2 - 7b + 6 < 0$ and $\eta^2 < \eta_0^2\}$, then the system (FNS) has a unique non-trivial closed orbit' was given in [K-S]. In [H1] the result that 'There is a positive constant $\eta_1 \leq \eta_0$ such that the system (FNS) has a unique non-trivial closed orbit for $|\eta| \leq \eta_1$' was given. Therefore the result of the above theorem is clearly stronger than those in [K-S] and [H1].

2. **Lemmas**

In this section we prepare some lemmas to be used in the next section to prove the Theorem. We consider the Liénard system of the following form
\[
\begin{cases}
\dot{x} = y - F(x) \\
\dot{y} = -g(x)
\end{cases}
\] (1)
where $F$ is continuously differentiable and $g$ is continuous. We assume the following conditions for the system (1):

[C3] $xg(x) > 0$ if $x \neq 0$,

[C4] There exist $a_1 < 0$ and $a_2 > 0$ such that $F(a_1) = F(a_2) = 0$, 

$xF(x) \leq 0$ for $a_1 < x < a_2$, $F(x)$ is nondecreasing for $x < a_1$ and $x > a_2$,

[C5] $\lim_{x \to \pm \infty} \int_0^x \{F'(\xi) + |g(\xi)|\} \, d\xi = \pm \infty$.

To prove the Theorem we shall use the following three lemmas.

**Lemma 1.** Assume that the system (1) satisfies the conditions [C3], [C4], [C5] and besides

[C6] $G(a_1) > G(a_2)$ and there exists a constant $\alpha \geq 0$ such that $\frac{F(x)}{G^\alpha(x)}$ is nondecreasing for $x \in (a_1, x_1) \cup (a_2, +\infty)$; moreover, there exists a constant $\delta > 0$ such that $\frac{F(x)}{G^\alpha(x)}$ is strictly increasing in $x$ with $0 < |x| < \delta$,

where $G(x) = \int_0^x g(\xi) \, d\xi$ and $x_1 < 0$ is a number satisfying the equation $G(a_2) = G(x_1)$.

Then the system (1) has a unique non-trivial closed orbit.

**Proof of Lemma 1.** Under the conditions [C3], [C4] and [C5] the system (1) has at least one non-trivial closed orbit. See [H1]. Moreover, by [Ze], under the conditions [C3], [C4] and [C6] the system (1) has at most one non-trivial closed orbit. \[\square\]

We can assume that the condition $\eta^2 < \eta_0^2$ in the Theorem holds with $\eta \geq 0$. The proof for the case $\eta < 0$ is essentially the same.

**Lemma 2.** Let

$$\Gamma(x) = \{2\eta_0^2 + 3(\frac{1}{b} - 1)\}x^2 + \eta(2(\eta_0^2 - \eta^2) + 3(\eta_0^2 + \frac{1}{b} - 1))x$$

$$- 3(\eta^2 - \eta_0^2)(\eta^2 + \frac{1}{b} - 1).$$

Then $\Gamma(\epsilon(\eta - \eta_0)) > 0$ if $\eta^2 < \eta_0^2$, where $\epsilon = \frac{3(\eta + \eta_0)(\eta^2 + \frac{1}{b} - 1)}{2\eta_0\{2(\eta_0^2 - \eta^2) + 3(\eta_0^2 + \frac{1}{b} - 1)\}}$. 

Proof of Lemma 2. If $\eta^2 < \eta_0^2$, we have

$$
\Gamma(\epsilon(\eta - \eta_0)) = \epsilon^2 (\eta - \eta_0)^2 \{2\eta_0^2 + 3(\frac{1}{b} - 1)\}
+ \epsilon \eta \{2(\eta_0^2 - \eta^2) + 3(\eta_0 + \frac{1}{b} - 1)\}(\eta - \eta_0) - 3(\eta^2 - \eta_0^2)(\eta^2 + \frac{1}{b} - 1)
= \epsilon^2 (\eta - \eta_0)^2 \{2\eta_0^2 + 3(\frac{1}{b} - 1)\} + 3(\frac{\eta}{2\eta_0} - 1)(\eta^2 - \eta_0^2)(\eta^2 + \frac{1}{b} - 1) > 0.
\square
$$

Lemma 3. Let

$$
g(x) = \frac{\rho b}{3} \left\{ x^3 + 3\eta x^2 + 3(\eta^2 + \frac{1}{b} - 1)x \right\} \text{ and } G(x) = \int_0^x g(\xi)d\xi.
$$

Then $G(a) - G(-a) \geq 0$ for every $a > 0$.

Proof of Lemma 3. Since $G(a) - G(-a) = \frac{2}{3}\rho b\eta a^3 \geq 0$, the proof is completed. \square

3. Proof of Theorem

We shall prove the Theorem by using the above three lemmas. We set $F(x) = \frac{1}{3} x^3 + \eta x^2 + (\eta^2 - \eta_0^2)x$. Then, if $\eta^2 < \eta_0^2$, we see easily that the system (FNS) satisfies the conditions [C3], [C4] and [C5]. We shall check the condition [C6] in Lemma 1. We have for $\eta^2 < \eta_0^2$

$$
a_1 = \frac{-3\eta - \sqrt{12\eta_0^2 - 3\eta^2}}{2} < 0 \quad \text{and} \quad a_2 = \frac{-3\eta + \sqrt{12\eta_0^2 - 3\eta^2}}{2} > 0.
$$

Then we get

$$
G(a_1) - G(a_2) = \frac{\rho b}{4} \{ \eta^3 + 2\eta_0^2 + 6(\frac{1}{b} - 1)\}\sqrt{12\eta_0^2 - 3\eta^2} > 0.
$$

If $\eta^2 < \eta_0^2$, since $0 < \epsilon < 1$, we have $a_2 > \frac{3}{2}(\eta_0 - \eta) > \epsilon(\eta_0 - \eta)$. Let $x_1 < 0$ be a number satisfying the equation $G(a_2) = G(x_1)$. From the above fact, the monotonicity of $G$ and Lemma 3, it follows that

$$
G(x_1) = G(a_2) > G(\epsilon(\eta_0 - \eta)) \geq G(\epsilon(\eta - \eta_0)).
$$

Using the fact that $a_1 < x_1 < \epsilon(\eta - \eta_0) < 0$, we shall show that $F(x)/G^\alpha(x)$ is nondecreasing for $x \in (a_1, \epsilon(\eta - \eta_0)) \cup (a_2, +\infty)$. This means that $F'(x)G(x) - \alpha F(x)g(x) \geq 0$ for $x \in (a_1, \epsilon(\eta - \eta_0)) \cup (a_2, +\infty)$.
From the calculation in [H1] we see that the above claim means that

$$
\Phi(x, \alpha) = (3 - 4\alpha)x^4 + 6\eta(3 - 4\alpha)x^3 + 3\left\{5(3 - 4\alpha)\eta^2 - (1 - 4\alpha)\eta_0^2 + 2(3 - 2\alpha)\frac{1}{b} - 1\right\}x^2 \\
+ 12\eta\left\{2(2 - 3\alpha)\eta^2 - (1 - 3\alpha)\eta_0^2 + 3(1 - \alpha)\frac{1}{b} - 1\right\}x + 18(\eta^2 - \eta_0^2)(\eta^2 + \frac{1}{b} - 1)(1 - 2\alpha)
$$

\[ \geq 0 \]

for \( x \in (a_1, \epsilon(\eta - \eta_0)) \cup (a_2, +\infty) \).

Let \( \alpha = \frac{3}{4} \). Thus we get the following expression which is of degree 2 in \( x \).

$$
\Phi(x, \frac{3}{4})
$$

\[ = 3 \left\{2\eta_0^2 + 3\frac{1}{b} - 1\right\}x^2 + \eta\left\{2(\eta_0^2 - \eta^2) + 3(\eta_0^2 + \frac{1}{b} - 1)\right\}x - 3(\eta^2 - \eta_0^2)(\eta^2 + \frac{1}{b} - 1) \]

\[ = 3\Gamma(x). \]

If \( \eta^2 < \eta_0^2 \), from the fact that \( \Gamma \) is a function of the degree 2, the inequality \( \Gamma(0) > 0 \) and Lemma 2, we conclude that \( \Phi(x, \frac{3}{4}) \geq 0 \) for \( x \in (a_1, \epsilon(\eta - \eta_0)) \cup (a_2, +\infty) \). Therefore the condition [C6] in Lemma 1 is satisfied. \( \square \)

4. A numerical example

We shall present the phase portrait of the following system as an example illustrating the application of the Theorem. We consider the system (FNS) with \( b = 1/2 \), \( \rho = 1 \) and \( \eta^2 = 3/8 \):

$$
\left\{ \begin{array}{l}
\dot{x} = y - \left(\frac{1}{3}x^3 + \frac{\sqrt{6}}{4}x^2 - \frac{1}{8}x\right) \\
\dot{y} = -\frac{1}{6}\left(x^3 + \frac{3\sqrt{6}}{4}x^2 + \frac{33}{8}x\right)
\end{array} \right. \tag{2}
$$

In this case, since \( \eta_0^2 = 1 - \rho b = 1/2 > \eta^2 \), the system (2) satisfies the condition in the Theorem. Thus we see that the system (2) has a unique non-trivial closed orbit (see the Figure below). We note that this system does not satisfy the condition in [H1] nor that of [K-S], either.
5. Appendix

Recently, in [H–T] the result that the system (FNS) has a unique non-trivial closed orbit if $\eta^2 = \eta_0^2 > \frac{1}{b} - 1$ was given by using some Lyapunov-type functions.

In [Su] it was shown that the system (FNS) has no non-trivial closed orbits if it satisfies the condition

$$\eta^2 \geq \eta_0^2 \text{ and } \eta^4 - 4\eta^2\eta_0^2 + \eta_0^4 + 2\left(\frac{1}{2} - 1\right)\eta^2 - 4\left(\frac{1}{b} - 1\right)\eta_0^2 + 4\left(\frac{1}{b} - 1\right)^2 \geq 0$$

or

$$2\{\eta_0^2 + \left(\frac{1}{b} - 1\right)\}^3 < \eta^2 \left(\eta^2 + 3\left(\frac{1}{b} - 1\right)\right)^2.$$

We do not know yet what happens in the region in the $(\eta, \eta_0)$-plane in which $\eta^2 > \eta_0^2$, but the condition of [Su] is not satisfied. But some numerical experiments
tell us that the system may have two non-trivial closed orbits if \((\eta, \eta_0)\) is in the above mentioned region. Thus we have a conjecture:

'The system (FNS) has either exactly two non-trivial closed orbits or no non-trivial closed orbits in the above mentioned region.'

REFERENCES


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