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The Continuation of Holomorphic Solutions for Convolution Equations in Complex Domains

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§1. Introduction

The problem of analytic continuation of the solutions is a very important issue in the theory of partial differential equations. In the case of partial differential equations with finite order, the first results for such a problem was obtained by Kiselman[Ki] in the relation with the characteristic set of operators. After that, Sébbar[S] extended them to the case of differential operators of infinite order with constant coefficients. Motivated by [S], Aoki[A] proved a local continuation theorem for differential operators of infinite order with variable coefficients, using his theory of exponential caluculus for pseudo-differential operators. In the case of convolution equations defined in tube domains invariants by any real translations, Ishimura and Y. Okada[I-YO1] showed that the directions to whom not all solution can be continued were estimated by the characteristic set of the operator by using the method developped by [Ki] and [S]. Similar problems can be found in the literature[B-G-V] and [V].

In this paper, we consider the problem of analytic continuation of the solutions of homogeneous equations in complex domains. Firstly, we define the characteristic set of convolution operators to be a natural generalization of the case of differential operators. Secondly, we will give the division lemma under the condition (S). Finally, we evaluate the directions to whom not all holomorphic solution is analytically extended by the characteristic set. We refer to [I-JO-YO] for more details and proofs.

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§2. The condition (S) and the characteristic set

Let U be an open set in $\mathbf{C}^n$ and $\mathcal{O}(U)$ be the space of holomorphic functions on U equipped with the topology of uniform convergence on compact subsets of U. For each nonzero analytic functional $T \in \mathcal{O}(\mathbf{C}^n)'$ carried by a compact convex set $M \subset \mathbf{C}^n$, we denote by $\hat{T}(\zeta)$ its Fourier-Borel transform:

$$\hat{T}(\zeta) = T_z(\exp < z, \zeta >)$$

which is an entire function of exponential type satisfying the following estimate:

Theorem 2.1.(Polyá-Ehrenpreis-Martineau) If $T \in \mathcal{O}(\mathbf{C}^n)'$ and T is carried by M, then $\hat{T}(\zeta)$ is an entire function and for every $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$|\hat{T}(\zeta)| \leq C \exp(H_M(\zeta) + \varepsilon|\zeta|) \qquad \text{for } \zeta \in \mathbf{C}^n, \tag{2.1}$$

where $H_M(\zeta) = \sup_{z \in M} \operatorname{Re} < z, \zeta >$ is the supporting function of M.

Let ω be an open set in $\mathbf{C}^n$. Throughout the remainder of this report, we suppose that $\omega + (-M) \subset \omega$. Then we define a continuous linear convolution operator

$$T* \; : \; \mathcal{O}(\omega + (-M)) \longrightarrow \mathcal{O}(\omega)$$

which is given by

$$(T * f)(z) = T_\zeta(f(z - \zeta)) \qquad z \in \omega,$$

and we consider the homogeneous convolution equation

$$T * f = 0.$$

We define the sphere at infinity S^{2n-1}_∞ by $(\mathbf{C}^n \backslash \{0\})/\mathbf{R}_+$ and we consider the compactification by directions:

$$\mathbf{D}^{2n} = \mathbf{C}^n \sqcup S^{2n-1}_\infty \quad \text{of} \quad \mathbf{C}^n \cong \mathbf{R}^{2n}.$$

For $\zeta \in \mathbf{C}^n \setminus \{0\}$, we denote by $\zeta\infty \in S^{2n-1}_\infty$ the class defined by ζ, that is,

$$\zeta\infty = (\overline{\{t\zeta \; ; \; t > 0\}} \text{ in } \mathbf{D}^{2n}) \cap S^{2n-1}_\infty.$$

In the sequel, we will take $s(\zeta) = \check{\hat{T}}(\zeta)$, where for a function $g(\zeta)$, we put $\check{g}(\zeta) = g(-\zeta)$.

By Lelong and Gruman[L-G], we define the growth indicator $h_s(\zeta)$ and the regularized growth indicator $h^*_s(\zeta)$ of $s(\zeta)$:

$$h_s(\zeta) = \limsup_{r \to \infty} \frac{\log |s(r\zeta)|}{r}, \tag{2.2}$$

$$h^*_s(\zeta) = \limsup_{\zeta' \to \zeta} h_s(\zeta'). \tag{2.3}$$

As in [I-YO1], and generalizing to the present case, we define the characteristic set of $T*$.

Definition 2.2. We set

$$\begin{aligned}\operatorname{Char}_\infty(T*) = \; & \text{the complement of } \{\rho\infty \in S^{2n-1}_\infty \; ; \text{ for every } \varepsilon > 0, \\ & \text{there exists } N > 0 \text{ and a conic neighborhood } \Gamma \text{ of } \rho \text{ such that} \\ & |\hat{T}(\zeta)| \geq \exp(h^*_{\hat{T}}(\zeta) - \varepsilon|\zeta|) \text{ for any } \zeta \in \Gamma \cap \{|\zeta| > N\}\}\end{aligned}$$

and call it the characteristic set of the operator $T*$.

Now we introduce the condition (S), originally due to T. Kawai[Ka] and was defined in a direction in [I-JO].

Definition 2.3. An entire function $s(\zeta) \in \mathcal{O}(\mathbf{C}^n)$ of exponential type is said to satisfy the condition (S) at the direction $\zeta_0 \in \mathbf{C}^n \setminus \{0\}$, if it satisfies the folloing condition:

$$\left\{ \begin{array}{l} \text{For every } \varepsilon > 0, \text{ there exists } N > 0 \text{ such that} \\ \text{for any } r > N, \text{ we have } \zeta \in \mathbf{C}^n \text{ satisfying} \\ |\zeta - \zeta_0| < \varepsilon, \quad \log|s(r\zeta)| \geq (h_s^*(\zeta_0) - \varepsilon)r. \end{array} \right.$$

Remark 2.4. [I-JO] showed that this condition (S) is nothing but the condition of regular growth, which is the classical notion in the theory of entire functions.

Remark 2.5. By (2.1) and (2.3), we have in general $h_{\check{T}}^*(\zeta) \leq H_M(\zeta)$. In the sequel, we will suppose $h_{\check{T}}^*(\zeta) \equiv H_M(\zeta)$. For open convex domains, this last condition with the condition (S) is, in a sense, necessarry and sufficient condition for solvability of inhomogeneous convolution equation $T * f = g$. See Krivosheev[Kr] for more precise statement.

§3. Division lemma

In this section we present some results which are used to prove the main theorem. First we recall a lemma due to Harnack, Malgrange and Hörmander:

Lemma 3.1. Let $F(\zeta), H(\zeta)$ and $G(\zeta) = H(\zeta)/F(\zeta)$ be three holomorphic functions in the open ball $B(0;R)$. If the inequalities $|F(\zeta)| < A$ and $|H(\zeta)| < B$ hold on $B(0;R)$. Then we have

$$|G(\zeta)| \leq BA^{\frac{2|\zeta|}{R-|\zeta|}}|F(0)|^{-\frac{R+|\zeta|}{R-|\zeta|}} \qquad \text{for all } \zeta \in B(0;R).$$

By using this lemma, we can show the following:

Lemma 3.2. Let s, φ and ψ be entire functions satisfying $s\varphi = \psi$, and M and K be two compact convex sets in $\mathbf{C}^n$. We also suppose that for any $\varepsilon > 0$, there are constants $A > 0$ and $B > 0$ such that

$$\begin{aligned} \log|s(\zeta)| &\leq A + H_M(\zeta) + \varepsilon|\zeta|, \\ \log|\psi(\zeta)| &\leq B + H_K(\zeta) + \varepsilon|\zeta|. \end{aligned}$$

Moreover we assume that s satisfies the condition (S) in any direction $\zeta_0 \in \mathbf{C}^n \setminus \{0\}$ and $h_s^*(\zeta) = H_M(\zeta)$ for any $\zeta \in \mathbf{C}^n$. Then for every $\varepsilon > 0$, there exists a constant $C > 0$ such that, setting $L = K + 3M$, we have

$$\log|\varphi(\zeta)| \leq C + H_L(\zeta).$$

For any open set $U \subset \mathbf{C}^n$, we put

$$\mathcal{N}(U) = \{f \in \mathcal{O}(U)\ ;\ T * f = 0\}$$

and equip it with the topology induced from $\mathcal{O}(U)$. By applying Lemma 3.2, we can prove the following proposition:

Proposition 3.3. Let ω and Ω be two open sets in $\mathbf{C}^n$ with $\omega \subset \Omega$. We suppose that T satisfies the condition (S) in every direction in $\mathbf{C}^n \setminus \{0\}$ and $h^*_{\hat{T}}(\zeta) \equiv H_M(\zeta)$. Then the restriction map:

$$\tau\ :\ \mathcal{N}(\Omega + (-M)) \longrightarrow \mathcal{N}(\omega + (-M))$$

has the dense image.

§4. Continuation of solutions of homogeneous equations

For $\mathrm{Char}_\infty(T*)$ and an open convex set $\omega \subset \mathbf{C}^n$, we set

$$\Omega = \text{the interior of } \Big(\bigcap_{\zeta\infty \in \mathrm{Char}_\infty(T*)^a} \{z \in \mathbf{C}^n\ ;\ \mathrm{Re} < z, \zeta > \leq H_\omega(\zeta)\}\Big), \tag{4.1}$$

where ^{a} means the antipodal: $D^a = -D$. By definition of Ω, we know that for any compact convex set $L \subset \Omega$, there exists a compact convex set $K \subset \omega$ such that

$$H_L(\zeta) \leq H_K(\zeta) \qquad \text{for any } \zeta\infty \in \mathrm{Char}_\infty(T*)^a \tag{4.2}$$

and so

$$H_{L+(-M)}(\zeta) \leq H_{K+(-M)}(\zeta) \qquad \text{for any } \zeta\infty \in \mathrm{Char}_\infty(T*)^a.$$

Lemma 4.1. We suppose $h^*_{\hat{T}}(\zeta) \equiv H_M(\zeta)$. Let K and L be two compact sets of $\mathbf{C}^n$ satisfying (4.2) and $p(\zeta)$ be an entire function with the estimate:

$$\log |p(\zeta)| \leq H_L(\zeta).$$

Then for any $\varepsilon > 0$, there exists a constant $C > 0$ and entire functions $q(\zeta)$ and $r(\zeta)$ which satisfy:

$$\begin{aligned} p(\zeta) &= \check{\hat{T}}(\zeta) q(\zeta) + r(\zeta), \\ \log |q(\zeta)| &\leq H_{L\cup K}(\zeta) - H_{-M}(\zeta) + \varepsilon|\zeta| + C, \\ \log |r(\zeta)| &\leq H_K(\zeta) + \varepsilon|\zeta| + C \end{aligned}$$

for any $\zeta \in \mathbf{C}^n$.

Now we can state our main theorem:

Theorem 4.2. Let $M \subset \mathbf{C}^n$ be a compact convex set and T be an analytic functional carried by M. We suppose that T satisfies the condition (S) in every direction in $\mathbf{C}^n \setminus \{0\}$ and

$h^*_{\hat{T}}(\zeta) \equiv H_M(\zeta)$. For any open convex set $\omega \subset \mathbf{C}^n$ with $\omega + (-M) \subset \omega$, let Ω be the open set defined by (4.1). Then any holomorphic solution f of the homogeneous convolution equation $T * f = 0$ defined on $\omega + (-M)$ is analytically continued to $\Omega + (-M)$.

proof. We will prove that the restriction map

$$\tau : \mathcal{N}(\Omega + (-M)) \longrightarrow \mathcal{N}(\omega + (-M))$$

is an isomorphism. For the space $\mathcal{N}(\omega + (-M))$, we will denote by $\mathcal{N}(\omega + (-M))'$ the dual space. By Proposition 3.3, τ is of dense image, therefore the transposed map

$${}^t\tau : \mathcal{N}(\omega + (-M))' \longrightarrow \mathcal{N}(\Omega + (-M))'$$

is injective. It is sufficient to prove ${}^t\tau$ is also surjective. By the Hahn-Banach theorem, any $S \in \mathcal{N}(\Omega + (-M))'$ has an extension $\tilde{S} \in \mathcal{O}(\Omega + (-M))'$. Then there exist a compact convex set $L \subset \Omega$ and a constant $C > 0$ such that

$$|\tilde{S}(\zeta)| \leq C \exp(H_{L+(-M)}(\zeta)) \quad \text{for any } \zeta \in \mathbf{C}^n.$$

We can take a compact convex set $K \subset \omega$ satisfying (4.2). By using the lemma 3.4 to $p = \tilde{S}$, $L + (-M)$ and $K + (-M)$, for any small $\varepsilon > 0$, there exist entire functions $q(\zeta)$, $r(\zeta)$ and a constant $C > 0$ such that

$$\begin{aligned} p(\zeta) &= \check{\hat{T}}(\zeta) q(\zeta) + r(\zeta), \\ \log |q(\zeta)| &\leq H_{L \cup K}(\zeta) + \varepsilon|\zeta| + C, \\ \log |r(\zeta)| &\leq H_{K+(-M)}(\zeta) + \varepsilon|\zeta| + C. \end{aligned}$$

Thus if $\varepsilon > 0$ is taken small enough, we find analytic functionals $Q \in \mathcal{O}(\Omega + (-M))'$ and $R \in \mathcal{O}(\omega + (-M))'$ corresponding to $q(\zeta)$ and $r(\zeta)$ i.e. $\hat{Q} = q$ and $\hat{R} = r$ such that $\hat{\tilde{S}} = \check{\hat{T}}\hat{Q} + \hat{R}$. Then for any $g \in \mathcal{N}(\Omega + (-M))$, we have

$$< S, g >=< \tilde{S}, g >=< Q, T * g > + < R, g >=< R, g >,$$

and this means $S = {}^t\tau(R|_{\mathcal{N}(\omega+(-M))})$. Q.E.D.

We conclude this section with the following example.

Example 4.3. Let $\{\lambda_1, \lambda_2, \cdots, \lambda_l\}$ be a finite set in $\mathbf{C}^n$ and M be its convex-hull and $p_1(\zeta), p_2(\zeta), \cdots, p_l(\zeta)$ be entire functions of minimal type. We denote by T the analytic functional of which Fourier-Borel transform is $\sum_{j=1}^{l} p_j(\zeta) \exp < \zeta, \lambda_j >$. Then T is carried by M. Furthermore by Ronkin[R] and by [I-JO], we know $h^*_{\hat{T}}(\zeta) \equiv H_M(\zeta)$ and that $\hat{T}(\zeta)$ satisfies the condition (S) in every direction in $\mathbf{C}^n \setminus \{0\}$. Thus this T satisfies all hypothesis of Theorem 4.2. In particular, in the case where p_j $(1 \leq j \leq l)$ are non-zero constants, we can prove that the characteristic set $\mathrm{Char}_\infty(T*)$ coincides with the set

$$\{\zeta\infty \in S^{2n-1}_\infty \,;\, \sharp\{j \,;\, \mathrm{Re} < \zeta, \lambda_j >= H_M(\zeta)\} \geq 2\}.$$

(See [I-YO3] for more details.)

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