<table>
<thead>
<tr>
<th>Title</th>
<th>Finite Element Analysis for Parametrized Nonlinear Equations around Turning Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Tsuchiya, Takuya</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1999), 1084: 111-123</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1999-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62781">http://hdl.handle.net/2433/62781</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
Finite Element Analysis for Parametrized Nonlinear Equations around Turning Points

Abstract. Nonlinear equations with parameters are called parametrized nonlinear equations. In this paper, a priori error estimates of finite element solutions of parametrized nonlinear elliptic equations on branches around turning points are considered. Existence of a finite element solution branch is shown under suitable conditions on an exact solution branch around a turning point. Also, some error estimates of distance between exact and finite element solution branches are given. It is shown that error of a parameter is much smaller than that of functions. Approximation of nondegenerate turning points is also considered. We show that if a turning point is nondegenerate, there exists a locally unique finite element nondegenerate turning point. At a nondegenerate turning point an elaborate error estimate of the parameter is proved.

1. Introduction.

Let $A$, $B$ be Banach spaces and $\Lambda \subset \mathbb{R}^n$ a bounded interval. Let $F : \Lambda \times A \rightarrow B$ be a smooth operator. The nonlinear equations

$$F(\lambda, u) = 0,$$

with parameter $\lambda \in \Lambda$ is called parametrized nonlinear equations.

In [17] and [18] a thorough theory of a priori error estimates of finite element solutions of the following parametrized strongly nonlinear problems has been developed:

$$F(\lambda, u) = 0, \quad (\lambda, u) \in \Lambda \times H^1_0(\Omega),$$

(1.1)

$$\langle F(\lambda, u), v \rangle := \int_{\Omega} \left[ \bar{a}(\lambda, x, u(x), \nabla u(x)) \cdot \nabla v(x) 
+ f(\lambda, x, u(x), \nabla v(x))v(x) \right] dx, \quad \forall v \in H^1_0(\Omega),$$

where

$$\bar{a}(\lambda, x, u(x), \nabla u(x)) = a(\lambda, x, u(x), \nabla u(x)) + \gamma(\lambda, x) \nabla u(x).$$
where $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) is a bounded domain with the piecewise $C^2$ boundary $\partial \Omega$, and $\tilde{a} : \Lambda \times \overline{\Omega} \times \mathbb{R}^{d+1} \to \mathbb{R}^d$, $f : \Lambda \times \overline{\Omega} \times \mathbb{R}^{d+1} \to \mathbb{R}$ are sufficiently smooth functions. Here, the equation (1.1) is called strongly nonlinear if $\tilde{a}(\lambda, x, y, z)$ ($\lambda \in \Lambda$, $x \in \Omega$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$) is nonlinear with respect to $z$. Otherwise, it is called mildly nonlinear.

Since the equation (1.1) is defined in divergence form, finite element solutions to (1.1) is defined in a natural way.

In [8], [9], and [13] Fink and Rheinboldt have shown that some subset of the solutions to (1.1) form an one-dimensional smooth manifold without boundaries, if the nonlinear operator defined by (1.1) is Fréchet differentiable and Fredholm of index 1. They have also shown that corresponding finite element solutions form an one-dimensional smooth manifold. In this paper we denote by $\mathcal{M}_0$ and $\mathcal{M}_h$ the exact solution manifold of (1.1) and the corresponding finite element solution manifold, respectively.

Here, a linear operator $P \in \mathcal{L}(A, B)$ is called Fredholm if (1) the dimension of Ker$P$ is finite, (2) Im$A \subset B$ is closed, (3) the dimension of Coker$A := B/\text{Im}A$ is finite. If $P \in \mathcal{L}(A, B)$ is Fredholm, its index ind$P$ is defined by ind$P := \dim\text{Ker}P - \dim\text{Coker}P$. Let $U \subset A$ be open and $F : U \to B$ Fréchet differentiable. $F$ is called Fredholm in $U$ if its Fréchet derivative $DF(u) \in \mathcal{L}(A, B)$ is Fredholm at any $u \in U$. It is shown that ind$DF(u)$ is constant in each connected component of $U$. Hence, we define the index of $F$ by ind$F := \text{ind}DF(u)$.

In [17] and [18], it is shown that, under reasonable conditions, for each compact subset $\overline{\mathcal{M}}_0 \subset \mathcal{M}_0$, there exists a locally unique compact subset $\overline{\mathcal{M}}_h \subset \mathcal{M}_h$ such that $\overline{\mathcal{M}}_0$ is approximated uniformly by $\overline{\mathcal{M}}_h$, if triangulation of $\Omega$ is sufficiently fine. Moreover, several a priori error estimates are obtained.

The aim of this paper is to refine the error analysis on branches around turning points. A point $(\lambda, u) \in \mathcal{M}_0$ is called a turning point if the partial Fréchet derivative $D_u F(\lambda, u) \in \mathcal{L}(A, B)$ at $(\lambda, u)$ is not an isomorphism.

To develop a refined error analysis around a turning point, we introduce a slightly different formulation of the problem from that in [17], and show a theorem which is similar to [18, Theorem 8.6] and [17, Corollary 7.8]. Next, we obtain an elaborate error estimate of parameter. In the following we explain the basic ideas of this paper.

In the error analysis of parametrized nonlinear equations, we have the following difficulty. Suppose that we are approaching a turning point during continuation process of a solution branch. Since we cannot fix the parameter $\lambda$ around a turning point in (1.1), $\lambda$ should be treated as an unknown parameter. Hence, correspondence of an approximated solution to an exact solution becomes ambiguous in such a situation.
Usually, this difficulty is overcome by the following manner. We introduce a (nonlinear, in general) functional $\rho : \Lambda \times A \rightarrow \mathbb{R}$, and consider the following problem:

\begin{equation}
(1.2) \quad H(\gamma, \lambda, u) := (\rho(\lambda, u) - \gamma, F(\lambda, u)) = (0, 0) \in \mathbb{R} \times A,
\end{equation}

where $H : \mathbb{R} \times \Lambda \times A \rightarrow \mathbb{R} \times B$. We expect that the partial Frechet derivative $D_{(\lambda, u)}H(\gamma, \lambda, u) \in \mathcal{C}(\mathbb{R} \times A, \mathbb{R} \times B)$ is an isomorphism at a turning point $(\lambda, u)$ and in its neighborhood. In Section 2, it will be shown that, if $D_{\lambda}F(\lambda, u) \neq 0$ and $\ker D_{\lambda}F(\lambda, u) \cap \ker D_{\rho}(\lambda, u) = \{(0, 0)\}$ at $(\lambda, u) \in \mathcal{M}_0$, then the above partial Frechet derivative is an isomorphism. If we could find a good definition of such $\rho$, then the solution branch would now be parametrized by $\gamma$.

Finite element solutions $(\lambda_h, u_h)$ would be defined by

\begin{equation}
(1.3) \quad H_h(\gamma, \lambda, u) := (\rho(\lambda_h, u_h) - \gamma, F_h(\lambda_h, u_h)) = (0, 0),
\end{equation}

where $F_h$ is an approximation of $F$. In this setting the correspondence of an exact solution $(\lambda, u)$ and an finite element solution $(\lambda_h, u_h)$ is represented by $\rho(\lambda_h, u_h) = \gamma = \rho(\lambda, u)$.

In the above setting we will show that, even around a turning point, there exists a locally unique finite element solution branch near an exact solution branch under suitable conditions. Also, some error estimates of distance between the exact and finite element solution branches are given.

Next, we will consider an elaborate error estimate of parameter $\lambda$. In error analysis of the finite element method (1.3) for (1.2) around a turning point, we would have error estimates such as

$$|\lambda - \lambda_h| + \|u - u_h\|_A \leq Ch^r.$$

In many practical computation, it is usually observed that the error $|\lambda - \lambda_h|$ is much smaller than $\|u - u_h\|_A$, or $Ch^r$.

A typical and well-known example of this phenomenon is finite element approximation of the eigenvalue problems:

\begin{equation}
(1.4) \quad -\Delta u = \lambda u, \quad u \in H_0^1(\Omega).
\end{equation}

Let $(\lambda, u)$ be an eigen-pair of (1.4) and $(\lambda_h, u_h)$ its finite element approximation. Suppose that the eigenvalue $\lambda$ is simple. Then we have an error estimate such as

$$|\lambda - \lambda_h| \leq C\|u - u_h\|_{H_0^2}^2,$$

where $C$ is a positive constant independent of $h$ (see, for example, [14, Chapter 6], [1]).

We will show that a similar estimate hold for the finite element solutions $(\lambda_h, u_h)$ of (1.3) under the condition that $D_{\lambda}F(\lambda, u)$ is self-adjoint. To obtain a similar estimate we introduce an auxiliary equation. Let $z$ and $z_h$ be the exact and finite element solutions to the auxiliary equation. We will show that the error $|\lambda - \lambda_h|$ is estimated as

$$|\lambda - \lambda_h| \leq C\|u - u_h\|_A(\|u - u_h\|_A + \|z - z_h\|_A)$$

around a turning point, where $C$ is a positive constant independent of $h$.

Occasionally, a turning point on the exact solution manifold $\mathcal{M}_0$ has a certain physical meaning, and, in such a case, computing its precise value will become important. If a turning point $(\lambda_0, u_0) \in \mathcal{M}_0$ is nondegenerate (see Section 3 for its definition), we can show that the
associated finite element solution manifold also has a locally unique nondegenerate turning point \((\lambda_0^\dagger, u_0^\dagger) \in \mathcal{M}_k\). The error \(|\lambda_0 - \lambda_0^\dagger|\) is estimated accurately by a similar manner as above.

In Section 2 and 3 we develop our theory in an abstract setting. Applications of the abstract theorems obtained in Section 2 and 3 to the strongly nonlinear elliptic boundary value problem (1.1) will be found in the original version of this paper.


In this section, we formulate our problem in an abstract setting, and show a theorem which claims existence of a locally unique solution branch of a discretized problem. The setting in this section is slightly different from that of [17].

For the stage of our analysis we first introduce functional spaces.

(A1) There are Banach spaces, \(V, W, \) and \(X_p (1 \leq p \leq \infty)\), where \(X_2\) is a Hilbert space, such that \(V \subset X_\infty \subset X_p (1 \leq p \leq \infty)\) and \(W \subset X'_1 \subset X_q' (1 \leq q \leq \infty)\). Here, \(X_q'\) is the dual space of \(X_q\). We suppose that all inclusions are continuous. We also suppose that \(X_r\) is dense in \(X_p\) if \(1 \leq p \leq r < \infty\).

Let \(F : \Lambda \times X_p \rightarrow X_q' (1/p + 1/q = 1)\) be a nonlinear map, where \(\Lambda \subset \mathbb{R}\) is an interval. We consider the parametrized nonlinear equation \(F(\lambda, u) = 0\). Since we will suppose that \(F\) is strongly nonlinear, the domain and the range should be taken carefully. In many cases, \(F\) is not Fréchet differentiable on \(\Lambda \times X_p, p < \infty\), and should be restricted to a certain subspace to make it differentiable.

We also need extensions and restrictions of the Fréchet derivatives \(DF(\lambda, v), D_vF(\lambda, v)\) etc. at \((\lambda, v)\). When we need to specify the domain of, say, \(D_vF(\lambda, v)\) clearly, we will write such as \(D_vF(\lambda, v) \in \mathcal{L}(P, Q)\). This means that \(D_vF(\lambda, v)\) now denotes its extension (or restriction) whose domain is \(P\) and range is in \(Q\).

Now, we take certain \(p \geq 2\) and \(q\) with \(1/p + 1/q = 1\), and fix them. We then assume the following:

(A2) The restriction of \(F\) to \(\Lambda \times X_\infty\), denoted by \(F\) again, is a Fréchet differentiable map from \(\Lambda \times X_\infty\) to \(X'_q\). For any \(\lambda \in \Lambda\) and \(v \in X_\infty\), the derivative \(DF(\lambda, v) \in \mathcal{L}(\mathbb{R} \times X_\infty, X_q')\) can be extended to \(DF(\lambda, v) \in \mathcal{L}(\mathbb{R} \times X_p, X_q')\) and it is locally Lipschitz continuous on \(\Lambda \times X_\infty\); i.e., for any bounded convex set \(\mathcal{O} \subset \Lambda \times X_\infty\) there exists a positive constant \(C_1(\mathcal{O})\) such that

\[
\|DF(\lambda_1, v) - DF(\lambda_2, w)\|_{\mathcal{L}(\mathbb{R} \times X_p, X_q')} \leq C_1(\mathcal{O})(|\lambda_1 - \lambda_2| + \|v - w\|_{X_\infty})
\]

for arbitrary \((\lambda_1, v), (\lambda_2, w) \in \mathcal{O}\).

(A3) We suppose that there exists an open subset \(S \subset \Lambda \times V\) in which \(F : S \rightarrow W\) is a Fredholm operator of index 1. We also suppose that, for each \((\lambda, u) \in S, DF(\lambda, u) \in \mathcal{L}(\mathbb{R} \times X_p, X_q')\) is a Fredholm operator of index 1 as well.

We define the subset \(\mathcal{R}(F, S) \subset S\) by

\[
\mathcal{R}(F, S) := \{(\lambda, u) \in S | DF(\lambda, u) \in \mathcal{L}(\mathbb{R} \times V, W) \text{ is onto}\}.
\]

The following lemma is valid:

**Lemma 2.1.** (1) For any \((\lambda, u) \in \mathcal{R}(F, S), \dim \ker D_uF(\lambda, u)\) is at most 1.
(2) For $(\lambda, u) \in \mathcal{R}(F, S)$, we have either

Case 1: $\ker D_u F(\lambda, u) = \{0\}$ and $D_\lambda F(\lambda, u) \in \text{Im}D_u F(\lambda, u)$, or

Case 2: $\dim \ker D_u F(\lambda, u) = 1$, and $D_\lambda F(\lambda, u) \not\in \text{Im}D_u F(\lambda, u)$. □

For the proof, see [18, Section 4].

We introduce a nonlinear functional $\rho : \Lambda \times X_p \to \mathbb{R}$ and assume that

(A4) The restriction of $\rho$ to $\Lambda \times X_\infty$, denoted by $\rho$ again, is Fréchet differentiable.

(A5) For $(\lambda, u) \in \Lambda \times X_\infty$, the Fréchet derivative $D\rho(\lambda, u) \in \mathcal{L}(\mathbb{R} \times X_\infty, \mathbb{R}) = \mathbb{R} \times X'_\infty$ can be extended to $D\rho(\lambda, u) \in \mathcal{L}(\mathbb{R} \times X_p, \mathbb{R})(= \mathbb{R} \times X'_p)$, and it is locally Lipschitz continuous on $\Lambda \times X_\infty$, i.e., for any bounded convex set $\mathcal{O} \subset \Lambda \times X_\infty$, there exists a positive constant $C_2(\mathcal{O})$ such that

$$\|D\rho(\lambda_1, v) - D\rho(\lambda_2, w)\|_{\mathbb{R} \times X'_p} \leq C_2(\mathcal{O})(|\lambda_1 - \lambda_2| + \|v - w\|_{X_\infty})$$

for any $(\lambda_1, v)$, $(\lambda_2, w) \in \mathcal{O}$.

(A6) Let $(\lambda, u) \in \mathcal{S}$ and $D_u F(\lambda, u) \in \mathcal{L}(X_p, X'_q)$. We suppose that if $D_u F(\lambda, u)\psi = f$ for $\psi \in X_p$ and $f \in W$, then $\psi \in V$.

**Lemma 2.2.** Assume that (A1)-(A6) are valid. Suppose that there is $(\lambda_0, u_0) \in \mathcal{R}(F, S)$ such that $D_\lambda F(\lambda_0, u_0) \not= 0 \in W$. From (A3), there exists $(\mu_0, \psi_0) \in \mathbb{R} \times V$ such that}$

$$\ker DF(\lambda_0, u_0) = \text{span}\{\mu_0, \psi_0\}. \quad \text{We assume that } D\rho(\lambda_0, u_0)(\mu_0, \psi_0) \not= 0 \in \mathbb{R}.$$

Define $G : \Lambda \times W \to \mathbb{R} \times V$ by $G(\lambda, u) := (\rho(\lambda, u) - \gamma, F(\lambda, u))$, where $\gamma \in \mathbb{R}$.

Then, $DG(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times W, \mathbb{R} \times V)$ is an isomorphism. Moreover, $DG(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times X_p, \mathbb{R} \times X'_p)$ is an isomorphism as well.

**Proof.** From the assumptions we find that $\ker DF(\lambda_0, u_0) \cap \ker D\rho(\lambda_0, u_0) = \{(0, 0)\}$. This implies that $\ker DG(\lambda_0, u_0)$ is trivial and $DG(\lambda_0, u_0)$ is one-to-one.

Since $DF(\lambda_0, u_0)$ is onto, for any $f \in W$, there is $(v, \varphi) \in \mathbb{R} \times V$ such that $DF(\lambda_0, u_0)(v, \varphi) = f$. Since $D\rho(\lambda_0, u_0)(\mu_0, \psi_0) \not= 0$, for any $t \in \mathbb{R}$, there is $\alpha \in \mathbb{R}$ such that $D\rho(\lambda_0, u_0)((v, \varphi) + \alpha(\mu_0, \psi_0)) = t$. This yields that $DG(\lambda_0, u_0)$ is onto. Therefore, $DG(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times V, \mathbb{R} \times W)$ is an isomorphism.

To show that $DG(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times X_p, \mathbb{R} \times X'_p)$ is an isomorphism, we first show that $DF(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times X_p, X'_p)$ is onto. Since $DF(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times X_p, X'_p)$ is Fredholm with index 1 by (A3), we only have to show that the dimension of $\ker DF(\lambda_0, u_0) \subset \mathbb{R} \times X_p$ is 1.

Let $(\mu, \psi) \in \mathbb{R} \times X_p$ be such that $DF(\lambda_0, u_0)(\mu, \psi) = 0 \in X'_p$. This is also written as $D_u F(\lambda_0, u_0)\psi = -\mu D_\lambda F(\lambda_0, u_0)$. Since $D_\lambda F(\lambda_0, u_0) \in W$ and (A6), we conclude that $\psi \in W$ and $\dim \ker(DF(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times X_p, X'_p)) = 1$.

Using this fact, we show that $DG(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times X_p, \mathbb{R} \times X'_p)$ is an isomorphism by the exactly same manner as above. □

**Corollary 2.3.** Assume that (A1)-(A6) are valid. Suppose that there exists $(\lambda_0, u_0) \in \mathcal{R}(F, S)$ such that $F(\lambda_0, u_0) = 0$, $\rho(\lambda_0, u_0) = \gamma_0$, and $D_\lambda F(\lambda_0, u_0) \not= 0$. Suppose also that $\ker DF(\lambda_0, u_0) \cap \ker D\rho(\lambda_0, u_0) = \{(0, 0)\}$. Define $H : \mathbb{R} \times \Lambda \times V \to \mathbb{R} \times W$ by $H(\gamma, \lambda, u) := (\rho(\lambda, u) - \gamma, F(\lambda, u))$.

Then, we have $H(\gamma_0, \lambda_0, u_0) = (0, 0)$ and $D_{(\lambda, u)}H(\gamma_0, \lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times V, \mathbb{R} \times W)$ is an isomorphism. Therefore, by the implicit function theorem, there exist a positive constant $\epsilon$ and a $C^1$ map $(\gamma_0 - \epsilon, \gamma_0 + \epsilon) \ni \gamma \mapsto (\gamma(\gamma), u(\gamma)) \in \Lambda \times V$ such that $(\gamma(\gamma_0), u(\gamma_0)) = (\lambda_0, u_0)$ and $H(\gamma(\gamma), u(\gamma)) = (0, 0)$ for any $\gamma$. That is, the solution manifold of the equation $F(\lambda, u) = 0$ is parametrized by $\gamma = \rho(\lambda, u)$ around $(\lambda_0, u_0)$. □
To define discretized solutions of $F(\lambda, u) = 0$, we introduce the finite-dimensional subspaces $S_h \subset X_\infty$ which are parametrized by $h$, $0 < h < 1$ with the following properties:

(A7) There exists a real $r \geq 0$ and a positive constant $C_3$ independent of $h$ such that

$$\|v_h\|_{X_\infty} \leq \frac{C_3}{h^r}\|v_h\|_{X_p}, \quad \forall v_h \in S_h.$$  

The relations of Banach spaces are depicted in the following:

$$\Lambda \times S_h \cap \Lambda \times V \subset \Lambda \times X_\infty \subset \mathbb{R} \times X_p \subset \mathbb{R} \times X_2$$

$$W \subset X'_1 \subset X'_q \subset X'_2$$

The finite element solution $(\lambda_h, u_h) \in \Lambda \times S_h$ is defined naturally by

$$(F(\lambda_h, u_h), v_h) = 0, \quad \forall v_h \in S_h,$$

where $\langle \cdot, \cdot \rangle$ is the duality pair of $X'_q$ and $X_2$. We derive an equivalent definition of the finite element solutions which is more convenient in the error analysis.

Let $Q \in \mathcal{L}(X_2, X'_2)$ be a self-adjoint operator, that is, $(Qv, u) = (Qu, v)$ for all $u, v \in X_2$. Suppose that there exists a positive constant $\alpha$ such that

$$(2.1) \quad (Qv, v) \geq \alpha\|v\|_{X_2}^2, \quad \forall v \in X_2.$$  

We define $\langle \cdot, \cdot \rangle_Q$ by $(u, v)_Q := (Qv, u)$. It is easy to show that $(\cdot, \cdot)_Q$ is an inner product and the norm $\|v\|_Q := (v, v)_Q^{1/2}$ is equivalent to the original norm $\|v\|_{X_2}$. It is also easy to show that $Q \in \mathcal{L}(X_2, X'_2)$ is an isomorphism.

We define the canonical projection $\tilde{P}_h : X_2 \to S_h$ by $(\psi - \tilde{P}_h\psi, v_h)_Q = 0$ for all $v_h \in S_h$. Obviously, we have that $(u, \tilde{P}_h v)_Q = (\tilde{P}_h u, v)_Q$ for all $u, v \in X_2$. As in [18, Section 6] it follows from the definitions that $(\lambda_h, u_h)$ is a finite element solution if and only if $(Q\tilde{P}_h Q^{-1}F(\lambda_h, u_h), v) = 0$ for all $v \in X_2$.

Following Fink and Rheinboldt ([8], [9], [13]) we define the approximation of $F(\lambda, u)$ by

$$(2.2) \quad F_h(\lambda, u) := (I - P_h)Q u + P_hF(\lambda, u), \quad P_h := Q\tilde{P}_h Q^{-1},$$

where $I$ is the identity of $X'_2$. It can be seen easily [13, Lemma 5.1] that $F_h(\lambda, u) = 0$ if and only if $u \in S_h$ and $(\lambda, u)$ is a finite element solution.

**Theorem 2.4.** Assume that (A1)-(A7) are valid. Suppose that there exists $(\lambda_0, u_0) \in \mathcal{R}(F, S)$ such that $F(\lambda_0, u_0) = 0$, $\rho(\lambda_0, u_0) = \gamma_0$, and $D\lambda F(\lambda_0, u_0) \neq 0$. Suppose also that $\text{Ker}DF(\lambda_0, u_0) \cap \text{Ker}D\rho(\lambda_0, u_0) = \{(0, 0)\}$. Then, by Corollary 2.3, there exist a positive constant $\varepsilon_0$ and a $C^1$ map $[\gamma_0 - \varepsilon_0, \gamma_0 + \varepsilon_0] \ni \gamma \mapsto (\lambda(\gamma), u(\gamma)) \in \Lambda \times V$ such that $(\lambda(\gamma_0), u(\gamma_0)) = (\lambda_0, u_0)$, $\gamma = \rho(\lambda(\gamma), u(\gamma))$, and $F(\lambda(\gamma), u(\gamma)) = 0$. We assume that $(\lambda(\gamma), u(\gamma)) \in \mathcal{R}(F, S)$ for all $\gamma \in [\gamma_0 - \varepsilon_0, \gamma_0 + \varepsilon_0]$. We also assume that there exists the projection $\Pi_h : X_p \to S_h$ for each $h > 0$ such that, for all $\gamma \in [\gamma_0 - \varepsilon_0, \gamma_0 + \varepsilon_0]$,

$$\lim_{h \to 0} h^{-r}\|u(\gamma) - \Pi_h u(\gamma)\|_{X_p} = 0,$$

$$\lim_{h \to 0} \|u(\gamma) - \Pi_h u(\gamma)\|_{X_\infty} = 0,$$
and the above convergences are uniform.

We, on the other hand, suppose that $D_uF(\lambda_0, u_0)$ is decomposed into $D_uF(\lambda, u) = Q + R$, where $Q \in \mathcal{L}(X_p, X_q')$ is the principal part which is self-adjoint and satisfies (2.1), and $R \in \mathcal{L}(X_p, X_q')$ is compact. The discretized nonlinear map $F_h : X_p \rightarrow X_q'$ and the projection $P_h : X_q' \rightarrow X_q'$ is defined by (2.2). We suppose that

$$\lim_{h \to 0} \|\psi - P_h\psi\|_{X_q'} = 0, \quad \forall \psi \in X_q'.$$

Then, for sufficiently small $h > 0$, there exist a positive constant $\epsilon_1 \leq \epsilon_0$ and a unique map $[\gamma_0 - \epsilon_1, \gamma_0 + \epsilon_1] \ni \gamma \mapsto (\lambda_h(\gamma), u_h(\gamma)) \in \Lambda \times S_h$ such that $F_h(\lambda_h(\gamma), u_h(\gamma)) = 0$ for all $\gamma \in [\gamma_0 - \epsilon_1, \gamma_0 + \epsilon_1]$. Moreover, we have the estimate

$$|\lambda(\gamma) - \lambda_h(\gamma)| + \|u_h(\gamma) - \Pi_hu(\gamma)\|_{X_p} \leq K_1\|u(\gamma) - \Pi_hu(\gamma)\|_{X_p},$$

for all $\gamma \in [\gamma_0 - \epsilon_1, \gamma_0 + \epsilon_1]$, where $K_1$ is a positive constant independent of $h$ and $\gamma$.

Proof. The proof of Theorem 2.4 is quite similar to those of [17, Theorem 7.7] and [18, Theorem 8.4]. Hence, we here skip the proof. $\square$

3. Elaborate Error Estimates of the Parameter $\lambda$.

In this section we give elaborate error estimates of the parameter $\lambda$. To do this we need more assumptions.

(A8) The nonlinear maps $F : \Lambda \times X_{\infty} \rightarrow X_1'$ and $\rho : \Lambda \times X_{\infty} \rightarrow \mathbb{R}$ are of $C^2$ class.

(A9) For any $(\lambda, u) \in S \subset \Lambda \times W$, $D_uF(\lambda, u) \in \mathcal{L}(X_2, X_2')$ is self-adjoint.

Now, let $(\lambda, u) \in \mathcal{R}(F, S)$ be a solution of $F(\lambda, u) = 0$ at which all assumptions of Theorem 2.4 and (A8), (A9) hold. Let $(\lambda_h, u_h) \in \Lambda \times S_h$ be the corresponding finite element solution with $\rho(\lambda_h, u_h) = \rho(\lambda, u)$.

We consider the following auxiliary problem: find $(\eta, z) \in \mathbb{R} \times X_p$ such that

$$\begin{align*}
((D_uF^0)z, v) &= \eta(D_u\rho^0, v), \quad \forall v \in X_p, \\
(D_\lambda F^0, z) - \eta D_\lambda \rho^0 &= 1,
\end{align*}$$

where $D_uF^0 := D_uF(\lambda, u)$, $D_u\rho^0 := D_u\rho(\lambda, u)$, etc.

Lemma 3.1. Suppose that all assumptions of Theorem 2.4 and (A8), (A9) hold. Then, the equation (3.1) has an unique solution $(\eta, z) \in \mathbb{R} \times X_p$.

Proof. Recall that we have either

- Case 1: $\ker D_\lambda F(\lambda, u) = \{0\}$ and $D_\lambda F(\lambda, u) \in \mathrm{Im} D_u F(\lambda, u)$, or
- Case 2: $\dim \ker D_\lambda F(\lambda, u) = 1$, and $D_\lambda F(\lambda, u) \notin \mathrm{Im} D_u F(\lambda, u)$.

Suppose that we are in Case 1. Then, $\ker D_\lambda F(\lambda, u) = \mathrm{span}\{(1, -(D_uF^0)^{-1}(D_\lambda F^0))\}$. By the assumption we have $D_\lambda \rho^0(1, -(D_uF^0)^{-1}(D_\lambda F^0)) \neq 0$, that is,

$$D_\lambda \rho^0 - (D_uF^0, (D_uF^0)^{-1}(D_\lambda F^0)) \neq 0.$$

Let $\eta := \langle(D_uF^0, (D_uF^0)^{-1}(D_\lambda F^0)) - D_\lambda \rho^0\rangle^{-1}$ and $z := \eta(D_uF^0)^{-1}(D_u\rho^0)$. Since $D_uF^0$ is self-adjoint by (A9), we have

$$\eta(D_\lambda F^0, (D_uF^0)^{-1}(D_u\rho^0)) = \eta(D_uF^0, (D_uF^0)^{-1}(D_\lambda F^0)).$$
and $\langle D_{\lambda}F^{0}, z \rangle - \eta D_{\lambda}\rho^{0} = \eta((D_{u}\rho^{0}, (D_{u}F^{0})^{-1}(D_{\lambda}F^{0})) - D_{\lambda}\rho^{0}) = 1$. Hence $(\eta, z)$ is a solution of (3.1). Uniqueness is proved by the same manner.

Now, suppose that we have Case 2. Then, there exists $\psi_{0} \in V$ such that $\ker DF(\lambda, u) = \text{span}\{(0, \psi_{0})\}$ and $\langle D_{u}\rho^{0}, \psi_{0} \rangle \neq 0$.

Since $DF(\lambda, u)$ is onto, there exists $(\theta, \phi) \in \mathbb{R} \times X_{p}$ such that

$$\theta(\langle D_{\lambda}F^{0}, v \rangle + \langle (D_{u}F^{0})\phi, v \rangle = \langle D_{u}\rho^{0}, v \rangle, \quad \forall v \in X_{2},$$

and $\theta$ is determined uniquely.

We claim that $D_{u}\rho^{0} \notin \text{Im}(D_{u}F^{0})$. If $D_{u}\rho^{0} \in \text{Im}(D_{u}F^{0})$, then there would exist $w \in X_{p}$ such that $(D_{u}F^{0})w = D_{u}\rho^{0}$. Hence, we have

$$0 \neq \langle D_{u}\rho^{0}, \psi_{0} \rangle = \langle (D_{u}F^{0})w, \psi_{0} \rangle = \langle (D_{u}F^{0})\psi_{0}, w \rangle = 0,$$

and obtain a contradiction. Therefore, we conclude that $D_{u}\rho^{0} \notin \text{Im}(D_{u}F^{0})$ and $\theta \neq 0$.

Letting $v := \psi_{0}$ in (3.2), we have $\theta(\langle D_{\lambda}F^{0}, \psi_{0} \rangle = \langle D_{u}\rho^{0}, \psi_{0} \rangle \neq 0$. Hence, we conclude $\langle D_{\lambda}F^{0}, \psi_{0} \rangle = \langle D_{u}\rho^{0}, \psi_{0} \rangle / \theta \neq 0$. We thus immediately notice that $(0, \alpha\psi_{0})$ with $\alpha := (D_{\lambda}F^{0}, \psi_{0})^{-1}$ is a solution of (3.1). Again, the uniqueness is shown by the same manner. \hfill \square

It is obvious that we may apply Theorem 2.4 to the equation (3.1) with the following setting:

$$F(\eta, z) := (D_{u}F^{0})z - \eta(D_{u}\rho^{0}),$$

$$\rho(\eta, z) := (D_{\lambda}F^{0}, z) - \eta(D_{\lambda}\rho^{0}),$$

and obtain

**Lemma 3.2.** For sufficiently small $h > 0$, there exists the unique finite element solution $(\eta_{h}, z_{h}) \in \mathbb{R} \times S_{h}$ of (3.1) such that

$$\langle (D_{u}F^{0})z_{h}, v_{h} \rangle = \eta_{h}(D_{u}\rho^{0}, v_{h}), \quad \forall v \in S_{h},$$

$$\langle D_{\lambda}F^{0}, z_{h} \rangle - \eta_{h}D_{\lambda}\rho^{0} = 1.$$  

Moreover, we have the estimate

$$|\eta - \eta_{h}| + \|z - z_{h}\|_{X_{p}} \leq C\|z - \Pi_{h}z\|_{X_{p}},$$

where $C$ is a positive constant independent of $h$. \hfill \square

Let $(\lambda, u) \in \mathcal{R}(F, S)$ is a solution of $F(\lambda, u) = 0$ which satisfies the assumptions of Theorem 2.4 and (A8), (A9), and $(\lambda_{h}, u_{h}) \in \Lambda \times S_{h}$ the corresponding finite element solution. By Taylor's theorem and $\langle F(\lambda_{h}, u_{h}), v_{h} \rangle = \langle F(\lambda, u), v_{h} \rangle = 0$ for any $v_{h} \in S_{h}$, we have

$$0 = (\lambda_{h} - \lambda)(D_{\lambda}F^{0}, v_{h}) + \langle (D_{u}F^{0})(u_{h} - u), v_{h} \rangle + \frac{1}{2}(\lambda_{h} - \lambda)^{2}\langle D_{\lambda\lambda}F^{0}, v_{h} \rangle$$

$$+ (\lambda_{h} - \lambda)(D_{\lambda\lambda}F^{0})(u_{h} - u), v_{h}) + \frac{1}{2}\langle (D_{uu}F^{0})(u_{h} - u)^{2}, v_{h} \rangle,$$

where

$$D_{\lambda\lambda}F^{0} := \int_{0}^{1}(1 - s)D_{\lambda\lambda}F(\lambda + s(\lambda_{h} - \lambda), u + s(u_{h} - u))ds,$$

$$(D_{\lambda\lambda}F^{0})(u_{h} - u) := \int_{0}^{1}(1 - s)D_{\lambda\lambda}F(\lambda + s(\lambda_{h} - \lambda), u + s(u_{h} - u))(u_{h} - u)ds,$$

$$(D_{uu}F^{0})(u_{h} - u)^{2} := \int_{0}^{1}(1 - s)D_{uu}F(\lambda + s(\lambda_{h} - \lambda), u + s(u_{h} - u))(u_{h} - u)^{2}ds.$$
Letting $v := u - u_h$ in (3.1), we obtain
\[
\langle (D_u F^0)(u - u_h), z \rangle = \eta \langle (D_{uu} F^0)(u - u_h), (u - u_h) \rangle.
\]
Since
\[
0 = \rho(\lambda_h, u_h) - \rho(\lambda, u)
\]
\[
= (\lambda_h - \lambda)(D_{\lambda} \rho^0) + (D_{uu} \rho^0)(u_h - u) + \frac{1}{2} (\lambda_h - \lambda)^2 (D_{\lambda \lambda} \rho^0)
\]
\[
+ (\lambda_h - \lambda)(D_{uu} \rho^0)(u_h - u) + \frac{1}{2} (D_{uu} \rho^0)(u_h - u)^2,
\]
where
\[
D_{\lambda \lambda} \rho^0 := \int_0^1 (1 - s) D_{\lambda \lambda} \rho(\lambda + s(\lambda_h - \lambda), u + s(u_h - u))ds,
\]
\[
(D_{uu} \rho^0)(u_h - u) := \int_0^1 (1 - s)(D_{uu} \rho(\lambda + s(\lambda_h - \lambda), u + s(u_h - u)), (u_h - u))ds,
\]
we have
\[
\langle (D_u F^0)(u - u_h), z \rangle = -\eta (\lambda - \lambda_h)(D_{\lambda} \rho^0) + \frac{\eta}{2} (\lambda_h - \lambda)^2 (D_{\lambda \lambda} \rho^0)
\]
\[
+ \eta (\lambda - \lambda_h)(D_{uu} \rho^0)(u_h - u) + \frac{\eta}{2} (D_{uu} \rho^0)(u_h - u)^2.
\]
(3.4)

It follows from (3.3) with $v_h := \eta_h$ (recall that $(\eta_h, z_h) \in \mathbb{R} \times S_h$ is the finite element solution of (3.1)) and (3.4) that
\[
(\lambda - \lambda_h)(\langle D_u F^0, z \rangle - \eta(D_{\lambda} \rho^0) + B_h) = \langle (D_u F^0)(u - u_h), (z - z_h) \rangle
\]
\[
+ \frac{1}{2} \langle (D_{uu} F^0)(u - u_h)^2, z_h \rangle - \frac{\eta}{2} (D_{uu} \rho^0)(u_h - u)\),
\]
where $\lim_{h \to 0} B_h = 0$. Therefore, we have proved the following theorem:

**Theorem 3.3.** Let $(\lambda, u) \in \mathcal{R}(F, S)$ be a solution of $F(\lambda, u) = 0$ which satisfies the assumptions of Theorem 2.4 and (A8), (A9). Let $(\lambda_h, u_h) \in \Lambda \times S_h$ be the corresponding finite element solution. Let $(\eta, z) \in \mathbb{R} \times X_p$ and $(\eta_h, z_h) \in \mathbb{R} \times S_h$ be the exact and the finite element solutions of (3.1).

Then, for sufficiently small $h > 0$, we have the following elaborate error estimate of $|\lambda - \lambda_h|$:
\[
|\lambda - \lambda_h| \leq C_h \Big| \langle (D_u F^0)(u - u_h), z - z_h \rangle + \frac{1}{2} \langle (D_{uu} F^0)(u - u_h)^2, z_h \rangle
\]
\[
- \frac{\eta}{2} (D_{uu} \rho^0)(u_h - u)\),
\]
where $D_u F^0 := D_u F(\lambda, u)$,
\[
(D_{uu} F^0)(u - u_h)^2 := \int_0^1 (1 - s) D_{uu} F(\lambda + s(\lambda_h - \lambda), u + s(u_h - u))(u - u_h)^2 ds,
\]
\[
(D_{uu} \rho^0)(u - u_h)^2 := \int_0^1 (1 - s)(D_{uu} \rho(\lambda + s(\lambda_h - \lambda), u + s(u_h - u))(u - u_h))(u - u_h), u - u_h ds,
\]
and $C_h$ is a positive constant such that $\lim_{h \to 0} C_h = 1$. □
Sometimes, one may want to compute a turning point itself. For such a purpose we are able to develop a similar analysis as above. Let \((\lambda_0, u_0) \in \mathcal{R}(F, S)\) be a turning point of the equation \(F(\lambda, u) = 0\) at which the assumptions of Theorem 2.4 and \((A8), (A9)\) hold. That is, \(F(\lambda_0, u_0) = 0\), \(DF(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times V, W)\) is onto, and \(D_u F(\lambda_0, u_0) \in \mathcal{L}(V, W)\) is not an isomorphism. In this case we have \(\text{dim Ker} D_u F(\lambda_0, u_0) = 1\) and \(D_\lambda F(\lambda_0, u_0) \not\in \text{Im} D_u F(\lambda_0, u_0)\). It then follows from the proof of Lemma 3.1 that (3.1) has an unique solution \((0, z_0) \in \mathbb{R} \times X_p\) at \((\lambda_0, u_0)\):

\[
\begin{align*}
\langle D_u F(\lambda_0, u_0) z_0, v \rangle &= 0, \quad \forall v \in X_p, \\
\langle D_\lambda F(\lambda_0, u_0), z_0 \rangle &= 1.
\end{align*}
\]

(3.5)

We consider the nonlinear map \(K: \Lambda \times V \times X_p \to \mathbb{R} \times W \times X'_q\) defined by

\[
K(\lambda, u, z) := \begin{pmatrix}
    \langle D_\lambda F(\lambda, u), z \rangle - 1 \\
    F(\lambda, u) \\
    D_u F(\lambda, u) z
\end{pmatrix}.
\]

At a turning point \((\lambda_0, u_0) \in \mathcal{R}(F, S)\) the equation \(K(\lambda, u, z) = (0, 0, 0)\) has the solution \((\lambda_0, u_0, z_0) \in \Lambda \times V \times X_p\). A turning point \((\lambda_0, u_0) \in \mathcal{R}(F, S)\) is called nondegenerate, if

\[
D_u u F(\lambda_0, u_0) \psi \psi_0 \not\in \text{Im} D_\lambda F(\lambda_0, u_0),
\]

where \(\{\psi_0\} \subset X_p\) is the basis of \(\text{Ker} D_u F(\lambda_0, u_0)\) (see [4, Section 4]). For a nondegenerate turning point, we have the following lemma. For the proof of the lemma, see [4], [15].

**Lemma 3.4.** Let \((\lambda_0, u_0) \in \mathcal{R}(F, S)\) be a tuning point at which the assumptions of Theorem 2.4 and \((A8), (A9)\) hold. Then, \((\lambda_0, u_0)\) is a nondegenerate tuning point if and only if the Fréchet derivative \(D K(\lambda_0, u_0, z_0) \in \mathcal{L}(\mathbb{R} \times V \times X_p, \mathbb{R} \times W \times X'_q)\) is an isomorphism, where \(z_0 \in X_p\) is the solution of (3.5) and the nonlinear map \(K\) is defined by (3.6). \(\Box\)

From Lemma 3.4, the results in [16] can be applied to the equation \(K(\lambda, u, z) = (0, 0, 0)\) at a nondegenerate turning point \((\lambda_0, u_0)\) and obtain the following lemma:

**Lemma 3.5.** Let \((\lambda_0, u_0) \in \mathcal{R}(F, S)\) is a nondegenerate tuning point. Then, for sufficiently small \(h > 0\), there exist a locally unique finite element solution \((\lambda^h_0, u^h_0, z^h_0) \in \mathbb{R} \times (S_h)^2\) such that

\[
\begin{align*}
\langle D_\lambda F_h(\lambda^h_0, u^h_0), z^h_0 \rangle &= 1, \\
F_h(\lambda^h_0, u^h_0) &= 0, \\
D_u F_h(\lambda^h_0, u^h_0) z^h_0 &= 0,
\end{align*}
\]

where \(F_h\) is the nonlinear map defined by (2.2). The finite element solution \((\lambda^h_0, u^h_0)\) is a nondegenerate turning point on the finite element solution manifold \(M_h\).

Moreover, we have the following error estimate:

\[
|\lambda_0 - \lambda^h_0| + ||u_0 - u^h_0||_{X_p} + ||z_0 - z^h_0||_{X_p} \leq C(||u_0 - \Pi_h u_0||_{X_p} + ||z_0 - \Pi_h z_0||_{X_p}),
\]

where \(C\) is a positive constant independent of \(h\), and \(\Pi_h : X_p \to S_h\) is the projection which appears in Theorem 2.4. \(\Box\)
Now, we develop a similar elaborate error estimate for $|\lambda_0 - \lambda_0^h|$. Again, let $(\lambda_0, u_0) \in \mathcal{R}(F, S)$ be a nondegenerate turning point which satisfies the assumptions of Theorem 2.4 and (A8), (A9), and $(\lambda_0^h, u_0^h) \in \Lambda \times S_h$ the corresponding finite element solution. By Taylor's theorem and $(F(\lambda_0^h, u_0^h), v_h) = \langle F(\lambda_0, u_0), v_h \rangle = 0$ for any $v_h \in S_h$, we have

$$0 = (\lambda_0^h - \lambda_0)D\lambda F(\lambda_0, u_0), v_h) + (D_u F(\lambda_0, u_0)(u_0^h - u_0), v_h)$$

$$+ \frac{1}{2}(\lambda_0^h - \lambda_0)^2(D_{\lambda\lambda} F^0, v_h) + (\lambda_0^h - \lambda_0)((D_u u F^0)(u_0^h - u_0), v_h)$$

$$+ \frac{1}{2}((D_{uu} F^0)(u_0^h - u_0)^2, v_h),$$

where

$$D_{\lambda\lambda} F^0 := \int_0^1 (1 - s)D_{\lambda\lambda} F(\lambda_0 + s(\lambda_0^h - \lambda_0), u_0 + s(u_0^h - u_0))ds,$$

$$(D_u u F^0)(u_0^h - u_0) := \int_0^1 (1 - s)D_u F(\lambda_0 + s(\lambda_0^h - \lambda_0), u_0 + s(u_0^h - u_0))ds,$$

$$(D_{uu} F^0)(u_0^h - u_0)^2 := \int_0^1 (1 - s)D_{uu} F(\lambda_0 + s(\lambda_0^h - \lambda_0), u_0 + s(u_0^h - u_0))(u_0^h - u_0)^2ds.$$

Letting $v := u_0 - u_0^h$ in (3.5), we obtain

$$\langle D_u F(\lambda_0, u_0)z_0, u_0 - u_0^h \rangle = \langle D_u F(\lambda_0, u_0)(u_0 - u_0^h), z_0 \rangle = 0.$$

Plugging this equation into (3.7) with $v_h := z_0^h$, we obtain

$$(\lambda_0 - \lambda_0^h)((D\lambda F(\lambda_0, u_0), z_0) + B_h) = \langle D_u F(\lambda_0, u_0)(u_0 - u_0^h), z_0 - z_0^h \rangle$$

$$+ \frac{1}{2}((D_{uu} F^0)(u_0 - u_0^h)^2, z_0^h),$$

where $\lim_{h \to 0} B_h = 0$. Therefore, we have proved the following theorem:

**Theorem 3.6.** Let $(\lambda_0, u_0) \in \mathcal{R}(F, S)$ be a nondegenerate turning point which satisfies the assumptions of Theorem 2.4 and (A8), (A9). Let $(\lambda_0^h, u_0^h) \in \Lambda \times S_h$ be the corresponding nondegenerate turning point on the finite element solution branch $\mathcal{M}_h$. Let $z_0 \in X_p$ and $z_0^h \in S_h$ be the exact and finite element solutions which appear in Lemma 3.4 and 3.5.

Then, for sufficiently small $h > 0$, we have the following elaborate error estimate of $|\lambda_0 - \lambda_0^h|:

$$|\lambda_0 - \lambda_0^h| \leq C_h \left| \langle D_u F(\lambda_0, u_0)(u_0 - u_0^h), z_0 - z_0^h \rangle + \frac{1}{2}((D_{uu} F^0)(u_0 - u_0^h)^2, z_0^h) \right|$$

where

$$(D_{uu} F^0)(u_0 - u_0^h)^2 := \int_0^1 (1 - s)D_{uu} F(\lambda_0 + s(\lambda_0^h - \lambda_0), u_0 + s(u_0^h - u_0))(u_0 - u_0^h)^2ds,$$

and $C_h$ is a positive constant such that $\lim_{h \to 0} C_h = 1. \square$

**Remark.** Apparently, Lemma 3.5 and Theorem 3.6 are very similar to [4, Theorem 7]. The main difference is the tools used in [4] and in this paper. In [4] the Liapunov-Schmidt reduction is used to parametrize solution branches around turning points. On the other hand, so-called
"bordering technique" is used throughout this paper. In [15], it is pointed out that bordering technique is closely related with the Liapunov-Schmidt reduction.

Employing bordering technique, our situation becomes simpler than that of [4]. For instance, in [4] $F$ should be $C^3$ map while $F$ is $C^2$ map in this paper. Also, we do not need the derivatives of $\lambda$ and $u$ with respect to the newly introduced parameter, which are used frequently in [4]. The second point will be advantageous when we try to apply the results in this section to a posteriori error estimation of the parameter $\lambda$. This point will be discussed elsewhere by the author. □

References


