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Euclidean Isometries of Julia Sets of Entire Functions

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In this paper, we investigate entire functions whose Julia sets have some symmetries. In §1 we treat polynomials and classify the groups of Euclidean isometries of Julia sets. With this classification we get some properties related to two polynomials with a same Julia set. In §2 we treat transcendental entire functions and show some properties for functions with a Julia set having either a rotation symmetry or translation invariance. The content of §1 is to be published in [**K**].

1 The case of polynomials

1.1 Examples

Let P be a given polynomial of degree at least two and $J(P)$ its Julia set. Julia([**J**]) showed that if two polynomials P and Q are commutative then it holds that $J(P) = J(Q)$. (In fact, this holds even when P and Q are rational functions). Then how about the opposite implication? For this question Baker and Eremenko([**BE**, p.229, Theorem 1]) answered that the converse is not necessarily true in general but it is true unless there exist rotational symmetries of $J(P)$. After that, Beardon([**Be1**]) showed the following:

Theorem A ([B1, p.576, Theorem 1]). *Let*

$$\begin{aligned} \mathfrak{F}(P) &:= \{Q \mid J(Q) = J(P)\}, \\ \mathcal{E} &: \text{ group of conformal Euclidean isometries}, \\ \Sigma(P) &:= \{\sigma \in \mathcal{E} \mid \sigma(J(P)) = J(P)\}. \end{aligned}$$

Then $Q \in \mathfrak{F}(P)$ *if and only if there is some* $\sigma \in \Sigma(P)$ *with* $P \circ Q = \sigma \circ Q \circ P$*: thus*

$$\mathfrak{F}(P) = \{Q \mid P \circ Q = \sigma \circ Q \circ P \ for\ some\ \sigma \in \Sigma(P)\}. \qquad \square$$

In what follows, under the condition $J(P) = J(Q)$ we consider how P and Q are different dynamically, that is, if P is conjugate to Q or not. For example, if $Q = P^n$ for some $n \in \mathbb{N}$, $n \geq 2$, it is well known that $J(P) = J(P^n)$. But in this case P and P^n are not conjugate each other, because $\deg P \neq \deg P^n$. Then, first let us consider the following problem:

Problem 1 : Let Q be a polynomial with $\deg Q = \deg P$ and $J(Q) = J(P)$. Then is Q conformally conjugate to P ?

Beardon([**Be2**]) investigated the set

$$\mathcal{F}(P) := \{Q \mid \deg Q = \deg P,\ J(Q) = J(P)\}$$

and in particular found that the answer for the Problem 1 is "no" in general. More precisely he proved the following:

Theorem B ([Be2, p.196, Theorem 1 & p.199, Theorem 2]).
(1) $\mathcal{F}(P) = \{\sigma \circ P \mid \sigma \in \Sigma(P)\}$.
(2) Suppose that $J(P)$ *is not a circle. Then there is a polynomial* $Q \in \mathcal{F}(P)$ *which is not conformally conjugate to* P *if and only if* $^{\#}\mathrm{Aut}(P) > 1$*, where*

$$\begin{aligned}\mathrm{Aut}(P) &:= \{\gamma \in \mathcal{E} \mid \gamma \circ P \circ \gamma^{-1} = P\} \\ &= \{\gamma \in \mathcal{E} \mid \gamma \circ P = P \circ \gamma\}. \qquad \square\end{aligned}$$

Example 1 ([Be2, p.195, §1]). Consider the following two polynomials:

$$P(z) := z(z^2+1), \quad Q(z) := -z(z^2+1)\ (= -P(z)).$$

Since $P \circ Q = Q \circ P$, we have $J(P) = J(Q)$ so $Q \in \mathcal{F}(P)$. But P and Q are not conformally conjugate each other, because Q has four distinct fixed points but P does not. Moreover P and Q cannot be conjugate each other in any sense so they are dynamically different.

But in some other cases, for a given P and $Q \in \mathcal{F}(P)$, P and Q are not conformally conjugate but can be anti-conformally conjugate as the example below shows. So P and Q have essentially the same dynamics in this case.

Example 2. Consider the following two polynomials:

$$P(z) := iz^3(z^2+1), \quad Q(z) := -iz^3(z^2+1).$$

It is easy to show that P and Q are never conformally conjugate but $\varphi^{-1} \circ P \circ \varphi(z) = Q(z)$, where $\varphi(z) := -\overline{z}$.

This situation can occur due to the existence of axial symmetries of the Julia set. So next let us investigate Euclidean isometries of Julia sets. Also in the sequel we consider the following problem:

Problem 2 : Is $Q \in \mathcal{F}(P)$ conjugate to P under some Euclidean isometry?

Remark 1. For further results on the polynomials with same Julia set, see [**AH**].

1.2 Euclidean isometries of Julia sets

Define

$$\widetilde{\Sigma}(P) := \text{the group of Euclidean isometries of } J(P),$$
$$\Sigma(P) := \text{the group of conformal Euclidean isometries of } J(P).$$

Here we say that σ is a *Euclidean isometry* of $J(P)$ if σ is an isometry of $\mathbb{C} \simeq \mathbb{R}^2$ with respect to the Euclidean metric on $\mathbb{R}^2$ and satisfies $\sigma(J(P)) = J(P)$. It is well known that σ is either a rotation centered about some point or a reflection with respect to some line. In the former case σ is conformal, whereas σ is anti-conformal in the latter case. $\Sigma(P)$ is a subgroup of $\widetilde{\Sigma}(P)$. It is known that $\Sigma(P)$ is a group of rotations centered about the centroid of P ([**Be1**, p.578, Theorem 5]). rotations centered about the centroid of P ([**Be1**, p.578, Theorem 5]). Here the *centroid* ζ *of* P is the center of gravity of all the solutions of $P(z) = w_0$, (which does not depend on w_0). If we denote $P(z) = \sum_{k=0}^{d} a_k z^k$, we have $\zeta = -a_{d-1}/(da_d)$. First we show the following:

Proposition 1. *If $J(P)$ has an axial symmetry, the axis of symmetry passes through the centroid of P.*

(Proof) : It is sufficient to show this in the case where the centroid of P is equal to the origin, since by a suitable linear conjugation we can normalize P so that its centroid is equal to the origin. Suppose that $J(P)$ is invariant under a reflection

$$\sigma(z) := \alpha\overline{(z-c)} + c, \quad (|\alpha| = 1),$$

that is, $\sigma(J(P)) = J(P)$. Here, σ is a reflection with respect to a line

$$L := \{\rho t + c \mid |\rho| = 1, \ \arg\rho = \frac{1}{2}\arg\alpha, \ t \in \mathbb{R}\}.$$

Consider

$$\begin{aligned} Q(z) := \sigma^{-1} \circ P \circ \sigma(z) &= \alpha\overline{(P(\alpha\overline{(z-c)} + c) - c)} + c \\ &= \alpha(\overline{P}(\overline{\alpha}(z-c) + \overline{c}) - \overline{c}) + c, \end{aligned}$$

where $\overline{P}(z) := \sum_{k=0}^{d} \overline{a_k} z^k$. By the definition of Q we have

$$J(Q) = \sigma^{-1}(J(P)) = \sigma(J(P)) = J(P).$$

Then from Theorem B (1) we have $Q = \sigma' \circ P$ for some $\sigma' \in \Sigma(P)$. This shows that the centroid of Q must be also the origin. Since the coefficient of z^{d-1} of $Q(z)$ is

$$\alpha\overline{a_d} \cdot \overline{\alpha}^{d-1} d(-\overline{\alpha}c + \overline{c}),$$

it follows that $\alpha\overline{c} = c$, which means that L passes through the origin. This completes the proof. □

As above in the proof of the Proposition 1, in what follows, we assume that the centroid of P is equal to the origin, that is, $a_{d-1} = 0$.

Theorem 2. *For $P(z) = \sum_{k=0}^{d} a_k z^k$, the Julia set $J(P)$ has an axial symmetry with the axis $\{\rho t \mid \rho = \exp(i\theta), \ 0 \le \theta < \pi, \ t \in \mathbb{R}\}$ if and only if either $P(z) = az^d$ $(a \neq 0)$ (in this case $J(P)$ is a circle) or there exists a λ with $\lambda^s = 1$ such that $a_k \rho^{k-1}/\sqrt{\lambda} \in \mathbb{R}$ $(k = 0, 1, \cdots, d)$, where $s := {}^{\#}\Sigma(P)$. In particular $J(P)$ is symmetric with respect to the real axis if and only if either $P(z) = az^d$ $(a \neq 0)$ or $P(z) = \sqrt{\lambda} R(z)$, where R is a real polynomial.*

(Proof) : First we consider the case where $\rho = 1$, that is, $J(P)$ is symmetric with respect to the real axis. Let $\varphi(z) := \overline{z}$, then the Julia set of $\varphi \circ P \circ \varphi^{-1} = \overline{P}$ is equal to $\varphi(J(P)) = J(P)$. From Theorem B (1) we have

$$\overline{P} = \sigma \circ P, \quad {}^{\exists}\sigma \in \Sigma(P).$$

Denote

$$a_k = r_k \exp(i\theta_k),\ (r_k \in \mathbb{R}),\quad \sigma(z) = \alpha z,\ (|\alpha| = 1),$$

then $\overline{P} = \sigma \circ P$ implies that

$$r_k \exp(-i\theta_k) = \alpha r_k \exp(i\theta_k),\ (k = 0, 1, \ldots, d).$$

Then $\alpha = \exp(-2i\theta_k)$ holds for every k with $a_k \neq 0$. Unless $P(z) = az^d\ (a \neq 0)$, it follows that $\alpha^s = 1$ where $s = {}^{\#}\Sigma(P)$, since $\sigma \in \Sigma(P)$. Hence from this we have $P(z) = \sqrt{\lambda}R(z)$, where $\lambda := \alpha^{-1}$ and R is some real polynomial.

In general cases, let $\psi(z) = \rho z$. Then the Julia set of $\psi^{-1} \circ P \circ \psi$ is equal to $\psi^{-1}(J(P))$ and from the assumption this is symmetric with respect to the real axis. Hence from the above observation we have

$$\psi^{-1} \circ P \circ \psi = \sqrt{\lambda}R,$$

unless $J(P)$ is a circle. By comparing the coefficients of both left and right hand sides, we can obtain the desired condition. □

Remark 2. Here we used the result by Beardon, but we can proof this result directly by using the Boettcher function of the superattractive basin at ∞. In [**S**, p.180, Theorem 4] a similar result is proved in the case where symmetry axis is equal to the real axis. But the form of the polynomial obtained in [**S**] is somehow different from ours.

1.3 Classification of $\widetilde{\Sigma}(P)$

Next we classify the group $\widetilde{\Sigma}(P)$ as follows:

Proposition 3. *$\widetilde{\Sigma}(P)$ is a closed subgroup of the group $\widetilde{\mathcal{E}}$ of all Euclidean isometries. There are four possibilities for $\widetilde{\Sigma}(P)$:*

(1) If it is one dimensional, then $\widetilde{\Sigma}(P)$ consists of all the rotations about the origin and all the reflections with axis passing through the origin.

(2) If it is discrete and contains only reflections, then $\widetilde{\Sigma}(P) = \{\mathrm{Id},\ \varphi\}$, where $\varphi(z) = \rho^2\overline{z}$ for a ρ with $|\rho| = 1$.

(3) If it is discrete and contains only rotations, then $\widetilde{\Sigma}(P) = \Sigma(P) = \{\sigma^i \mid i = 0, 1, \cdots s-1\}$ is a cyclic group, where $\sigma(z) = \mu z$ with $\mu = \exp(2\pi i/s)$ and $s = {}^{\#}\Sigma(P)$.

(4) If it is discrete and contains both reflections and rotations, then $\widetilde{\Sigma}(P)$ *is a dihedral group of order* $2s$*, where* $s = {}^{\#}\Sigma(P) \geq 2$.

We omit the proof, since it is easy. We shall call $\widetilde{\Sigma}(P)$ of *type* x ($x =$ I, II, III, IV) according to the four possibilities (1) $\sim$ (4) above. Note that the case where $J(P)$ has no symmetries is included in type III.

Corollary 4. *(1)* $\widetilde{\Sigma}(P)$ *is of type I if and only if* $P(z) = az^d$ $(a \neq 0)$.
(2) $\widetilde{\Sigma}(P)$ *is of type II if and only if there exists a* $\gamma \in \mathcal{E}$ *such that* $\gamma \circ P \circ \gamma^{-1}$ *is a real polynomial and* $s = {}^{\#}\Sigma(P) = 1$.
(3) $\widetilde{\Sigma}(P)$ *is of type III if and only if* $P(z) = z^a P_1(z^s)$*, where* a *and* s *are maximal with* $0 \leq a < d$ *and* $s = {}^{\#}\Sigma(P) \geq 1$*, and for any* $\gamma \in \mathcal{E}$ *and any* λ *with* $\lambda^s = 1$, $\dfrac{\gamma \circ P \circ \gamma^{-1}}{\sqrt{\lambda}}$ *is not a real polynomial.*
(4) $\widetilde{\Sigma}(P)$ *is of type IV if and only if there exist a* $\gamma \in \mathcal{E}$ *and a* λ *with* $\lambda^s = 1$ *such that* $\dfrac{\gamma \circ P \circ \gamma^{-1}}{\sqrt{\lambda}}$ *is a real polynomial and* $P(z) = z^a P_1(z^s)$*, where* a *and* s *are maximal with* $0 \leq a < d$ *and* $s = {}^{\#}\Sigma(P) \geq 2$.

(Proof) : This can be obtained immediately from Proposition 3 and [**Be1**, p.578, Theorem 5]. □

1.4 Main Result

Finally in this section we give the answer to Problem 2. Now we consider how many $Q \in \mathcal{F}(P)$ is conformally or anti-conformally conjugate to P. Define $g :=$ G.C.M.$(d-1, s)$, where $d = \deg P$, $s = {}^{\#}\Sigma(P)$ in the case that $\widetilde{\Sigma}(P)$ is not of type I. When $\widetilde{\Sigma}(P)$ is of type IV, let $l \in \mathbb{N}$ be the smallest integer such that $\lambda = \mu^l$, where λ is in the above Corollary (4) and $\mu := \exp(2\pi i/s)$.

Main Theorem.
(1) If $\widetilde{\Sigma}(P)$ *is of type I, then any* $Q \in \mathcal{F}(P)$ *is conformally conjugate to* P.
(2) If $\widetilde{\Sigma}(P)$ *is of type II, then* $\mathcal{F}(P) = \{P\}$.
(3) If $\widetilde{\Sigma}(P)$ *is of type III, then* s/g *of* $Q \in \mathcal{F}(P)$ *are conformally conjugate*

to P.

(4) If $\widetilde{\Sigma}(P)$ is of type IV, then whether $g \mid l$ or $g \nmid l$ is determined independently of the choice of λ. Moreover the following holds:

(i) If $g \mid l$, then s/g of $Q \in \mathcal{F}(P)$ are conformally conjugate to P. These Qs are also anti-conformally conjugate to P.

(ii) If $g \nmid l$, then s/g of $Q \in \mathcal{F}(P)$ are conformally conjugate to P and other s/g of different $Q \in \mathcal{F}(P)$ are anti-conformally conjugate to P.

(Proof) : (1) From Corollary 4 (1) we have $P(z) = az^d$ $(a \neq 0)$, and any $Q \in \mathcal{F}(P)$ has the form $Q(z) = \alpha P(z)$ $(|\alpha| = 1)$. Then it is easy to see that

$$\varphi^{-1} \circ P \circ \varphi = Q, \quad \varphi(z) = \nu z \ (\nu^{d-1} = \alpha).$$

(2) This is a direct consequence from Theorem B (1).

(3) This part is essentially contained in [**Be2**, p.199], but for the completeness we include the proof here. Let $\sigma(z) = \mu z$, $\mu = \exp(2\pi i/s)$, then if $Q \in \mathcal{F}(P)$ is conformally conjugate to P, the conformal conjugacy φ between P and Q keeps $J(P) = J(Q)$ invariant. Then we have $\varphi \in \Sigma(P)$ and hence

$$Q = \sigma^{-k} \circ P \circ \sigma^k = \sigma^{(d-1)k} \circ P, \ ^{\exists}k \in \mathbb{N}.$$

Hence for our purpose it is sufficient to show how many $Q = \sigma^{(d-1)k} \circ P$ are different. From the standard argument of elementary number theory, we can conclude that we have s/g of different $Q \in \mathcal{F}(P)$ which is conformally conjugate to P.

(4) First we consider the case where $J(P)$ is symmetric with respect to the real axis. From Theorem 2 we have

$$P(z) = \sqrt{\lambda} R(z), \ (\lambda = \mu^l),$$

where $R(z) = z^a R_1(z^s)$ and R_1 is a real polynomial. Define

$$\varphi_k(z) := \mu^k \overline{z}, \ (k = 0, 1, \ldots, s-1).$$

Suppose that P and $Q \in \mathcal{F}(P)$ are anti-conformally conjugate each other, then anti-conformal conjugacy φ between P and Q is equal to one of the φ_ks, since φ keeps $J(P) = J(Q)$ invariant. Then we have

$$\varphi_k^{-1} \circ P \circ \varphi_k(z) = \mu^{(1-a)k-l} P(z).$$

So if $g|l$, we have

$$\{\mu^{(1-a)k-l}P\}_{k=0}^{s-1} = \{\mu^{(d-1)k}P\}_{k=0}^{s-1},$$

and hence these s/g of different polynomials are both conformally and anti-conformally conjugate to P. If $g \not| l$ we have

$$\{\mu^{(1-c)k-l}P\}_{k=0}^{s-1} \cap \{\mu^{(d-1)k}P\}_{k=0}^{s-1} = \emptyset,$$

and hence we obtain the result. □

The following is an immediate corollary of the main theorem, so we omit the proof.

Corollary 5. *Every $Q \in \mathcal{F}(P)$ is conjugate to P by a Euclidean isometry if and only if one of the following holds:*

(i) $\widetilde{\Sigma}(P)$ *is of type I,*

(ii) $\widetilde{\Sigma}(P)$ *is of type II,*

(iii) $\widetilde{\Sigma}(P)$ *is of type III and* $g = 1$*,*

(iv) $\widetilde{\Sigma}(P)$ *is of type IV and* $g = 2$ *and* l *is odd.* □

2 The case of transcendental entire functions

Let $f(z)$ be a transcendental entire function. In what follows, we observe some easy facts concerning with Julia sets which are invariant under some conformal Euclidean isometries.

2.1 Rotation symmtries

Proposition 6.

(1) If $f(z) = z^a f_1(z^s)$ $(a,\ s \in \mathbb{N},\ s \geq 2)$*, where* $f_1(z)$ *is a transcendental entire function, then* $\sigma(J(f)) = J(f)$ *holds, where* $\sigma(z) = \lambda z$ $(\lambda^s = 1)$*. That is,* $J(f)$ *has a rotation symmetry.*

(2) $J(f)$ *has* $SO(2)$ *symmetry, that is,* $\sigma(J(f)) = J(f)$ *holds for any* $\sigma(z) = \lambda z$ $(|\lambda| = 1)$ *if and only if* $J(f) = \mathbb{C}$*.*

(Outline of the Proof) : (1) By the assumption and the definition of Julia set, one can show that $z \in J(f)$ if and only if $\sigma(z) \in J(f)$.

(2) This is obtained by the maximal principle. □

Opposite implication of Propositon 6 (1) does not hold. Consider $f(z) = e^z$. Since $J(e^z) = \mathbb{C}$, it is obvious that $J(e^z)$ has a rotation symmetry. On the other hand, it is easy to check that any conjugate function of e^z by a translation cannot be written in the form of $z^a f_1(z^s)$ $(a,\ s \in \mathbb{N},\ s \geq 2)$. We conjecture that opposite implication is true provided that $J(f) \neq \mathbb{C}$.

2.2 Translation invariance

Since the Julia set of a polynomial is a compact subset of $\mathbb{C}$, it is impossible that it has a translation invariance. But since the Julia set of a transcendental entire function is an unbounded closed subset of $\mathbb{C}$, it can be translation invariant. For example, it is easy to see that a periodic function with period $c \neq 0$ have a Julia set which is invariant under the translation $\gamma(z) = z + c$. In general, the following holds.

Proposition 7.

(1) If $f(z)$ satisfies $f(z+c) = f(z) + nc$ $(n \in \mathbb{N},\ c \neq 0)$, then $\gamma(J(f)) = J(f)$ holds, where $\gamma(z) = z + c$. That is, $J(f)$ has a translation invariance.

(2) $f(z)$ satisfies $f(z+c) = f(z) + nc$ $(n \in \mathbb{N},\ c \neq 0)$ if and only if $f(z) = g(\exp(2\pi i z/c)) + nz$, where $g(z)$ is a holomorphic function on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

(Outline of the Proof) : (1) By the assumption and the definition of Julia set, one can show that $z \in J(f)$ if and only if $\gamma(z) \in J(f)$.

(2) Define $h(z) := f(z) - nz$, then we have

$$h(z+c) = f(z+c) - n(z+c) = f(z) - nz = h(z).$$

Hence $h(z)$ is a periodic function of period c and it is easy to see that we can express $h(z) = g(\exp(2\pi i z/c))$, where $g(z)$ is a holomorphic function on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. □

Opposite implication of Propositon 7 (1) does not hold. It is easy to construct a counter-example with the function $f(z) = \lambda z e^z$, where λ is a suitable constant. But we conjecture again that opposite implication is true provided that $J(f) \neq \mathbb{C}$.

Incidentally, in the case of rational functions there is a following result:

Theorem ([Bo, p.1, Theorem 1]). *Let f be a rational function of degree at least two such that $J(f)+1 = J(f)$ and such that infinity is either periodic or preperiodic. Then $J(f)$ is either $\widehat{\mathbb{C}}$ or a horizontal line.*

We end this paper with the following conjecture:

Conjecture. *Let f be a transcendental entire function and $\nu(z) = \mu z$ ($\mu = \exp(2\pi i/s)$) or $z + c$. Then $\nu(J(f)) = J(f)$ holds if and only if $J(f) = \mathbb{C}$ or $f(\nu(z)) = \nu^l(f(z))$ ($l \in \mathbb{N}$) hold.*

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