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C-algebras and their applications to reflection groups and conformal field theories

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The aim of this lecture is to present the concept of C-algebra and to illustrate its applications in two contexts: the study of reflection groups and their folding on the one hand, the structure of rational conformal field theories on the other. For simplicity the discussion is restricted to finite Coxeter groups and conformal theories with a $\widehat{sl}(2)$ current algebra, but it may be extended to a larger class of groups and theories associated with $\widehat{sl}(N)$.

1. Introduction

The purpose of this talk is to present the notion of C-algebra, a concept that appears particularly suited in the discussion of various topics of current interest in mathematics and mathematical physics: rational conformal field theories (rcft), topological field theories, singularity theory and related problems. The concept was originally developed in relation with finite groups and the algebras of their characters and classes (whence the "C"): this exposes clearly one of the key features of these algebras, namely the pattern of two dual algebras. More generally, (the precise definition will be given in sect. 3), C-algebras are associative, commutative algebras with a finite number of generators. They come in dual pairs, endowed with different multiplication laws, one algebra being generated by the idempotents of the other. We shall illustrate and apply this concept in two different contexts: the association between rational conformal field theories and graphs on the one hand; the folding of root systems and Dynkin diagrams on the other. In both cases, generalized Dynkin diagrams are the central objects, and pairs of algebras that are naturally associated with these graphs are C-algebras. The study of the C-subalgebras (to be also defined below) then enables one to understand the relationship between rcft and graph –how to construct one object from the other– and to understand the folding of root systems, Dynkin diagrams and reflection groups.

Because certain positivity properties play an important role in the discussion of C-algebras, we start with a presentation of such properties that are empirically observed in different contexts but do not seem to have been given enough attention.

For the sake of brevity, all the discussion will be restricted to the simplest –and best understood– case: rcft associated with $sl(2)$, "minimal" topological field theories, simple singularities, ordinary Coxeter-Dynkin diagrams, etc. There is ample evidence, however, –and a few proofs–, that the present considerations extend to a much larger context.

2. Three empirical facts

Consider the prepotential $F(\mathbf{t})$ of one of the ADE singularities. Here $\mathbf{t} = (t^1, \cdots, t^n)$, where n is the rank of the associated ADE algebra (the Milnor number of the singularity); the t^j are the flat coordinates in the versal deformation of the singularity. $F(\mathbf{t})$ satisfies the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations [1], which express the associativity of the algebra with structure constants ${C_{ij}}^k(\mathbf{t})$, where $C_{ijk}(\mathbf{t}) = \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k}$, and indices are raised and lowered with the t-independent tensor $\eta_{kl} = C_{1kl}$, and its inverse η^{jk}. The prepotential F is a quasihomogeneous polynomial of degree $2(h+1)$ if we assign to the variables t^i the degrees $\deg(t^i) = (h+1-\lambda_i)$, where λ_i is the i-th Coxeter exponent of the ADE algebra: these exponents are supposed to be labelled in increasing order: $\lambda_1 = 1 < \lambda_2 \leq \cdots \leq \lambda_{n-1} < \lambda_n = h-1$, h the Coxeter number. It is convenient to change notations, labelling the t's with the value of λ_i, hence replacing t^i with t^{λ_i}, and accordingly to denote the structure constants $C_{\lambda_i\lambda_j\lambda_k}$ or $C_{\lambda\mu\nu}$, for λ, μ, ν exponents. The expressions of the prepotentials for the various ADE cases have been listed in the literature ([2-3] and further references therein).

Now, by inspection, we observe the following

Fact 1 : *For the A_n, D_{2n}, E_6 and E_8 cases, there exists a choice of flat coordinates for which all the coefficients of F are real positive. For D_{2n+1} and E_7 there is no such choice.*

Two remarks are in order.

First, why is it meaningful to look at reality and positivity properties in a problem that looks intrinsically complex?

Secondly it is curious that this splitting of the ADE classification scheme into the same two sub-families appears also in other contexts. Let us quote

1) the structure of the modular invariant partition function of conformal field theories with a $\widehat{sl}(2)$ current algebra. The latter are known to follow an ADE classification scheme [4,5]. The question is to know if this partition function, which is a certain sesquilinear form with non negative integer coefficients, may or may not be written as a sum of blocks

$$Z = \sum \mathcal{N}_{\lambda\bar{\lambda}} \chi_\lambda \bar{\chi}_{\bar{\lambda}} \qquad \mathcal{N}_{\lambda\bar{\lambda}} \in \mathbb{N} \tag{2.1a}$$

$$\stackrel{?}{=} \sum_i \Big| \sum_{\lambda \in \hat{T}_\alpha} \chi_\lambda \Big|^2 . \tag{2.1b}$$

For example, the cases labelled by D_{10} and E_7 read respectively

$$Z^{(D_{10})} = |\chi_1 + \chi_{17}|^2 + |\chi_3 + \chi_{15}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 + 2|\chi_9|^2 \tag{2.2a}$$

$$\begin{aligned} Z^{(E_7)} &= |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 + |\chi_9|^2 + ((\chi_3 + \chi_{15})\chi_9^* + \text{complex conj.}) \\ &= |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 + |\chi_9 + \chi_3 + \chi_{15}|^2 - |\chi_3 + \chi_{15}|^2 \end{aligned} \tag{2.2b}$$

(for more details and explanation of notations, see below sect. 5).

2) the positivity of the structure constants of the "Pasquier algebras" to be discussed below;

3) the existence or non-existence of a "flat connection" on the path algebra on the Dynkin diagram [6];
4) the positivity of the coefficients of the prepotential just discussed;
5) the positivity properties of the coefficients of the factors of the Poincaré polynomial of the local (or "chiral") ring of the singularity. Let us discuss briefly this latest aspect, as it does not seem to be generally known. For any of the ADE singularities, let p denote the minimal number of non morsian variables X_i that enter the singular polynomial. Let us write the Poincaré polynomial in the form

$$P(t) = \prod_{i=1}^{p} \frac{(1 - t^{h-\deg(X_i)})}{(1 - t^{\deg(X_i)})} \tag{2.3}$$

in terms of the degrees of the variables X_i and of the Coxeter number h, equal to the degree of the singular polynomial.

	h	p	$\{\deg(X_i)\}$	exponents λ
A_n	$n+1$	1	1	$1, 2, \cdots, n$
$D_{\ell+2}$	$2(\ell+1)$	2	$2, \ell$	$1, 3, \cdots, 2\ell+1, \ell+1$
E_6	12	2	3,4	$1, 4, 5, 7, 8, 11$
E_7	18	2	4,6	$1, 5, 7, 9, 11, 13, 17$
E_8	30	2	6,10	$1, 7, 11, 13, 17, 19, 23, 29$

It is then an easy and amusing exercise to check that $P(t)$ may be written as a product of p factors with positive coefficients only in the first subfamily. (I owe this observation to M. Bauer [7]). This is somehow the multiplicative counterpart of the property 1) mentionned above.

The interesting thing is that the simultaneous occurrence of several of these properties seem to extend beyond the ADE case discussed here. The status of these various occurences is however not the same. I think it is fair to say that 1) is the best understood, as it is related to a structural property of the underlying conformal field theory. 2) is related to 4) as we shall see soon, but I doubt that 4) may be extended beyond the case of simple singularities, as the prepotential is then no longer a polynomial. Finally it seems that 5) does not generalize: for some singularities believed to be in correspondence with some conformal field theory, property 5) may fail while 1) and 2) are true (for example, the singularity associated with the fusion potential of $\widehat{sl}(4)_4$).

In fact we are not going to make use of Fact 1 for generic $\mathbf{t}$, but only for a particular case, obtained by the so-called Chebishev specialization. This refers to the deformation of the ADE singularity for which all the flat coordinates but the one, t^n, with the smallest degree $\deg(t^n) = 2$, (the largest exponent $\lambda_n = h-1$), i.e. the "less relevant" in the language of physics, is kept non zero. As it is the only parameter in the homogeneous deformed polynomial $W(X_1, \cdots, X_p, t^n)$, one may rescale it to $t^n = 1$. The origin of the denomination is that for the A_n case, the deformed polynomial reads then $W_{A_n}(X_1; t^n = 1) = T_{n+1}(X_1)$, with $T_{n+1}(x)$ the degree $n+1$ Chebishev polynomial of first kind, $T_{n+1}(x) = 2\cos(n+1)\theta$ if $x = 2\cos\theta$.

We also need some notations on the ADE Dynkin diagrams. Let G_{ab} denote the adjacency matrix of the Dynkin diagram under consideration: $a, b = 1, \cdots, n$ label the vertices. The corresponding Cartan matrix is $C_{ab} = 2\delta_{ab} - G_{ab}$. The eigenvectors $\psi^{(\lambda)}$ and eigenvalues of these symmetric matrices are indexed by the Coxeter exponents λ,

$$G_{ab}\psi_b^{(\lambda)} = 2\cos\frac{\pi\lambda}{h}\psi_a^{(\lambda)} \ . \tag{2.4}$$

The $\psi^{(\lambda)}$ may be chosen orthonormal.

Then we can state the

Fact 2 : *The structure constants of the chiral ring in the Chebishev specialization are diagonalized by the* $\psi^{(\lambda)}$

$$M_{\lambda\mu}{}^{\nu} := C_{\lambda\mu}{}^{\nu}(t^n = 1) = \sum_a \frac{\psi_a^{(\lambda)}\psi_a^{(\mu)}\psi_a^{(\nu)*}}{\psi_a^{(1)}} \tag{2.5}$$

Here I have introduced the notation M to be used in the forthcoming discussion. In the denominator of the right hand side, there appears the exponent 1, that yields the largest eigenvalue of the matrix G. By the Perron-Frobenius theorem, all the components of $\psi^{(1)}$ are non vanishing and of the same sign.

Fact 2 is not a surprise in the A_n case, where it follows from the combined work of Verlinde [8] and Gepner [9]. Indeed the above structure constants reduce then to the fusion coefficients of the $\widehat{sl}(2)$ algebra, for a value of the level (central extension) equal to $k = n-1$, and the latter are known to have an interpretation in terms of the chiral ring of a topological field theory. For the other D and E cases, the observation was made (in essence, not quite in these terms) by Lerche and Warner [10], and made more systematic and extended in [3].

The previous formula suggests to consider also the dual algebra (we shall see below that the word "dual" is legitimate), with structure constants

$$N_{ab}{}^{c} := \sum_{\lambda} \frac{\psi_a^{(\lambda)}\psi_b^{(\lambda)}\psi_c^{(\lambda)*}}{\psi_1^{(\lambda)}} \tag{2.6}$$

where the sum runs over the exponents λ of the case at hand. This definition depends on a choice of a vertex denoted 1 for which all the $\psi_1^{(\lambda)}$ are non-vanishing. Such a vertex exists for all the ADE cases. There may remain, however, some arbitrariness in the choice of that vertex 1 and also, in the case D_{2n} for which an exponent occurs with multiplicity 2, in the choice of the basis $\psi_a^{(\lambda)}$. Now comes the

Fact 3 : *For the A_n, D_{2n}, E_6 and E_8 cases, there exists a choice of vertex 1 and of the basis $\psi_a^{(\lambda)}$ such that the structure constants $M_{\lambda\mu}{}^{\nu}$ and $N_{ab}{}^{c}$ are all non negative. For the cases D_{2n+1} and E_7, there exists no such choice.*

Note that the non-negativity of the M is a simple consequence of Fact 1 $\cap$ Fact 2. For the ADE cases, the numbers N turn out to be integers (with an adequate choice of 1 and the basis). The interpretation of these numbers in the various contexts in which they occur (conformal field theories, topological theories and singularities, lattice models) has remained elusive so far. In contrast, the M that are in general non integers but rather algebraic numbers, have such an interpretation: they give the structure constants of the chiral ring of the Chebishev specialization, as just explained; in the context of conformal field theories and integrable lattice models, they give the coupling constants of field operators [11], [8], [12]. It is in that context that this algebra was first introduced by Pasquier [11], whence the name of Pasquier algebras that I give to the pair of M and N algebras.

3. C-algebras

3.1. Definitions and examples

The appropriate language to discuss these Pasquier algebras is that of *C-algebras*, ("C" for character), introduced in the 40's by Kawada and recently reviewed and revived by Bannai and Ito [13].

Definition : *An algebra $\mathfrak{A}$ over $\mathbb{C}$ with a given basis $x_1, \cdots, x_n$, is a C-algebra if it satisfies the following axioms:*

i) *it is a commutative and associative algebra with* real *structure constants* ${p_{ab}}^c$, *i.e.* $x_a.x_b = \sum_c {p_{ab}}^c x_c$;

ii) *it has an identity element, denoted* x_1, *i.e.* ${p_{1a}}^b = \delta_{ab}$;

iii) *there is an involution on the generators* $x_a \mapsto x_{\bar{a}}$ *that is an automorphism of the algebra, i.e.* ${p_{ab}}^c = {p_{\bar{a}\bar{b}}}^{\bar{c}}$;

iv) ${p_{ab}}^1 = k_a \delta_{a\bar{b}}$, *with* k_a *a real positive number* $k_a > 0$;

v) *the* k_a *form a one-dimensional representation of the algebra.*

Among the various consequences of these axioms, is the fact that $\mathfrak{A}$ is semi-simple. There are n one-dimensional representations of the algebra, that we label by an index λ taking n values : $x_a \mapsto p_a(\lambda) \in \mathbb{C}$. The value $\lambda = 1$ refers to the special representation of axiom v): $p_a(1) = k_a$. If e_λ denote the corresponding idempotents, one may decompose $x_a = \sum_\lambda p_a(\lambda) e_\lambda$. The matrix $p_a(\lambda)$ is invertible, let $q_\lambda(a)$ denote the matrix such that $\sum_\lambda p_a(\lambda) q_\lambda(b) = \kappa \delta_{ab}$, $\kappa := \sum_a k_a$. More explicitly, the matrices P_a of elements ${(P_a)_b}^c = \sqrt{\frac{k_c}{k_b}} {p_{ab}}^c$ form a representation of the algebra $\mathfrak{A}$. They are normal and commuting, and thus diagonalizable in a common orthonormal basis $\psi_a^{(\lambda)}$. All $\psi_1^{(\lambda)}$ and $\psi_a^{(1)}$ are non vanishing and may thus be chosen real positive. One may write

$$\begin{aligned}
{p_{ab}}^c &= \sqrt{\frac{k_a k_b}{k_c}} \sum_\lambda \frac{\psi_a^{(\lambda)} \psi_b^{(\lambda)} \psi_c^{(\lambda)*}}{\psi_1^{(\lambda)}} \\
\sqrt{k_a} &= \frac{\psi_a^{(1)}}{\psi_1^{(1)}} \\
p_a(\lambda) &= \frac{\psi_a^{(\lambda)} \psi_a^{(1)}}{\psi_1^{(\lambda)} \psi_1^{(1)}} \\
q_\lambda(a) &= \frac{\psi_a^{(\lambda)*} \psi_1^{(\lambda)}}{\psi_a^{(1)} \psi_1^{(1)}}
\end{aligned} \tag{3.1}$$

and let $\hat{k}_\lambda$ be such that

$$\sqrt{\hat{k}_\lambda} = \frac{\psi_1^{(\lambda)}}{\psi_1^{(1)}} \ . \tag{3.2}$$

One may then show that the dual $\hat{\mathfrak{A}}$ of $\mathfrak{A}$, defined as the set of linear maps from $\mathfrak{A}$ into $\mathbb{C}$, is endowed with a structure of C-algebra: its basis is labelled by the λ, its one dimensional representations are provided by the $q_\lambda(a)$, among which $q_\lambda(1) = \hat{k}_\lambda$ are positive, and the structure constants of the algebra are

$${q_{\lambda\mu}}^\nu = \sqrt{\frac{\hat{k}_\lambda \hat{k}_\mu}{\hat{k}_\nu}} \sum_a \frac{\psi_a^{(\lambda)} \psi_a^{(\mu)} \psi_a^{(\nu)*}}{\psi_a^{(1)}} \ . \tag{3.3}$$

The k_a and $\hat{k}_\lambda$ are called the Krein parameters of the algebras. They satisfy $\kappa = \sum_a k_a = \sum_\lambda \hat{k}_\lambda = 1/\psi_1^{(1)\,2}$.

Alternatively, one may regard this dual $\hat{\mathfrak{A}}$ as a second C-algebra structure on $\mathfrak{A}$, with basis κe_λ and idempotents x_a. To recapitulate, $\mathfrak{A}$ is endowed with a pair of dual C-algebra structures, one with multiplication ., structure constants $p_{ab}{}^c$ in the basis x_a, and idempotents e_λ, and the other with multiplication $\circ$, structure constants $q_{\lambda\mu}{}^\nu$ in the basis κe_λ and idempotents x_a

$$\begin{aligned} x_a . x_b &= \sum_c p_{ab}{}^c x_c \ , \qquad & e_\lambda . e_\mu &= \delta_{\lambda\mu} e_\lambda \\ \kappa e_\lambda \circ \kappa e_\mu &= \sum_\nu q_{\lambda\mu}{}^\nu \, \kappa e_\nu \ , \qquad & x_a \circ x_b &= \delta_{ab} x_a \ . \end{aligned} \tag{3.4}$$

Examples:

1. Character and class algebras of a finite group. Let Γ be a finite group, C_a denote its equivalence classes, (ρ) its irreducible representations, $\chi^{(\rho)}$ their characters, $\chi_a^{(\rho)}$ the value of these characters on class a; $a = 1$ refers to the class of the identity, $\rho = 1$ to the identity representation; $d_\rho = \chi_1^{(\rho)}$ is the dimension of representation ρ. One has two dual algebras

$$\begin{aligned} C_a C_b &= C_{ab}{}^c C_c \\ \chi^{(\lambda)} \chi^{(\mu)} &= K^{\lambda\mu}{}_\nu \chi^{(\nu)} \ . \end{aligned} \tag{3.5}$$

Introducing the $\widehat{\chi}_a^\lambda = \sqrt{\frac{|C_a|}{|\Gamma|}} \chi_a^{(\lambda)}$, orthonormal by virtue of the standard orthogonality and completeness relations of characters, one may write

$$\begin{aligned} p_{ab}{}^c &= C_{ab}{}^c = \sqrt{\frac{|C_a||C_b|}{|C_c|}} \sum_\lambda \frac{\widehat{\chi}_a^{(\lambda)} \widehat{\chi}_b^{(\lambda)} \widehat{\chi}_c^{(\lambda)*}}{\widehat{\chi}_1^{(\lambda)}} \\ q_{\lambda\mu}{}^\nu &= \frac{d_\lambda d_\mu}{d_\nu} K^{\lambda\mu}{}_\nu = \frac{d_\lambda d_\mu}{d_\nu} \sum_a \frac{\widehat{\chi}_a^{(\lambda)} \widehat{\chi}_a^{(\mu)} \widehat{\chi}_a^{(\nu)*}}{\widehat{\chi}_a^{(1)}} \end{aligned} \tag{3.6}$$

The two dual algebras have integer Krein parameters $k_a = |C_a|$, $\hat{k}_\lambda = d_\lambda^2$ with the well known relation $|\Gamma| = \sum k_a = \sum \hat{k}_\lambda = \sum d_\lambda^2$.

2. The Pasquier algebras introduced above are obviously a pair of dual C-algebras. The structure constants $p_{ab}{}^c$ and $q_{\lambda\mu}{}^\nu$ are respectively proportional to $N_{ab}{}^c$ and $M_{\lambda\mu}{}^\nu$, as indicated in (3.1) and (3.3). In that case, in contrast with example 1, the Krein parameters are not integers. Among these Pasquier algebras, there are the fusion algebras of affine

algebras $\hat{\mathfrak{g}}$. In that case, the two dual algebras are in fact isomorphic: this is due to the fact that according to the Verlinde formula, the diagonalizing matrix is the *symmetric* unitary matrix S of modular transformations of the affine characters [8]. Also, in that case, the Krein parameters are equal to $\hat{k}_\lambda = \left(\frac{S_{1\lambda}}{S_{11}}\right)^2$, that is $\hat{k}_\lambda = D_\lambda^2$, the square of the quantum dimension of the corresponding representation of $\hat{\mathfrak{g}}$. This is thus a quantum deformation of the finite group situation of the previous example.

3.2. C-subalgebras

One then defines C-subalgebras of a C-algebra :

Definition : *Given a C-algebra with a basis $\{x_a\}$, $a = 1, \cdots, n$, a C-subalgebra is a C-algebra generated by a subset of the x_a, $a \in T$, $T \subset \{1, \cdots, n\}$, closed under multiplication, i.e. if $a, b \in T$, ${p_{ab}}^c \neq 0$ only if $c \in T$.*

Note that this condition implies that T contains 1 and is stable under the involution $a \mapsto \bar{a}$ [13].

We shall be mainly interested in the situation where the two dual algebras have non negative structure constants. Then there is a remarkable theorem that tells us that the existence of a C-subalgebra in $\mathfrak{A}$ implies the existence of a C-subalgebra in the dual. More precisely, suppose $\mathfrak{A}$ has a C-subalgebra $\mathfrak{A}_T$ associated with a subset T. One may then define an equivalence relation $a \sim b$ if $\exists c \in T \;:\; {p_{ac}}^b \neq 0$, and there is a partition of the set $\{1, 2, \cdots n\}$ into equivalence classes, T_i, $i = 1, \cdots, p$, $T_1 \equiv T$. Let $\rho = \sum_{a\in T} k_a$ and let $X_i := \sum_{a \in T_i} x_a$. One also defines the subset $\hat{T}$ of the dual basis by the decomposition of $X_1 = \sum_T x_a$ into idempotents $X_1 = \rho \sum_{\lambda \in \hat{T}} e_\lambda$.

Theorem (Bannai-Ito [13], theorem 9.9): *Consider a C-algebra $\mathfrak{A}$ with non negative structure constants ${p_{ab}}^c$ and ${q_{\lambda\mu}}^\nu$. With the notations just introduced,*

i) the $\frac{1}{\rho}X_i$, $i = 1, \cdots, p$, generate themselves a C-algebra, called the quotient C-algebra $\mathfrak{A}/\mathfrak{A}_T$, with a product inherited from $\mathfrak{A}$;

ii) the κe_λ, for $\lambda \in \hat{T}$, generate a C-subalgebra $\hat{\mathfrak{A}}_{\hat{T}}$ of the dual algebra $\hat{\mathfrak{A}}$;

iii) these two C-algebras are dual to one another.

Thus one has a dual pattern of subsets T and $\hat{T}$, of C-subalgebras $\mathfrak{A}_T$ and $\hat{\mathfrak{A}}_{\hat{T}}$, and of quotients $\mathfrak{A}/\mathfrak{A}_T$ and $\hat{\mathfrak{A}}/\hat{\mathfrak{A}}_{\hat{T}}$ with the isomorphisms $\widehat{\mathfrak{A}/\mathfrak{A}_T} \cong \hat{\mathfrak{A}}_{\hat{T}}$ and vice versa $\hat{\mathfrak{A}}/\hat{\mathfrak{A}}_{\hat{T}} \cong \widehat{\mathfrak{A}_T}$.

One proves also that all X_i may be expanded on the e_λ, $\lambda \in \hat{T}$, and conversely. Recalling that $x_a = \sum_\lambda p_a(\lambda) e_\lambda$ and $\kappa e_\lambda = \sum_a q_\lambda(a) x_a$, with expressions of $p_a(\lambda)$ and $q_\lambda(a)$ given in (3.1), we find that

$$\begin{aligned} &\sum_{a\in T_i} p_a(\lambda) = 0 \quad \text{if } \lambda \notin \hat{T} \\ \text{thus} \quad &\sum_{a\in T_i} \psi_a^{(\lambda)} \psi_a^{(1)} = 0 \quad \text{if } \lambda \notin \hat{T} \\ \text{for } \lambda \in \hat{T} \quad &q_\lambda(a) = \frac{\psi_a^{(\lambda)*}}{\psi_a^{(1)}} \frac{\psi_1^{(\lambda)}}{\psi_1^{(1)}} \quad \text{independent of } a \in T_i \ . \end{aligned} \tag{3.7}$$

These two conditions may be conveniently assembled into a single one

$$\forall \lambda, \forall T_i, \forall a \in T_i \qquad \sum_{b\in T_i} \psi_b^{(\lambda)} \psi_b^{(1)} = \delta_{\lambda\in\hat{T}} \frac{\psi_a^{(\lambda)}}{\psi_a^{(1)}} \sum_{b\in T_i} (\psi_b^{(1)})^2 \ , \tag{3.8}$$

a form that will be useful in the sequel. It is also easy to write explicitly the expressions of the structure constants of the quotient algebras. For example, from $X_i = \sum_{a\in T_i} x_a$ it follows that $\frac{1}{\rho} X_i . \frac{1}{\rho} X_j = \sum_k \mathbf{p}_{ij}{}^k \frac{1}{\rho} X_k$ with

$$\mathbf{p}_{ij}{}^k = \frac{1}{\rho} \sum_{c\in T_k} p_{ab}{}^c \ , \ \forall a \in T_i, b \in T_j \ . \tag{3.9}$$

In the following two sections, I shall present two applications of this theorem. The first deals with reflection groups and their folding, the second with conformal field theories. The first starts with C-subalgebras of the M algebra (subject to an additional constraint), the second with those of the N algebra.

4. Folding of *ADE* Dynkin diagrams

4.1. The problem

It is well known that non simply laced Dynkin diagrams (of type B_n, C_n, F_4, G_2) may be obtained by folding the simply laced ones, using the symmetries of the original diagram. The extension to Coxeter diagrams of H or I type, associated with the non-crystallographic Coxeter groups, seems more recent [14,15]. In all these works, one is given a simply laced Dynkin diagram describing the scalar products of a set of simple roots $\{\alpha_a\}$, $a = 1, \cdots, n$, according to

$$(\alpha_a, \alpha_b) = 2\delta_{ab} - G_{ab} \tag{4.1}$$

(G the adjacency matrix as in (2.4)). Then a certain partition is found of this set into subsets $\{\alpha_a, a \in T_i\}$ of mutually orthogonal roots

$$(\alpha_a, \alpha_b) = 0 \qquad \text{if} \quad a, b \in T_i \ . \tag{4.2}$$

Let S_a denote the reflection in the hyperplane orthogonal to α_a through the origin, and G the group generated by all the S_a, $a = 1, \cdots, n$. Then one forms the products

$$R_i = \prod_{a \in T_i} S_a \tag{4.3}$$

in which the order is immaterial, since the α are orthogonal within the same T_i, and thus the S_a commute. The group G' generated by the R_i is clearly a subgroup of G. Since G is a Coxeter group (of finite order), G' is also of finite order, hence in the $A - I$ list. The corresponding Coxeter diagram thus results from identifying the vertices of a same block T_i, while the superscript of an edge $i - j$, which yields the order of the element $R_i R_j$ may be computed easily in terms of the original S_a. One finds empirically the adequate foldings of the A, D, E diagrams necessary to manufacture all the others (see Fig. 1). For example, the order 5 of the product $R_2 R_3$ in the diagram H_3, *i.e.* the smallest power m s.t. $(S_2 S_3 S_4 S_6)^m = I$ is simply the order of the Coxeter element of the A_4 Coxeter group generated by these four reflections.

As far as I can see, this procedure is, however, empiric, and doesn't say which folding does the job and in which subspace of the original n-dimensional space the subgroup acts. In the fairly different context of topological field theories (tft), a parallel observation was made. Starting from the so-called minimal tft's labelled by ADE Dynkin diagrams, i.e. solutions of the WDVV equations of the type mentionned in sect. 2, one finds that there are other solutions obtained by restriction of the latter. In such a restriction, only a subset of the flat coordinates t is kept non-vanishing. These non-vanishing t's are labeled by the Coxeter exponents of some non simply laced Coxeter groups [16,17]. These restrictions are consistent with the algebra of the tft, in the sense that they correspond to a sub-algebra of the $C_{\lambda\mu}{}^{\nu}(\mathbf{t})$. If we consider the Chebishev specialization and recall Fact 2 of sect. 2, this means that the Pasquier algebra M of the original ADE diagram admits a sub-algebra, whose generators are labelled by the exponents of a Coxeter group of type $B, C, F, G - I$ [17].

In fact there is a strong connection between the two observations, and through the theory of C-algebras, one is able to answer the previous objection and *determine* the folding through the study of the subalgebras of M type.

4.2. From a M-subalgebra to a subgroup

Consider a simply laced ADE Dynkin diagram such that the structure constants M and N are non negative (see Fact 3 of sect. 2). Recall that all Dynkin diagrams may be 2-coloured, *i.e.* their vertices may be assignend a $\mathbb{Z}_2$ grading τ, the "colour", such that $G_{ab} = 0$ if $\tau(a) = \tau(b)$. Now suppose that a subalgebra of the M algebra has been found, *i.e.* a subset $\hat{T}$ of exponents such that

$$\lambda, \mu \in \hat{T} \qquad {M_{\lambda\mu}}^{\nu} \neq 0 \Rightarrow \nu \in \hat{T} \; ; \tag{4.4}$$

the subset $\hat{T}$ of exponents is assumed to be stable under $\lambda \mapsto h - \lambda$. The positivity condition tells us that we are in the conditions of the theorem of sect. 3.2. Because here we start from a C-subalgebra of the M (or q) algebra, the theorem has to be transposed to its dual version, namely

(i) there is a partition of the set of exponents into equivalence classes $\hat{T}_\alpha$,

$$\mu \sim \nu \quad \text{if} \quad \exists \lambda \in \hat{T} \, , \; {M_{\lambda\mu}}^{\nu} \neq 0 \; ; \tag{4.5}$$

(ii) there exists a special subset T of the dual set of vertices that contains 1;

(iii) the set T enables one to define a dual equivalence relation: $b \sim c$ if $\exists a \in T$ such that ${N_{ab}}^{c} \neq 0$, and hence a partition of the set of vertices into equivalence classes T_i;

(iv) the relation (3.8) is satisfied.

Now the assumption made above that $\hat{T}$ is stable under $\lambda \mapsto h - \lambda$ implies that: (i) the same is true for each class $\hat{T}_\alpha$; (ii) the class T contains only vertices a satisfying $\tau(a) = \tau(1)$; (iii) more generally all the vertices within a same class T_i have the same colour τ and thus the corresponding roots are mutually orthogonal. These are trivial consequences of the symmetry of the ψ

$$\psi_a^{(h-\lambda)} = (-1)^{\tau(a)} \psi_a^{(\lambda)} \tag{4.6}$$

I now claim that with this pattern of subalgebras one may associate a subgroup of G; it is again described by a graph, whose vertices are in one-to-one correspondence with the classes T_i and whose set of exponents is $\hat{T}$. This subgroup is generated by reflections in the hyperplanes orthogonal to some β, that are some linear (real) combinations of the roots α:

$$\beta_i = \mathcal{N}_i \sum_{a \in T_i} \psi_a^{(1)} \alpha_a \; ; \tag{4.7}$$

the normalisation is adjusted so that $(\beta_i, \beta_i) = 2$, namely

$$\mathcal{N}_i^2 \sum_{a \in T_i} (\psi_a^{(1)})^2 = 1 \tag{4.8}$$

(since the α_a, $a \in T_i$ are mutually orthogonal). One verifies, using (3.8), that the product $\prod_{a \in T_i} S_a$ has the same action as the reflection R_i in the hyperplane orthogonal to β_i, in the subspace spanned by the β [18].

The scalar products of two distinct roots β_i and β_j is non positive, as follows from the same property for the original simple positive roots α_a and from the positivity of the $\psi_a^{(1)}$

$$(\beta_i, \beta_j) = \mathcal{N}_i \mathcal{N}_j \sum_{\substack{a \in T_i \\ b \in T_j}} (\alpha_a, \beta_b) \psi_a^{(1)} \psi_b^{(1)} \leq 0 \,.$$

The metric defined on the original roots may be diagonalized by the ψ

$$g_{ab} = (\alpha_a, \alpha_b) = \sum_{\text{exponents } \lambda} g^{(\lambda)} \psi_a^{(\lambda)} \psi_b^{(\lambda)\,*} \,, \tag{4.9}$$

with $g^{(\lambda)} = 2 - 2\cos\frac{\pi\lambda}{h}$. From the expressions of the new roots β_i it is easy to compute the new metric, making use again of (3.8)

$$\begin{aligned} g_{ij} = (\beta_i, \beta_j) &= \mathcal{N}_i^{-1} \mathcal{N}_j^{-1} \sum_{\lambda \in \hat{T}} g^{(\lambda)} \frac{\psi_a^{(\lambda)}}{\psi_a^{(1)}} \frac{\psi_b^{(\lambda)\,*}}{\psi_b^{(1)}} \qquad \forall\, a \in T_i \;, \forall\, b \in T_j \\ &= \sum_{\lambda \in \hat{T}} g^{(\lambda)} \Psi_i^{(\lambda)} \Psi_j^{(\lambda)\,*} \end{aligned} \tag{4.10}$$

in terms of the new eigenvectors

$$\begin{aligned} \lambda \in \hat{T} \qquad \Psi_i^{(\lambda)} &= \mathcal{N}_i^{-1} \frac{\psi_a^{(\lambda)}}{\psi_a^{(1)}} \qquad \forall\, a \in T_i \\ &= \mathcal{N}_i \sum_{a \in T_i} \psi_a^{(\lambda)} \psi_a^{(1)} \,. \end{aligned} \tag{4.11}$$

These eigenvectors form an orthonormal system of rank $|\hat{T}|$.

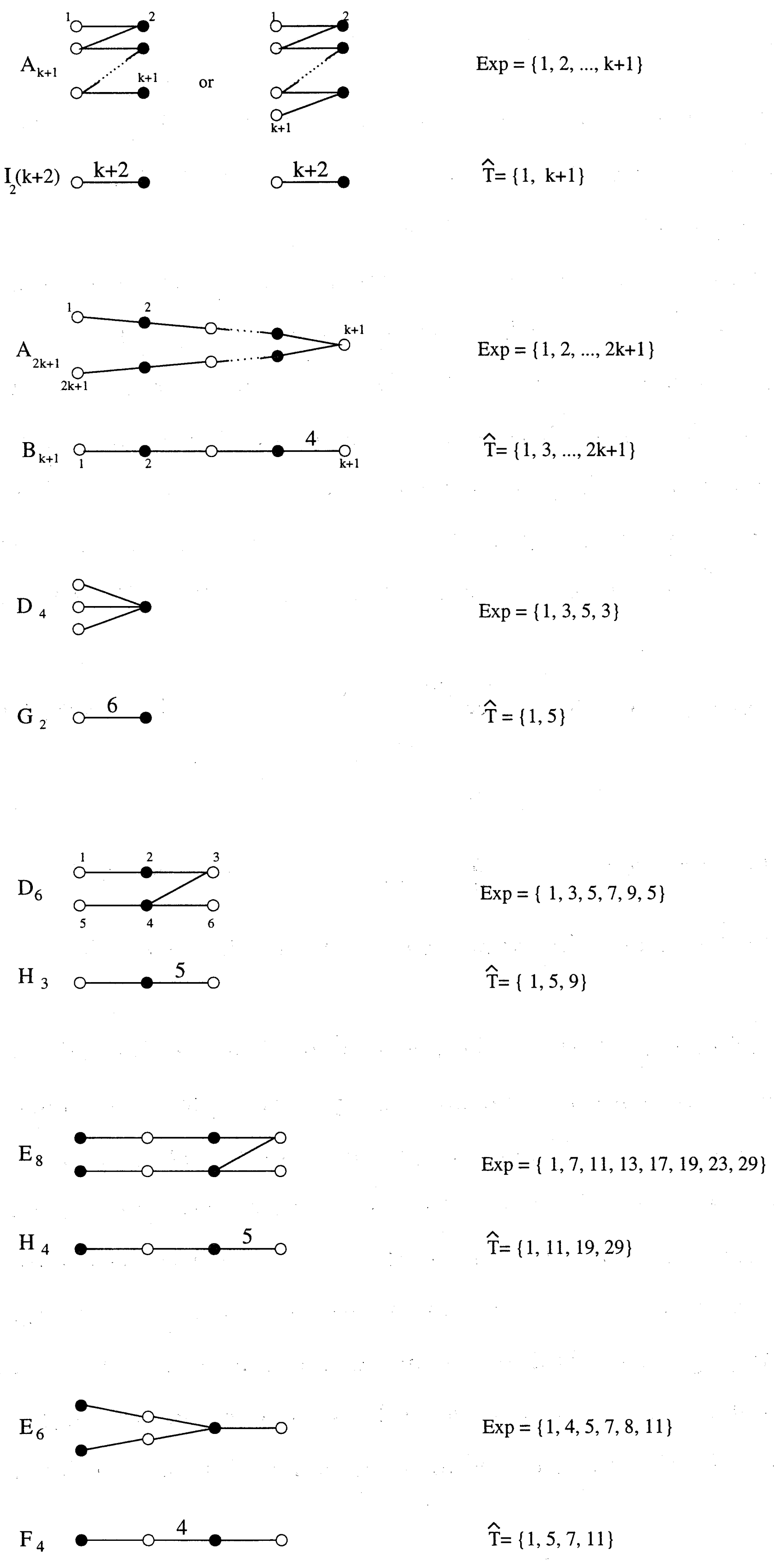

Fig. 1: The folding of ADE Dynkin diagrams of positive type. Classes T_i of vertices encompass nodes on the same vertical.

4.3. Discussion

The reader may wonder what happens in (4.7) if the Perron-Frobenius eigenvector $\psi^{(1)}$ is changed into another eigenvector. In fact, this has the effect of giving roots of the folded diagram that are simple but not positive.

The result of the procedure is presented in fig. 1. For each simply laced Dynkin diagram of type A, $D_{2\ell}$, E_6 or E_8, a systematic search of subalgebra of the M algebra, satisfying the invariance of $\hat{T}$ under $\lambda \mapsto h - \lambda$ has been carried out. All cases are not exhibited in the Figures, as there is some redundancy. For example, any diagram of the previous type admits a subalgebra associated with $\hat{T} = \{1, h-1\}$. This corresponds to folding all vertices of a given colour onto one another, resulting in a 2-vertex graph of type $I_2(h)$. This has been represented only for $A_{k+1} \mapsto I_2(k+2)$ or $D_4 \mapsto G_2 \equiv I_2(6)$.

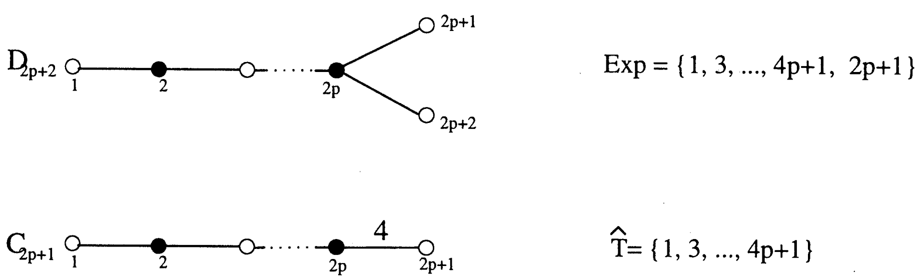

Fig. 2: A case of folding which is discarded by the assumption of positivity

By inspection of fig. 1, the reader may convince herself or himself that the procedure is exhaustive, in the sense that all non simply laced Coxeter diagrams, or all Coxeter groups, have been obtained. In fact, one possible folding of D_{2p+2} into C_{2p+1} (fig. 2) does not appear in the present discussion. To expose the corresponding M subalgebra of the D_{2p+2} diagram requires indeed to change the basis of eigenvectors ψ into another one, in which positivity is lost [12]. In the present case, because of the isomorphism of B_n and C_n Coxeter-Weyl groups, this does not hinder the exhaustivity, but we may expect that the extension of the method to more general cases may require relaxing the hypothesis of positivity. We refer the reader to [18] for a discussion of the appropriate extension of the present method.

5. Dynkin diagrams and RCFT

I shall be more concise on this part as it has already been expounded elsewhere [12,19]. As recalled above in sect 2, conformal field theories with a $\widehat{sl}(2)$ current algebra have been classified according to an ADE scheme. This manifests itself first in the form of their modular invariant genus 1 partition function, written as a sesquilinear form of characters $\chi_\lambda(q)$, $q = e^{2i\pi\tau}$, of the affine $\widehat{sl}(2)$ algebra at a given level k, with the integrable weights λ labelled by integers $1 \le \lambda \le k+1$. One proves [4-5] that the possible expressions of that partition function

$$Z = \sum \mathcal{N}_{\lambda\bar\lambda}\chi_\lambda(q)\bar\chi_{\bar\lambda}(\bar q) \qquad \mathcal{N}_{\lambda\bar\lambda} \in \mathbb{N} \tag{5.1}$$

are such that the *diagonal* terms $\lambda = \bar\lambda$ are the Coxeter exponents of one of the ADE Dynkin diagrams of Coxeter number $h = k+2$.

As alluded to in sect 2, the A, $D_{2\ell}$, E_6 and E_8 cases –and only those– are such that Z is a sum of blocks $Z = \sum_\alpha |\sum_{\lambda\in\hat T_\alpha} \chi_\lambda|^2$:

$$\begin{aligned}
Z^{(A_n)} &= \sum_{\lambda=1}^{n} |\chi_\lambda|^2 && k+2 = n+1, \\
Z^{(D_{2\ell})} &= \sum_{\lambda=1,3,\cdots,2\ell-3} |\chi_\lambda + \chi_{4\ell-2-\lambda}|^2 + 2|\chi_{2\ell-1}|^2 && k+2 = 4\ell-2 \\
Z^{(E_6)} &= |\chi_1+\chi_7|^2 + |\chi_4+\chi_8|^2 + |\chi_5+\chi_{11}|^2 && k+2 = 12 \\
Z^{(E_8)} &= |\chi_1+\chi_{11}+\chi_{19}+\chi_{29}|^2 + |\chi_7+\chi_{13}+\chi_{17}+\chi_{23}|^2 && k+2 = 30\ .
\end{aligned} \tag{5.2}$$

This pattern reflects the existence of an underlying "extended" chiral algebra, containing the current algebra $\widehat{sl}(2)$ as a subalgebra. The combinations $\hat\chi_\alpha = \sum_{\lambda\in\hat T_\alpha}\chi_\lambda$ that appear in (5.2) are characters of the extended algebra decomposed into irreducible characters of $\widehat{sl}(2)$. Let us denote $S_{\lambda\mu}$, resp $\mathbf{S}_{\alpha\beta}$, the matrices of modular transformations of the two sets of characters

$$\begin{aligned}
\chi_\lambda(\tilde q) &= \sum_\mu S_{\lambda\mu}\chi_\mu(q) \\
\hat\chi_\alpha(\tilde q) &= \sum_\beta \mathbf{S}_{\alpha\beta}\hat\chi_\beta(q)\ ,
\end{aligned} \tag{5.3}$$

where $\tilde q = e^{\frac{-2i\pi}{\tau}}$. One has $\mathbf{S}_{\alpha\beta} = \sum_{\lambda\in\hat T_\alpha} S_{\lambda\mu}$, $\forall\mu\in\hat T_\beta$. The quantum dimensions of the representations are the ratios $\mathbf{D}_\alpha = \mathbf{S}_{\alpha 1}/\mathbf{S}_{11}$ and $D_\lambda = S_{\lambda 1}/S_{11}$.

It has been observed in [12,19] that there is a second manifestation of the ADE diagrams hidden in the structure of the operator algebra. For the theories (5.2), one proves that the fusion coefficients $\mathbf{N}_{\alpha\beta}{}^{\gamma}$ of the extended algebra satisfy

$$\mathbf{N}_{\alpha\beta}{}^{\gamma} = \sqrt{\frac{\mathbf{D}_\alpha}{D_\lambda}}\sqrt{\frac{\mathbf{D}_\beta}{D_\mu}} \sum_{\nu\in\hat{T}_\gamma} M_{\lambda\mu}{}^{\nu}\sqrt{\frac{D_\nu}{\mathbf{D}_\gamma}} \,, \qquad \forall\lambda\in\hat{T}_\alpha,\ \mu\in\hat{T}_\beta \,, \tag{5.4}$$

where M are the structure constants of the Pasquier algebra of the relevant Dynkin diagram. (For the sake of simplicity, we assume here and in the rest of the discussion that none of the exponents has a multiplicity larger than 1: this excludes the $D_{\rm even}$ case. The cases with multiplicities require a more elaborated labelling, see [19]).

This equation has several interesting consequences. First, since the matrix $\mathbf{N}$ is diagonalized by the $\mathbf{S}$ matrix, according to the Verlinde formula, it follows from (5.4) that $Y_\lambda := \frac{\mathbf{S}_{\alpha\delta}}{\mathbf{S}_{1\delta}}\sqrt{\frac{D_\lambda}{\mathbf{D}_\alpha}}$, where $\hat{T}_\alpha$ is the block containing λ and δ is any representation of the extended algebra, forms a one-dimensional representation of the M algebra, i.e. $Y_\lambda Y_\mu = \sum_\nu M_{\lambda\mu}{}^{\nu} Y_\nu$, and may thus be identified with some $\frac{\psi_d^{(\lambda)}}{\psi_d^{(1)}}$, for some vertex d

$$\frac{\psi_d^{(\lambda)}}{\psi_d^{(1)}} = \frac{\mathbf{S}_{\alpha\delta}}{\mathbf{S}_{1\delta}}\sqrt{\frac{D_\lambda}{\mathbf{D}_\alpha}} \,. \tag{5.5}$$

In particular, the Krein parameter of the Pasquier algebra reads

$$\hat{k}_\lambda = D_\lambda \mathbf{D}_\alpha, \qquad \text{if} \quad \lambda\in\hat{T}_\alpha \tag{5.6}$$

to be compared with the formula $\hat{k}_\lambda = D_\lambda^2$ of sect. 3.1, example 2, valid for the fusion algebras, i.e. for the A cases for which the blocks T_α contain only one exponent. Let T denote the subset of vertices d for which (5.5) holds. Each of them may be identified with a weight δ of the extended algebra. Further analysis [19] reveals that:

1) $\forall d\in T$, δ the corresponding extended weight, and for $\lambda\in\hat{T}_\alpha$ one has

$$\frac{\psi_d^{(\lambda)}}{\psi_1^{(\lambda)}} = \frac{\mathbf{S}_{\delta\alpha}}{\mathbf{S}_{1\alpha}} \qquad \text{and} \qquad \psi_1^{(\lambda)} = S_{1\lambda}\mathbf{S}_{1\alpha} \,; \tag{5.7}$$

2) one is precisely in the conditions of sect. 3.2: the set T defines a C-subalgebra of the N algebra. In the cases of (5.2) discussed here, the M and N structure constants are non negative (see Fact 3 of sect. 1). One may apply the theorem of Bannai and Ito: the dual subalgebra is associated with a special set $\hat{T}$ which is the block of the identity representation

and it defines a partition of the set of exponents into classes $\hat{T}_\alpha$. Finally equation (5.4) may be seen to be equivalent to equation (3.9) (or rather its dual), if one takes into account the change of normalization between the $q_{\lambda\mu}{}^{\nu}$ and $M_{\lambda\mu}{}^{\nu}$ structure constants and the explicit expressions of the Krein parameters (5.6).

Thus behind the modular invariants (5.4), there is again a structure of C-algebras and subalgebras. This had been first pointed out in [20], and then the more systematic discussion of [19] has shown that this follows from the basic equation (5.4), and that it yields a way to *determine* the expressions of some eigenvectors from conformal data (quantum dimensions).

6. Conclusion and perspectives

The purpose of this lecture was to present the concept of C-algebra and to illustrate its utility in two contexts: the discussion of reflection groups and their foldings on the one hand, and the structure of conformal field theories, on the other.

Note that these two seemingly disparate problems are in fact related in the framework of 2-dimensional topological field theories. For those theories, or at least for those that are obtained by twisting a $\mathcal{N}=2$ superconformal coset field theory, one has two approaches at one's disposal: the discussion of the (super)conformal field theory following lines analogous to the discussion of sect. 5; and the analysis of the Witten-Dijkgraaf-Verlinde-Verlinde equations [1], for which Dubrovin [16] has shown the appearance of monodromy groups generated by reflections. In fact the concept of C-algebra seems to be underlying in a natural way the whole discussion of topological field theories.

Note also that in the two discussions of the previous sections, the same C-algebras (based on the Pasquier algebra of the Dynkin diagrams) have been used in two different ways: in one case (folding), we have been looking at the C-subalgebras of the M algebra (subject to some constraint); in the other (rcft), it is rather some subalgebra of the N algebra that has determined the special set T of vertices, and by duality the blocks T_α.

One issue that requires clarification is the role of positivity. We have from the start restricted our attention to the subcases of the ADE list that have certain positivity properties (see sect. 2). The main benefit has been the possibility to use the theorem of Bannai and Ito (sect. 3.2). It is possible to relax the positivity assumption in the discussion of folding of graphs and groups: what is really crucial is eq. (3.8), see [18]. In the case of rcft, it is less clear how to proceed and what replaces (5.4). In that case, however, we know that any theory with a non block diagonal modular invariant (e.g. (2.2b)) may be obtained from a block diagonal one ((2.2a) in that case) by an automorphism of the fusion algebra [21]. The proper incorporation of that fact in the present considerations remains to be done.

As already mentionned, the very good news is that all this discussion is not limited to the $sl(2)$-ADE cases to which I have restricted myself here for simplicity. On the contrary, both the folding of generalized Dynkin diagrams associated with $sl(N)$ and the block structure of $\widehat{sl}(N)$ RCFT may be discussed in quite general terms. The C-algebra method enables one to find in a fairly systematic way the possible foldings of these generalized diagrams that respect some general properties, and in the second context, it gives non trivial relations between conformal data (fusion coefficients and quantum dimensions) and eigenvectors of the adjacency matrices. It may even enable one to *construct* the graph from these data. See [18] for the former subject and [19] for the latter.

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