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Examples of convolution equations in tube domains

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In the present paper, we study convolution equations of type $\mu * u = f$, where the kernel μ is a hyperfunciton with compact support, the given data f and the unknown function u are holomorhpic functions in the tube domain $U \in \mathbb{C}^n$ with the form $U = \mathbb{R}^n \times \sqrt{-1}\Omega$ by an open convex subset $\Omega \subset \mathbb{R}^n$. First we recall the notions of the condition (S) due to T. Kawai [3] and the characteristics Char($\mu*$) (see [2]), which are deeply related to the existence and the continuation of holomorphic solutions. After that we give some examples in non-local operator case.

1 The condition (S) and the characteristics

Let μ be a hyperfunction with compact support on $\mathbb{R}^n$, and Ω be a convex open set in $\mathbb{R}^n$. We consider a convolution equation:

$$\mu * u = f \qquad f \in \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\Omega).$$

We denote by $\hat{\mu}(\zeta)$ the Fourier-Borel transform of μ defined by

$$\hat{\mu}(\zeta) := \langle \mu, e^{x\zeta} \rangle_x = \int_{\mathbb{R}^n} \mu(x) e^{x\zeta} dx.$$

$\hat{\mu}$ is an entire function of exponential type, precisely $\hat{\mu}$ satisfies

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0, \quad |\hat{\mu}(\zeta)| \leq C_\varepsilon \exp(H_K(\zeta) + \varepsilon |\zeta|),$$

where $K = \operatorname{supp} \mu$ and

$$H_K(\zeta) := \sup_{x \in K} \operatorname{Re} x\zeta$$

is the supporting function of K.

For the convolution operator $\mu *$, we introduce the condition (S) due to Prof. Kawai and the notion of characteristics of $\hat{\mu}$.

Definition 1.1. $\hat{\mu}$ satisfies the condition (S) if and only if for any $\varepsilon > 0$, there exists $N = N_\varepsilon > 0$, such that for any $\eta \in \mathbb{R}^n$ with $|\eta| > N$, we can find $\zeta \in \mathbb{C}^n$ satisfying:

- $\left|\zeta - \sqrt{-1}\eta\right| < \varepsilon |\eta|$,
- $|\hat{\mu}(\zeta)| > -\varepsilon |\eta|$.

Definition 1.2. We define the characteristics $\mathrm{Char}(\mu *) \subset \sqrt{-1}S^{n-1}$ by: the vector $\sqrt{-1}\rho \in \sqrt{-1}S^{n-1}$ with $(|\rho| = 1)$, belongs to $\mathrm{Char}(\mu *)$ if and only if there exists a sequence $\{\zeta_\nu\}_\nu \subset \mathbb{C}^n$ satisfying:

- $\hat{\mu}(\zeta\nu) = 0$ for any ν,
- $|\zeta_\nu| \to \infty$ as $\nu \to \infty$,
- $\zeta_\nu / |\zeta_\nu| \to \sqrt{-1}\rho$ as $\nu \to \infty$.

Moreover we define the polar enveloping of Ω and $U = \mathbb{R}^n \times \sqrt{-1}\Omega$. We put

$$\langle \Omega \rangle_\mu := \text{ the interior of } \bigcap_{\eta \in \mathrm{Char}(\mu *)} \{y \in \mathbb{R}^n; y\eta < \sup_{y' \in \Omega} y'\eta\}$$

and

$$\langle U \rangle_\mu := \mathbb{R}^n \times \sqrt{-1}\langle \Omega \rangle_\mu.$$

Under these notations, we recall our result about existence and continuation problem (see [2]), which tell us the importance of the condition (S) and the notion of characteristics.

Theorem 1.3. *If $\hat{\mu}$ satisfies* (S), *then for any open convex subset $\Omega \subset \mathbb{R}^n$, $\mu * : \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\Omega) \to \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\Omega)$ is surjective. Conversely, assume that $\mu * : \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\Omega) \to \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\Omega)$ is surjective for a bounded open convex subset $\Omega \subset \mathbb{R}^n$ with C^2-boundary. Then $\hat{\mu}$ satisfies* (S).

Theorem 1.4. *Assume that* $\hat{\mu}$ *satisfies* (S). *We put*

$$\mathrm{Sol}^{\mu}(U) := \{u \in \mathcal{O}(U); \mu * u = 0\}$$

for $U = \mathbb{R}^n \times \sqrt{-1}\Omega$. *Then the restriction map* $\mathrm{Sol}^{\mu}(\langle U \rangle_{\mu}) \to \mathrm{Sol}^{\mu}(U)$ *is surjective.*

Remark that a kind of the converse statement of this theorem is also true (see [6]).

2 The case of differential-difference equations

Let us consider the case that $\mathrm{supp}\,\mu$ consists of finite points. We set

$$\mathrm{supp}\,\mu = \{\lambda_1, \ldots, \lambda_\ell\}$$

with $\lambda_j \in \mathbb{R}^n$, and $\lambda_i \neq \lambda_j$ for $i \neq j$.

By the standard structure theorem of hyperfunctions, we can find a family $\{P_j(\zeta)\}_{j=1,\ldots,\ell}$ of entire functions of infra-exponential type such that

$$\mu(x) = \sum_{j=1}^{\ell} P_j(D)\delta(x - \lambda_j),$$

where $P_j(D)$'s are the differential operators of infinite order with constant coefficients defined by P_j's. Thus we have:

$$\mu * u = \sum_{j=1}^{\ell} P_j(D)u(z - \lambda_j)$$

and the convolution equation is differential-difference equation of infinite order.

In this case, we give

Theorem 2.1. *Let* μ *be a hyperfunction whose support consists of finite points. Then* $\hat{\mu}$ *satisfies* (S). *Moreover if* $\#\,\mathrm{supp}\,\mu > 2$, *then* $\mathrm{Char}(\mu *) = \sqrt{-1}S^{n-1}$.

Corollary 2.2. *Differential-difference equations in tube domains are always solvable. Moreover all pure imaginary vectors are characteristic except the case the equations coincides with a differenctial equation under a suitable translation.*

This theorem can be proved by the theory of entire functions of completely regular growth and the asymptotic estimate of zeros of entire functions of this form due to Ronkin [7].

3 An example of elliptic operator

In this section, we give an example of non-local elliptic operator in the case $n = 1$.

For positive constant a and b with $(a < b)$, we will construct a hyperfunction μ with the following properties:

1. the convex full of $\operatorname{supp}\mu$ coincides with the segment $[a, b]$,
2. $\hat{\mu}$ satisfies (S),
3. $\operatorname{Char}(\mu *) = \emptyset$.

Moreover we give a remark that $\hat{\mu}$ is not of completely regular growth for any direction $\zeta_0 \in \mathbb{C} \setminus \sqrt{-1}\mathbb{R}$.

Take a sequence $\{a_n\}_{n=1,2,\ldots}$ in $\mathbb{R}$ with

- $\{|a_n|\}_n$ is strictly increasing,
- $n/|a_n| \to 0$ as $n \to \infty$,
- $\sum_n 1/|a_n|$ diverges,
- $\limsup_{N\to\infty} \sum_{n<N} 1/a_n = b$, and $\liminf_{N\to\infty} \sum_{n<N} 1/a_n = a$.

For example, $a_n = \epsilon_n \cdot n \log n$ with $\epsilon_n = \pm 1$ satisfies the conditions for any a and b, if we choose the signs ϵ_n suitably according to a and b.

Put

$$\delta(r) := \sum_{|a_n|<r} \frac{1}{a_n},$$

$$f(\zeta) := \prod_n \left(1 - \frac{\zeta}{a_n}\right) e^{\zeta/a_n},$$

then we can show that f is a Fourier-Borel transform of a hyperfunction μ satisfying the condition 1 and 3. We can also show the condition 2, by the estimate

$$\left|f(\sqrt{-1}\eta)\right| \geq 1 \quad \text{for any } \sqrt{-1}\eta \in \sqrt{-1}\mathbb{R}.$$

For this μ, we have

- the convolution equation $\mu * u = f$ is always solvable in any tube domain,
- any solution u of $\mu * u = 0$ can be continued analytically to $\mathbb{C}$.

References

[1] R. Ishimura, *A remark on the characteristic set for convolution equations*, Mem. Fac. Sci. Kyushu Univ., **46** (1992) 195–199.

[2] R. Ishimura and Y. Okada, *Existence and continuation of holomorphic solutions for convolution equations in tube domains*, Bull. Soc. Math. Fr., **122** (1994) 413–433.

[3] T. Kawai, *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math., **17** (1970) 467–517.

[4] B. Ja. Levin, *Distribution of zeros of entire functions*, Transl. Math. mono. vol.5, AMS, Providence, (1964).

[5] Y. Okada, *Solvability of convolution operators*, Publ. RIMS. Kyoto Univ., **30** (1994) 167–190.

[6] Okada Y., *On the characteristics for convolution equations in tube domains*, preprint.

[7] L. I. Ronkin, *Functions of completely regular growth*, MIA, Kluwer, (1992).