数理解析研究所講究録
1088 巻 1999 年 68-78

On the computation of Stokes multipliers via Hyperasymptotics

A. B. Olde Daalhuis

Department of Mathematics and Statistics, King's Buildings,
University of Edinburgh, Edinburgh, EH9 3JZ, UK
adri@maths.ed.ac.uk

Abstract. In this paper we explain how the hyperasymptotic expansion of late terms in divergent asymptotic expansions can be used to compute all the Stokes multipliers to arbitrary precision.

1. Introduction

We shall investigate the computation of the Stokes multipliers of the solutions of differential equations of the form

$$\frac{d^n w}{dz^n}+f_{n-1}(z)\frac{d^{n-1}w}{dz^{n-1}}+\cdots+f_0(z)w=0, \tag{1.1}$$

in which the coefficients $f_m(z)$, $m=0,1,\cdots,n-1$ can be expanded in power series

$$f_m(z)=\sum_{s=0}^{\infty}\frac{f_{sm}}{z^s}, \tag{1.2}$$

that converge on an open annulus $|z|>a$, and the point at infinity is an irregular singularity of rank 1. Formal series solutions in descending powers of z are given by

$$e^{\lambda_j z}z^{\mu_j}\sum_{s=0}^{\infty}a_{sj}z^{-s}, \qquad j=1,2,\cdots,n. \tag{1.3}$$

The constants λ_j, μ_j and a_{sj} are found by substituting into the differential equation and equating coefficients after setting $a_{0j}=1$. In this way we obtain the characteristic equation

$$\sum_{m=0}^{n}\lambda_j^m f_{0m}=0, \tag{1.4}$$

where we take $f_{0n}=1$, to compute λ_j. The constants μ_j are given by

$$\mu_j=-\left(\sum_{m=0}^{n-1}\lambda_j^m f_{1m}\right)\Big/\left(\sum_{m=1}^{n}m\lambda_j^{m-1}f_{0m}\right). \tag{1.5}$$

1991 *Mathematics Subject Classification.* Primary: 34E05; secondary: 34A20.

Key words and phrases. Asymptotic expansions, Borel-Laplace transform, connection formulaes, differential equations, exponential improvement, hyperasymptotics, hyperterminants, irregular singularity, remainder term, Stokes multiplier.

For the coefficients a_{sj} we obtain the recurrence relation

$$(s-1)a_{s-1,j}\sum_{m=1}^{n} m\lambda_j^{m-1}f_{0,m}=\sum_{t=2}^{s}a_{s-t,j}\sum_{p=0}^{t}(\mu_j+t-s)_p\sum_{m=p}^{n}\binom{m}{p}\lambda_j^{m-p}f_{t-p,m}, \tag{1.6}$$

where Pochhammer's symbol $(\alpha)_p$ is defined by $(\alpha)_p=\Gamma(\alpha+p)/\Gamma(\alpha)$.

We shall impose the restriction

$$\lambda_j\neq\lambda_k,\qquad j\neq k. \tag{1.7}$$

This restriction ensures that the left-hand side of (1.6) does not vanish.

The set up of this paper is as follows. In §2 we define the solutions of (1.1), which will be the Borel-Laplace transforms of (1.3), Stokes multipliers, some important numbers that will determine the optimal number of terms in the hyperasymptotic expansions, and the hyperterminants. We finish §2 with the hyperasymptotic expansion of the coefficients a_{sk}, as $s\to\infty$.

§3 is a short section in which we explain why we need hyperasymptotic expansions at all.

In §4 we illustrate how the hyperasymptotic expansion of a_{sk} can be used to compute all the Stokes multipliers to arbitrary precision. We will see that all the numerical work can be reduced to solving linear systems of equations.

We finish this papers with some remarks in §5.

2. Definitions and lemmas

We define

$$\left.\begin{aligned}\theta_{kj}&=\mathrm{ph}\,(\lambda_j-\lambda_k),\\ \lambda_{kj}&=\lambda_k-\lambda_j,\\ \mu_{kj}&=\mu_k-\mu_j,\end{aligned}\right\}\quad j\neq k,\qquad\text{and}\quad \tilde{\mu}=\max\{\Re\mu_1,\cdots,\Re\mu_n\}. \tag{2.1}$$

We call

$$\eta\in\mathbb{R}\text{ is }\textit{admissible}\iff\eta\neq\theta_{kj}\ (\mathrm{mod}\ 2\pi),\qquad 1\leqslant j,k\leqslant n,\ j\neq k. \tag{2.2}$$

For an example with $n=3$ see Figure 1.

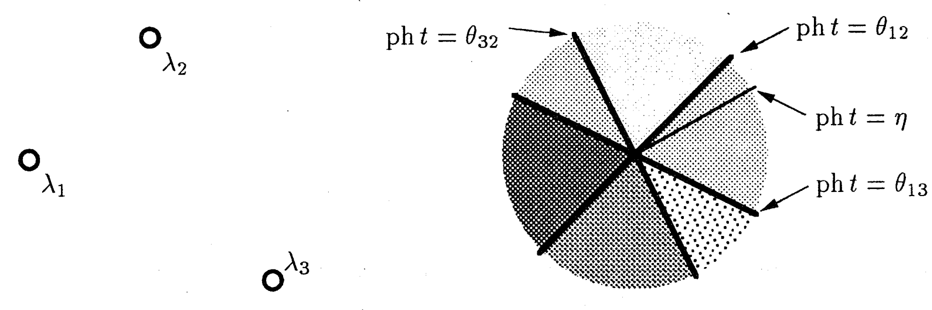

Figure 1. An example of admissible and non admissible directions.

For fixed admissible η we consider a t-plane together with parallel cuts from each λ_k to ∞ along the ray $\mathrm{ph}\,(t-\lambda_k)=\eta$. See Figure 2. If we specify for $k=1,2,\cdots,n$,

$$\log(t-\lambda_k)=\log|t-\lambda_k|+i\eta, \tag{2.3}$$

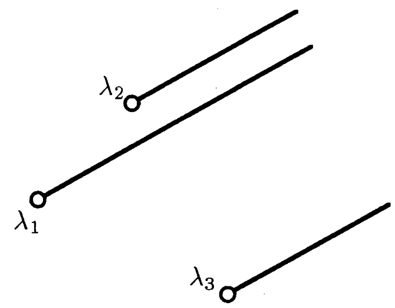

Figure 2. Cuts for $\mathcal{P}_\eta$.

for all t such that $\mathrm{ph}\,(t-\lambda_k)=\eta$, then we denote the t-plane with these cuts and choices of logarithms by $\mathcal{P}_\eta$. Thus $\log(t-\lambda_k)$ is continuous within $\mathcal{P}_\eta$, and is defined by (2.3) on $\mathrm{ph}\,(t-\lambda_k)=\eta$.

Let η be admissible. Then we define

$$\eta^- = \inf\{\check{\eta}<\eta \mid \tilde{\eta} \text{ is admissible for all } \tilde{\eta}\in(\check{\eta},\eta]\}, \tag{2.4a}$$
$$\eta^+ = \sup\{\hat{\eta}>\eta \mid \tilde{\eta} \text{ is admissible for all } \tilde{\eta}\in[\eta,\hat{\eta})\}, \tag{2.4b}$$
$$\mathcal{I}_\eta = (\eta^-,\eta^+). \tag{2.4c}$$

Note that $\eta^\pm$ are not admissible. In the example η^- is the value θ_{13} (mod 2π) for which $|\eta-\theta_{13}|$ is least, and η^+ is the value θ_{12} (mod 2π) for which $|\theta_{12}-\eta|$ is least.

With these definitions for $\eta^\pm$ we define the z-sectors

$$\mathcal{S}(\eta)=\{z \mid \Re(ze^{i\eta})<-a \text{ and } \frac{\pi}{2}-\eta^+<\mathrm{ph}\,z<\frac{3\pi}{2}-\eta^-\}, \tag{2.5}$$

The main tools that we will use in this paper are Theorems 1 and 2 of [1]. If we translate the results of these theorems to our notation we obtain:

Lemma 1. *The function $y_k(t)$ defined by*

$$y_k(t)=\sum_{p=0}^{\infty} a_{pk}\Gamma(\mu_k+1-p)(t-\lambda_k)^{p-\mu_k-1}, \qquad |t-\lambda_k|<\min_{j\neq k}|\lambda_j-\lambda_k|, \tag{2.6}$$

is analytic in $\mathcal{P}_\eta$, satisfies

$$y_k(t)=\frac{K_{jk}}{1-e^{-2\pi i\mu_k}}y_j(t)+reg(t-\lambda_j), \qquad j\neq k, \tag{2.7}$$

where the K_{jk} are constants, and can be continued analytically along every path that does not intersect any of the points $\lambda_1,\cdots,\lambda_n$. Furthermore, if S is any sector in the t-plane of the form $S=\{|t|>R,\ \alpha<\mathrm{ph}\,t<\beta\}$ with $0<\beta-\alpha<2\pi$ and $R>\max|\lambda_j|$, then

$$\lim_{t\to\infty} e^{-(a+\varepsilon)|t|}y_k(t)=0, \qquad t\in S, \tag{2.8}$$

for $\varepsilon>0$ arbitrary.

In (2.7) $reg(t-\lambda_j)$ denotes a function that is regular (or analytic) in a neighbourhood of $t=\lambda_j$.

Lemma 2. *Let $\eta\in\mathbb{R}$ be admissible. If we define*

$$w_k(z,\eta)=\frac{1}{2\pi i}\int_{\gamma_k(\eta)} e^{zt}y_k(t)\,dt, \tag{2.9}$$

where $\gamma_k(\eta)$ *is the contour in* $\mathcal{P}_\eta$ *from* ∞ *along the left-hand side of the cut* $\mathrm{ph}\,(t-\lambda_k)=\eta$, *around* λ_k *in the positive sense, and back to* ∞ *along the right-hand side of the cut, then* $w_k(z,\eta)$ *is a solution of* (1.1), $w_k(z,\tilde{\eta})=w_k(z,\eta)$ *for all* $\tilde{\eta}\in\mathcal{I}_\eta$, *and*

$$w_k(z,\eta)\sim e^{\lambda_k z}z^{\mu_k}\sum_{s=0}^{\infty}a_{sk}z^{-s}, \tag{2.10}$$

as $z\to\infty$ *in* $\mathcal{S}(\eta)$.

For each admissible η we have n solutions $w_1(z,\eta),\cdots,w_n(z,\eta)$. Since (1.1) is a linear ordinary differential equation of order n, for each admissible $\tilde{\eta}$ and $k\in\{1,\cdots,n\}$ there are connection coefficients $C_{jk}(\tilde{\eta},\eta)$ such that

$$w_k(z,\tilde{\eta})=C_{1k}(\tilde{\eta},\eta)w_1(z,\eta)+\cdots+C_{nk}(\tilde{\eta},\eta)w_n(z,\eta). \tag{2.11}$$

If $\tilde{\eta}\in\mathcal{I}_\eta$, then $C_{jk}(\tilde{\eta},\eta)=\delta_{jk}$. Hence, the connection coefficients can change only when we cross a non admissible direction. The corresponding directions in the z-plane are generally known as Stokes lines. To compute all the connection coefficients it suffices to compute the connection coefficients of two neighbouring intervals $\mathcal{I}_\eta$.

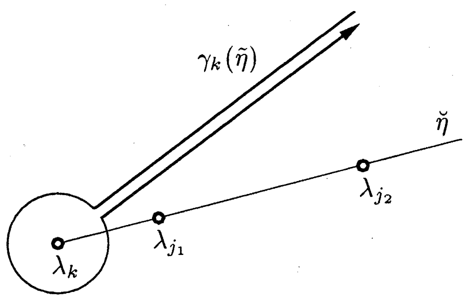

Figure 3. $\gamma_k(\tilde{\eta})$ before the rotation.

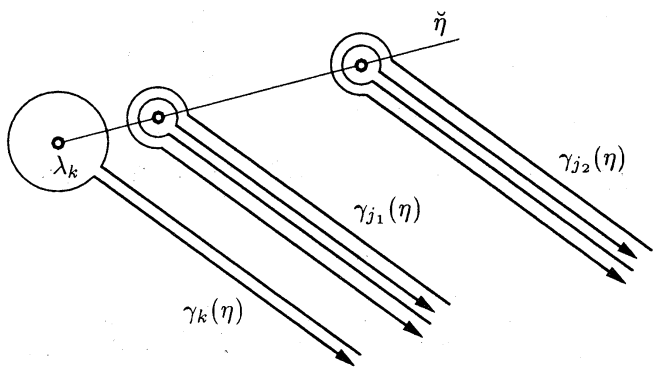

Figure 4. $\gamma_k(\tilde{\eta})$ after the rotation.

Take $\eta<\tilde{\eta}$ in two neighbouring intervals $\mathcal{I}_\eta$ and $\mathcal{I}_{\tilde{\eta}}$, and let $\breve{\eta}$ be the non admissible direction between η and $\tilde{\eta}$. Fix $k\in\{1,\cdots,n\}$ and let $j_1,\cdots,j_p$ be all the $j\neq k$ such that $\theta_{kj}=\breve{\eta}\pmod{2\pi}$. See Figure 3. If we rotate the contour $\gamma_k(\tilde{\eta})$ across the Stokes line at $\mathrm{ph}\,(t-\lambda_k)=\breve{\eta}$ we obtain the contour $\gamma_k(\eta)$ plus for each j_l contours $\gamma_{j_l}(\eta)$ and $\tilde{\gamma}_{j_l}(\eta)$. The contour $\tilde{\gamma}_{j_l}(\eta)$ is the inner contour of the two contours encircling λ_{j_l} in Figure 4, it is contour $\gamma_{j_l}(\eta)$ with the opposite direction of integration and it lies on the Riemann sheet $\log(t-\lambda_k)\in[\eta+2\pi,\eta+4\pi)$; furthermore $\log(t-\lambda_j)\in[\eta,\eta+2\pi)$, $j\neq k$. Hence, with (2.7) we obtain

$$\begin{aligned} w_k(z,\tilde{\eta}) &= w_k(z,\eta)+\sum_{l=1}^{p}\frac{1-e^{-2\pi i\mu_k}}{2\pi i}\int_{\gamma_{j_l}(\eta)}e^{zt}y_k(t)\,dt \\ &= w_k(z,\eta)+\sum_{l=1}^{p}\frac{K_{j_lk}}{2\pi i}\int_{\gamma_{j_l}(\eta)}e^{zt}y_{j_l}(t)\,dt \\ &= w_k(z,\eta)+\sum_{l=1}^{p}K_{j_lk}w_{j_l}(z,\eta). \end{aligned} \tag{2.12}$$

The constants K_{jk} are called the *Stokes multipliers*, and if we can compute the Stokes multipliers, then we can compute all the connection coefficients.

The Stokes multipliers play an important role in the definitions of the following numbers. Let

$$\alpha_k^{(m)} = \min\left\{ |\lambda_k - \lambda_{j_0}| + |\lambda_{j_0} - \lambda_{j_1}| + \cdots + |\lambda_{j_{m-1}} - \lambda_{j_m}| \,\middle|\, j_0 \neq k, K_{j_0 k} \neq 0, j_l \neq j_{l-1}, K_{j_l j_{l-1}} \neq 0 \right\}. \tag{2.13}$$

If $G = (V, E)$ is a directed graph with vertices $V = \{\lambda_1, \cdots, \lambda_n\}$ and edges $E = \{(\lambda_p, \lambda_q) \mid 1 \leqslant p, q \leqslant n, p \neq q,\ K_{qp} \neq 0\}$, then $\alpha_k^{(m)}$ is the length of the shortest directed path of m steps starting at λ_k.

In definition (2.13) we assume that we can determine whether $K_{pq} \neq 0$ or $K_{pq} = 0$. Usually we do not have this knowledge. Hence, we have to use the definition

$$\alpha_k^{(m)} = \min\left\{ |\lambda_k - \lambda_{j_0}| + |\lambda_{j_0} - \lambda_{j_1}| + \cdots + |\lambda_{j_{m-1}} - \lambda_{j_m}| \,\middle|\, j_0 \neq k, j_l \neq j_{l-1} \right\}. \tag{2.13a}$$

To define the hyperterminants we shall use the notation

$$\int_\lambda^{[\eta]} = \int_\lambda^{\infty e^{i\eta}}, \qquad \eta \in \mathbb{R}. \tag{2.14}$$

Let l be a nonnegative integer, $\Re M_j > 1$, $\sigma_j \in \mathbb{C}$, $\sigma_j \neq 0$, $j = 0, \cdots, l$. Then

$$\begin{aligned} F^{(0)}(z) &= 1, \\ F^{(1)}\left(z; \begin{matrix} M_0 \\ \sigma_0 \end{matrix}\right) &= \int_0^{[\pi-\theta_0]} \frac{e^{\sigma_0 t_0} t_0^{M_0-1}}{z - t_0}\, dt_0, \\ F^{(l+1)}\left(z; \begin{matrix} M_0, \cdots, M_l \\ \sigma_0, \cdots, \sigma_l \end{matrix}\right) &= \int_0^{[\pi-\theta_0]} \cdots \int_0^{[\pi-\theta_l]} \frac{e^{\sigma_0 t_0 + \cdots + \sigma_l t_l} t_0^{M_0-1} \cdots t_l^{M_l-1}}{(z - t_0)(t_0 - t_1)\cdots(t_{l-1} - t_l)}\, dt_l \cdots dt_0, \end{aligned} \tag{2.15}$$

where $\theta_j = \operatorname{ph}\sigma_j$, $j = 0, 1, \cdots, l$. In the case $\operatorname{ph}\sigma_j = \operatorname{ph}\sigma_{j+1} \pmod{2\pi}$ we have to make the choice between the t_j-contour being on the 'left' or 'right' of the t_{j+1}-contour. We make the choice via the definition

$$F^{(l+1)}\left(z; \begin{matrix} M_0, \cdots, M_l \\ \sigma_0, \cdots, \sigma_l \end{matrix}\right) = \lim_{\varepsilon \downarrow 0} F^{(l+1)}\left(z; \begin{matrix} M_0, & M_1, & \cdots, & M_{l-1}, & M_l \\ \sigma_0 e^{-l\varepsilon i}, & \sigma_1 e^{-(l-1)\varepsilon i}, & \cdots, & \sigma_{l-1} e^{-\varepsilon i}, & \sigma_l \end{matrix}\right). \tag{2.16}$$

The multiple integrals converge when $-\pi - \theta_0 < \operatorname{ph} z < \pi - \theta_0$. In [12] it is explained how to compute these hyperterminants.

The main tool to compute the Stokes multipliers will be the following theorem. The proof of this theorem is given in §7 of [11].

Theorem 1. *Let l be an arbitrary nonnegative integer, then*

$$
\begin{aligned}
a_{N_k^{(0)}k} = &- \sum_{k_1 \neq k} \frac{K_{k_1 k}}{2\pi i} \left\{ \sum_{s=0}^{N_{k_1}^{(1)}-1} a_{sk_1} (-1)^{N_k^{(0)}-s} e^{\mu_{k_1 k}\pi i} \lambda_{k_1 k}^{s-N_k^{(0)}-\mu_{k_1 k}} \Gamma(N_k^{(0)} - s + \mu_{k_1 k}) \right. \\
&+ \sum_{k_2 \neq k_1} \frac{K_{k_2 k_1}}{2\pi i} \left\{ \sum_{s=0}^{N_{k_2}^{(2)}-1} a_{sk_2} F^{(2)} \left(0; \begin{array}{cc} (N_k^{(0)}+1) - N_{k_1}^{(1)} + \mu_{k_1 k} + 1, & N_{k_1}^{(1)} - s + \mu_{k_2 k_1} \\ \lambda_{k_1 k}, & \lambda_{k_2 k_1} \end{array} \right) \right. \\
&\ddots \\
&+ \sum_{k_l \neq k_{l-1}} \frac{K_{k_l k_{l-1}}}{2\pi i} \left\{ \sum_{s=0}^{N_{k_l}^{(l)}-1} a_{sk_l} F^{(l)} \left(0; \begin{array}{cc} (N_k^{(0)}+1) - N_{k_1}^{(1)} + \mu_{k_1 k} + 1, \cdots, \\ \lambda_{k_1 k}, \quad \cdots, \end{array} \right. \right. \\
&\qquad \left. \begin{array}{cc} N_{k_{l-2}}^{(l-2)} - N_{k_{l-1}}^{(l-1)} + \mu_{k_{l-1}k_{l-2}} + 1, & N_{k_{l-1}}^{(l-1)} - s + \mu_{k_l k_{l-1}} \\ \lambda_{k_{l-1}k_{l-2}}, & \lambda_{k_l k_{l-1}} \end{array} \right) \\
&\left. \left. \left. \right\} \cdots \right\} \right\} + r_k^{(l)}(N_k^{(0)}).
\end{aligned}
\tag{2.17}
$$

If we take

$$
N_{k_j}^{(j)} = \frac{\beta_{k_j}^{(j)}}{\beta_k^{(0)}} N_k^{(0)} + \mathcal{O}(1), \qquad j = 1 \cdots l, \tag{2.18}
$$

where

$$
\beta_k^{(0)} = \alpha_k^{(l)} \qquad \text{and} \qquad \beta_{k_j}^{(j)} = \max\left(0, \beta_{k_{j-1}}^{(j-1)} - |\lambda_{k_j k_{j-1}}|\right), \qquad j = 1 \cdots l, \tag{2.19}
$$

then

$$
r_k^{(l)}(N_k^{(0)}) = \Gamma(N_k^{(0)}) \left(\alpha_k^{(l)}\right)^{-N_k^{(0)}} \mathcal{O}\left(\left(N_k^{(0)}\right)^{\tilde{\mu}-\mu_k+(l+2)/2}\right), \tag{2.20}
$$

as $N_k^{(0)} \to \infty$.

3. Why do we need hyperasymptotics?

In the case $l = 1$, Theorem 1 reads

$$
\begin{aligned}
a_{N_k^{(0)}k} = \sum_{k_1 \neq k} \frac{K_{k_1 k}}{2\pi i} \sum_{s=0}^{N_{k_1}^{(1)}-1} a_{sk_1} (-1)^{N_k^{(0)}-s} e^{\mu_{k_1 k}\pi i} \lambda_{k_1 k}^{s-N_k^{(0)}-\mu_{k_1 k}} \Gamma(N_k^{(0)} - s + \mu_{k_1 k}) \\
+ r_k^{(1)}(N_k^{(0)}).
\end{aligned}
\tag{3.1}
$$

By taking $N_{k_1}^{(1)} = \left(\beta_{k_1}^{(1)}/\beta_k^{(0)}\right) N_k^{(0)}$, we obtain an optimally truncated asymptotic expansion, which can also be obtained via Darboux's method. See §II.6 in [15]. Since we can use (1.6) to compute the coefficients a_{pq} in (3.1), and since we have estimate (2.20), we can use (3.1) to approximate some of the Stokes multipliers.

Note that in (3.1), we multiply $K_{k_1 k}$ times a factor Q_{k_1} which is of order $\lambda_{k_1 k}^{-N_k^{(0)}} \Gamma(N_k^{(0)} + \mu_{k_1 k})\mathcal{O}(1)$, as $N_k^{(0)} \to \infty$. When we compare the order estimate of Q_{k_1} with (2.20), in the case $l = 1$, we see that in the case $|\lambda_{k_1k}| \geqslant \alpha_k^{(1)}$ the order estimate of Q_{k_1} is smaller than $r_k^{(1)}(N_k^{(0)})$. Since, in the case $l=1$, we take $\beta_{k_1} = \max(0, \alpha_k^{(1)} - |\lambda_{k_1k}|)$, this observation is integrated in the definition of β_{k_1}, and hence, in the definition of $N_{k_1}^{(1)}$.

Hence, only Stokes multipliers K_{k_1k} for which $|\lambda_{k_1k}| < \alpha_k^{(1)}$, can be computed via (3.1). If we want to compute K_{k_1k} for which $|\lambda_{k_1k}| \geqslant \alpha_k^{(1)}$, we have to use (2.17) up to level l, where l is a positive integer such that $|\lambda_{k_1k}| < \alpha_k^{(l)}$. Note that, since we have $\alpha_k^{(l)} \geqslant (l+1)\min_{p\neq q}|\lambda_{pq}|$, we know that a positive integer l exists such that $|\lambda_{k_1k}| < \alpha_k^{(l)}$.

In the next section, we illustrate how hyperasymptotic expansion (2.17) can be used to compute all Stokes multipliers.

4. An example

We use the example

$$w^{(4)}(z) - 3w^{(3)}(z) + \left(\tfrac{9}{4} + \tfrac{1}{2}z^{-2}\right)w^{(2)}(z) - \left(3 + \tfrac{3}{4}z^{-2}\right)w'(z) + \left(\tfrac{5}{4} + \tfrac{9}{16}z^{-2}\right)w(z) = 0, \tag{4.1}$$

and our goal will be to compute K_{13}, K_{23} and K_{43} up to 8 digits precision. Hence, we take $k=3$ in Theorem 1. In this example we have $n=4$ and

$$\lambda_1 = \tfrac{1}{2}, \quad \lambda_2 = \tfrac{5}{2}, \quad \lambda_3 = i, \quad \lambda_4 = -i, \quad \mu_j = 0, \quad j = 1, \cdots, 4. \tag{4.2}$$

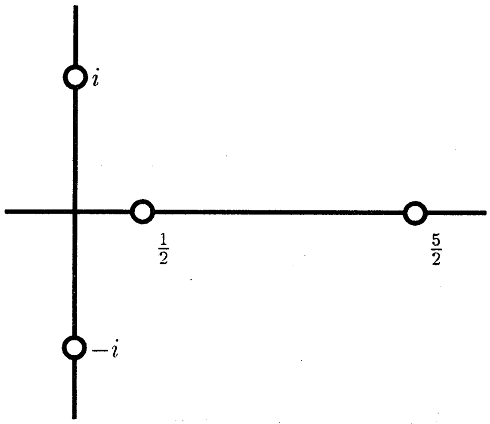

Figure 5. The distribution of the λ's.

4.1. Computing Stokes multipliers via the level 1 version of Theorem 1. We will first use Theorem 1 in the case $l=1$. Since,

$$\lambda_{13} = \tfrac{1}{2} - i, \qquad \lambda_{23} = \tfrac{5}{2} - i, \qquad \lambda_{43} = -2i, \tag{4.3}$$

we obtain from (2.13a) $\alpha_3^{(1)} = |\lambda_{13}| + |\lambda_{41}| = \sqrt{5} = 2.236\cdots$. Hence,

$$\beta_3^{(0)} = \alpha_3^{(1)} = 2.236\cdots, \quad \begin{cases} \beta_1^{(1)} = \max\left(0, \beta_3^{(0)} - |\lambda_{13}|\right) = \tfrac{1}{2}\sqrt{5} = 1.118\cdots, \\ \beta_2^{(1)} = \max\left(0, \beta_3^{(0)} - |\lambda_{23}|\right) = 0, \\ \beta_4^{(1)} = \max\left(0, \beta_3^{(0)} - |\lambda_{43}|\right) = \sqrt{5} - 2 = 0.236\cdots. \end{cases} \tag{4.4}$$

Hence, in the case $l=1$ we will be able to compute Stokes multipliers K_{13} and K_{43} to the required precision. The factor multiplying K_{13} is of order $|\lambda_{13}|^{-N_3^{(0)}}\Gamma(N_3^{(0)})$ and the factor multiplying K_{43} is of order $|\lambda_{43}|^{-N_3^{(0)}}\Gamma(N_3^{(0)})$. If we compare these factors with (2.20), which is in this case $r_3^{(1)}(N_3^{(0)}) = \left(\alpha_3^{(1)}\right)^{-N_3^{(0)}}\Gamma(N_3^{(0)})\mathcal{O}\left(\left(N_3^{(0)}\right)^{3/2}\right)$, we see that we need $N_3^{(0)}$ such that approximately

$$\left(\frac{|\lambda_{13}|}{\alpha_3^{(1)}}\right)^{N_3^{(0)}} \leqslant 10^{-9}, \qquad \text{and} \qquad \left(\frac{|\lambda_{43}|}{\alpha_3^{(1)}}\right)^{N_3^{(0)}} \leqslant 10^{-9}. \tag{4.5}$$

The reader can check that $N_3^{(0)} = 186$ is the first satisfactory number of terms. Note that the high number of terms is due to $|\lambda_{43}|/\alpha_3^{(1)}$ being so close to unity.

If we take in (3.1) $N_3^{(0)} = 186$, $N_1^{(1)} = 93$, $N_4^{(1)} = 44$ and compute the coefficients a_{pq} via (1.6) then we obtain

$$\begin{aligned}(1.00133073255\cdots - i0.97320367636\cdots)\times 10^{331}&\\ =&\frac{K_{13}}{2\pi i}(0.61708613155\cdots - i3.95787812069\cdots)\times 10^{331}\\ &-\frac{K_{43}}{2\pi i}(4.19763793947\cdots + i0.00004407518\cdots)\times 10^{284}\end{aligned} \tag{4.6a}$$

And if we take $N_3^{(0)} = 187$, $N_1^{(1)} = 93$, $N_4^{(1)} = 44$ we obtain

$$\begin{aligned}-(2.19311035862\cdots + i0.76592313764\cdots)\times 10^{333}&\\ =&-\frac{K_{13}}{2\pi i}(6.34843047388\cdots - i2.02641047191\cdots)\times 10^{333}\\ &-\frac{K_{43}}{2\pi i}(0.00004055835\cdots - i3.90383078089\cdots)\times 10^{286}\end{aligned} \tag{4.6b}$$

We combine the two results and obtain

$$\begin{aligned}K_{13} &= -1.31673553004 + 1.75027074192i,\\ K_{43} &= 0.35534060046 - 2.11723774454i.\end{aligned} \tag{4.7}$$

One reason to use the level 2 version of Theorem 1 is to compute K_{23}. We will see that a second reason might be that in the level 2 version of Theorem 1 we need less terms.

Remark 1. It follows from the 10^{331} and 10^{284} in (4.6a) that we have to perform our computation to 56 significant figures. Hence, by computing K_{43} to 8 significant figures we obtain K_{13} to 55 significant figures.

4.2. Computing Stokes multipliers via the level 2 version of Theorem 1. In the case $l=2$ we have $\beta_3^{(0)} = \alpha_3^{(2)} = \frac{3}{2}\sqrt{5}$ and

$$\beta_3^{(0)} = 3.354\cdots, \quad \begin{cases} \beta_1^{(1)} = 2.236\cdots, & \begin{cases}\beta_2^{(2)} = 0.236\cdots,\\ \beta_3^{(2)} = 1.118\cdots,\\ \beta_4^{(2)} = 1.118\cdots,\end{cases}\\ \beta_2^{(1)} = 0.661\cdots, & \quad \beta_j^{(2)} = 0, \quad j=1,3,4,\\ \beta_4^{(1)} = 1.354\cdots, & \begin{cases}\beta_1^{(2)} = 0.236\cdots,\\ \beta_j^{(2)} = 0, \quad j=2,3.\end{cases}\end{cases} \tag{4.8}$$

To compute the three Stokes multipliers to the required precision we need $N_3^{(0)}$ such that approximately

$$\left(\frac{|\lambda_{13}|}{\alpha_3^{(2)}}\right)^{N_3^{(0)}} \leqslant 10^{-9}, \qquad \left(\frac{|\lambda_{23}|}{\alpha_3^{(2)}}\right)^{N_3^{(0)}} \leqslant 10^{-9}, \qquad \text{and} \qquad \left(\frac{|\lambda_{43}|}{\alpha_3^{(2)}}\right)^{N_3^{(0)}} \leqslant 10^{-9}. \tag{4.9}$$

Now $N_3^{(0)} = 95$ is the first satisfactory number of terms. According to (2.18) and (4.8) we have to take

$$\begin{aligned} N_1^{(1)} &= 64, \quad \begin{cases} N_2^{(2)} = 7, \\ N_3^{(2)} = 32, \\ N_4^{(2)} = 32, \end{cases} \\ N_2^{(1)} &= 19, \qquad N_j^{(2)} = 0, \quad j = 1,3,4, \\ N_4^{(1)} &= 39, \quad \begin{cases} N_1^{(2)} = 7, \\ N_j^{(2)} = 0, \quad j = 2,3. \end{cases} \end{aligned} \tag{4.10}$$

The level 2 version of Theorem 1 is in this example:

$$\begin{aligned} a_{N_3^{(0)}3} = &- \frac{K_{13}}{2\pi i}\left\{ \sum_{s=0}^{N_1^{(1)}-1} a_{s1}(-)^{N_3^{(0)}-s}(\tfrac{1}{2}-i)^{s-N_3^{(0)}}\Gamma(N_3^{(0)}-s) \right. \\ &\qquad + \frac{K_{21}}{2\pi i}\sum_{s=0}^{N_2^{(2)}-1} a_{s2}F^{(2)}\left(0;\begin{matrix} N_3^{(0)}-N_1^{(1)}+2, & N_1^{(1)}-s \\ \frac{1}{2}-i, & 2 \end{matrix}\right) \\ &\qquad + \frac{K_{31}}{2\pi i}\sum_{s=0}^{N_3^{(2)}-1} a_{s3}F^{(2)}\left(0;\begin{matrix} N_3^{(0)}-N_1^{(1)}+2, & N_1^{(1)}-s \\ \frac{1}{2}-i, & i-\frac{1}{2} \end{matrix}\right) \\ &\qquad \left. + \frac{K_{41}}{2\pi i}\sum_{s=0}^{N_4^{(2)}-1} a_{s4}F^{(2)}\left(0;\begin{matrix} N_3^{(0)}-N_1^{(1)}+2, & N_1^{(1)}-s \\ \frac{1}{2}-i, & -i-\frac{1}{2} \end{matrix}\right)\right\} \\ &- \frac{K_{23}}{2\pi i}\sum_{s=0}^{N_2^{(1)}-1} a_{s2}(-)^{N_3^{(0)}-s}(\tfrac{5}{2}-i)^{s-N_3^{(0)}}\Gamma(N_3^{(0)}-s) \\ &- \frac{K_{43}}{2\pi i}\left\{ \sum_{s=0}^{N_4^{(1)}-1} a_{s4}(-)^{N_3^{(0)}-s}(-2i)^{s-N_3^{(0)}}\Gamma(N_3^{(0)}-s) \right. \\ &\qquad \left. + \frac{K_{14}}{2\pi i}\sum_{s=0}^{N_1^{(2)}-1} a_{s1}F^{(2)}\left(0;\begin{matrix} N_3^{(0)}-N_1^{(1)}+2, & N_1^{(1)}-s \\ -2i, & i+\frac{1}{2} \end{matrix}\right)\right\} \\ &+ r_3^{(2)}(N_3^{(0)}). \end{aligned} \tag{4.11}$$

Although our goal is the computation of K_{13}, K_{23} and K_{43}, other Stokes multipliers appear in (4.11). We will compute the other Stokes multipliers via the level 1 version of Theorem 1 and then use their values in (4.11). To compute the Stokes multipliers K_{21}, K_{31} and K_{41} we use the level 1 hyperasymptotic expansion of $a_{N_1^{(1)},1}$, $a_{N_1^{(1)}+1,1}$ and $a_{N_1^{(1)}+2,1}$. If we take the value of $N_1^{(1)}$ that is given in (4.10), then we compute the three Stokes multipliers exactly to the precision that is required in (4.11). The details of the computations are very similar to §4.1, and the result is

$$
\begin{aligned}
K_{21} &= 0.326930886422i, \\
K_{31} &= -0.335184742943 - 0.173943737206i, \\
K_{41} &= 0.335184742943 - 0.173943737206i.
\end{aligned}
\tag{4.12}
$$

The level 1 hyperasymptotic expansion of $a_{N_4^{(1)}-1,4}$ yields

$$K_{14} = 1.31673552831 + 1.750270074369i. \tag{4.13}$$

If we substitute (4.12) and (4.13) into (4.11) and use (1.6) then the only unknowns are K_{13}, K_{23} and K_{34}. By taking (4.10) and $N_3^{(0)} = 95, 96, 97$, and ignoring $r_3^{(2)}(N_3^{(0)})$, we obtain three linear equations with three unknowns. The solution of this system is

$$
\begin{aligned}
K_{13} &= -1.31673553004 + 1.750270074192i, \\
K_{23} &= 0.959989170107 - 0.3257646597 48i, \\
K_{43} &= 0.35534060049 - 2.11723774458i.
\end{aligned}
\tag{4.14}
$$

Remark 2. Note that by first computing K_{21}, K_{31}, K_{41} and K_{14} via level 1 versions of Theorem 1, and then using their values in (4.11), the numerical computations are reduced to solving linear systems of equations.

Remark 3. In §4.1 we needed $N_3^{(0)} = 186$ coefficients a_{s3}, and in §4.2 we needed $N_3^{(0)} = 95$ of these coefficients. Hence, by increasing the level l the number of coefficients that we need to achieve the required precision decreases.

5. Conclusions and generalisations

The method that we described in this paper can also be used to compute Stokes multipliers for integrals with saddles. In the case of integrals the Stokes multipliers can only have a finite number of values, which means that in order to determine the exact value of Stokes multipliers for integrals with saddles we need to approximate them only to a very low precision.

The Stokes multipliers for integrals with saddles contain the following important information:

$$K_{jk} \neq 0 \quad \Longleftrightarrow \quad \text{saddle point } j \text{ is adjacent to saddle point } k.$$

For more details see [2], [3], [5] and [11].

There are several results in the literature ([4], [6]–[10] and [13]) on the computation of Stokes multipliers. Our results can be seen as a direct generalisation of those in [13]. Many of the other results are of the form (3.1), but with the right-hand side replaced by its dominant term. With additional terms available on the right-hand side we have a more powerful way of computing the Stokes multipliers. As is explained in §3 and illustrated in §4, it is in general not possible to compute all the Stokes multipliers from (3.1). Theorem 1 is a generalisation of (3.1), and with this expansion we can compute the 'difficult' Stokes multipliers as well.

Other analytical methods for computing the difficult Stokes multipliers are based on conformal mappings in the t-plane (Borel-plane). See, for example, [10]. However, construction of the correct conformal mappings is still a difficult problem.

A numerical method for computing all the Stokes multipliers is discussed in [13] and [14]. This method is based on direct numerical integration of the differential equation.

Acknowledgment. The author is pleased to acknowledge the support of the Research Institute for Mathematical Sciences and the Japan Society for the Promotion of Science for his stay at RIMS in Kyoto, Japan, in January 1998.

References

1. W. Balser, W. B. Jurkat and D. A. Lutz, *On the reduction of connection problems for differential equations with an irregular singular point to ones with only regular singularities I*, SIAM J. Math. Anal. **12** (1981), 691–721.
2. M. V. Berry and C. J. Howls, *Hyperasymptotics for integrals with saddles*, Proc. Roy. Soc. London, Ser. A**434** (1991), 657–675.
3. W. G. C. Boyd, *Error bounds for the method of steepest descents*, Proc. Roy. Soc. London, Ser. A**440** (1993), 493–518.
4. B. L. J. Braaksma, *Multisummability and Stokes multipliers of linear meromorphic differential equations*, J. Differential Equations **92** (1991), 45–75.
5. C. J. Howls, *Hyperasymptotics for multidimensional integrals, exact remainder terms and the global connection problem*, Proc. Roy. Soc. London, Ser. A**453** (1997), 2271–2294.
6. G. K. Immink, *A note on the relationship between Stokes multipliers and formal solutions of analytic differential equations*, SIAM J. Math. Anal. **21** (1990), 782–792.
7. W. B. Jurkat, D. A. Lutz and A. Peyerimhoff, *Birkhoff invariants and effective calculations for meromorphic linear differential equations. I*, J. Math. Anal. Appl. **53** (1976), 438–470.
8. ———, *Birkhoff invariants and effective calculations for meromorphic linear differential equations, II*, Houston J. Math. **2** (1976), 207–238.
9. M. Loday-Richaud, *Calcul des invariants de Birkhoff des systèmes d'ordre deux*, Funkcial. Ekvac. **33** (1990), 161–225.
10. D. A. Lutz and R. Schäfke, *Calculating connection coefficients for meromorphic differential equations*, [Manuscript.]
11. A. B. Olde Daalhuis, *Hyperasymptotic solutions of higher order linear differential equations with a singularity of rank one*, Proc. Roy. Soc. London, Ser. A**454** (1998), 1–29.
12. ———, *Hyperterminants II*, J. Comp. Appl. Math. (1998). (In the press.)
13. A. B. Olde Daalhuis and F. W. J. Olver, *On the calculation of Stokes multipliers for linear differential equations of the second order*, Methods Appl. Anal. **2** (1995), 348–367.
14. ———, *On the asymptotic and numerical solution of linear ordinary differential equations*, SIREV (1998). (In the press.)
15. R. Wong, Asymptotic Approximations of Integrals, Academic Press, New York, (1989).